# RATIONAL CURVES ON SMOOTH CUBIC HYPERSURFACES

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Abstract. We prove that the space of rational curves of a fixed degree on *any* smooth cubic hypersurface of dimension at least four is irreducible and of the expected dimension. Our methods also show that the space of rational curves of a fixed degree on a *general* hypersurface in $\mathbb{P}^n$ of degree $2d \leq \min(n+4, 2n-2)$ and dimension at least three is irreducible and of the expected dimension.

Contents



## 1. Introduction

The arithmetic and geometric properties of a variety (e.g., existence of sections in families, weak approximation, unirationality) are closely tied to the geometry of the space of rational curves on that variety. Unfortunately, even for hypersurfaces, some of the basic properties of the space of rational curves, such as the dimension and the number of irreducible components, are not known. The purpose of this paper is to prove the irreducibility and calculate the dimension of the space of rational curves for all smooth cubic hypersurfaces and general Fano hypersurfaces of low degree. We also formulate some conjectures about the dimension and the irreducibility of the space of rational curves on general Fano hypersurfaces.

Let $X$ be a smooth hypersurface in $\mathbb{P}^n$ over the field of complex numbers $\mathbb{C}$. The Kontsevich moduli space $\overline{\mathcal{M}}_{0,m}(X,e)$ parameterizes maps $(C, p_1, \dots, p_m, f)$ such that

(1) The domain $C$ of the map $f$ is a proper, connected, at-worst-nodal curve of arithmetic genus zero;
(2) $p_1, \dots, p_m$ are $m$ distinct, smooth points on $C$ called marked points;
(3) $f : C \to X$ is a stable map (i.e., every irreducible component of $C$ mapped to a point by $f$ has at least three nodes or marked points) such that $f^*\mathcal{O}_X(1)$ has degree $e$.

The Kontsevich moduli space provides a very useful "compactification" of the space of rational curves of degree $e$ on $X$. The following is the main theorem of this note.

**Theorem 1.1.** *Let $X$ be a smooth cubic hypersurface in $\mathbb{P}^n$ for $n > 4$. Then $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible of the expected dimension*
$$\dim(\overline{\mathcal{M}}_{0,0}(X,e)) = e(n-2) + n - 4$$
*for every $e \geq 1$.*

In his thesis, the second author analyzed the Kontsevich moduli spaces of rational curves when the target is a smooth cubic threefold.

**Theorem 1.2** (Theorem 62 [S])**.** *Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold. Then $\overline{\mathcal{M}}_{0,0}(X,e)$ has two irreducible components $R_e$ and $N_e$ each of the expected dimension $2e$. A general point of $R_e$ parameterizes a smooth rational curve of degree $e$ on $X$. A general point of $N_e$ parameterizes a degree $e$ cover of a line on $X$.*

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The proof of Theorem 1.1 is by induction on the degree $e$. The space of lines on a cubic hypersurface of dimension at least three is smooth, irreducible and of the expected dimension. Moreover, the evaluation morphism

$$ev_1 : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$$

from pointed lines to $X$ is flat in the complement of finitely many points. When $ev_1$ is not flat, its fiber dimension increases by at most one. By Mori's bend-and-break argument, every component of the space of rational curves of degree $e$ contains reducible elements. Inductively, it follows that

$$ev_e : \overline{\mathcal{M}}_{0,1}(X, e) \to X$$

is flat in the complement of finitely many points. By a dimension count, we conclude that any irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e)$ contains the locus of maps with reducible domains as a divisor. We thus compute the dimension of $\overline{\mathcal{M}}_{0,0}(X, e)$ inductively. Finally, we show that every irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e)$ contains the locus of maps from a chain of $e$ rational curves to a chain of $e$ lines and that the space is smooth at a general such chain. It follows that $\overline{\mathcal{M}}_{0,0}(X, e)$ is irreducible.

Theorem 1.1 is interesting because the statement holds for *every* smooth cubic hypersurface. In [HRS], the authors prove that if $X$ is a general hypersurface of degree $d < (n + 1)/2$, then $\overline{\mathcal{M}}_{0,0}(X, e)$ is irreducible of the expected dimension $\dim(\overline{\mathcal{M}}_{0,0}(X, e)) = e(n + 1 - d) + n - 4$. However, there are many smooth hypersurfaces in this range for which $\overline{\mathcal{M}}_{0,0}(X, e)$ is reducible with components of larger than the expected dimension. For example, let $d \geq 4$, $e \geq 3$ and $X$ be a smooth hypersurface with a conical hyperplane section (such as the Fermat hypersurface $\sum_{i=0}^{n} x_i^d = 0$). Then $\overline{\mathcal{M}}_{0,0}(X, e)$ has components containing maps from a domain curve with $e + 1$ components consisting of $e$-rational curves $E_i$ attached at $e$ points to a single rational curve $C$ mapping the curves $E_i$ to $e$-lines passing through the vertex of the conical hyperplane section and contracting $C$ to the vertex. The dimension of the locus of such maps is $e(n - 3) + e - 3$. Note that this is larger than $\max(e(n + 1 - d) + n - 4, 0)$ if $e$ is large enough. Hence, the Kontsevich moduli space may be reducible with components that have larger than the expected dimension.

Our argument will use the assumption that $X$ has degree three in two ways. We need the degree of the hypersurface to be small enough so that for every point $p \in X$, every component of the space of degree $e$ rational curves passing through $p$ has reducible elements. Mori's bend-and-break argument guarantees this for rational curves of degree $e$ on a hypersurface of degree $d < n + 1$ if

$$\frac{e-1}{e}(n + 1) > d.$$

We also need the degree to be three to argue that the evaluation morphism is flat

$$ev_1 : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$$

in the complement of finitely many points and that where the map is not flat the fiber dimension increases exactly by one. The rest of the argument is formal and was developed in the second author's thesis [S]. While the irreducibility and the dimension estimates do not hold for every smooth hypersurface of higher degree, one may expect it to hold for a general Fano hypersurface. Here we state three conjectures that arose in conversations with Joe Harris with the hope of making them more widely known.

**Conjecture 1.3.** *Let $X \subset \mathbb{P}^n$ be a general Fano hypersurface of dimension at least three. Let $R_e(X)$ denote the closure of the locus in the Hilbert scheme parameterizing smooth rational curves of degree $e$ on $X$. Then $R_e(X)$ is irreducible of dimension $e(n + 1 - d) + n - 4$.*

[HRS] proves Conjecture 1.3 when $d < (n + 1)/2$ and shows that when $d \leq n - 1$, Conjecture 1.3 is implied by the following conjecture.

**Conjecture 1.4.** *Let $X \subset \mathbb{P}^n$ be a general hypersurface of dimension at least three and degree $d \leq n - 1$. Then the evaluation morphism*

$$ev_e : \overline{\mathcal{M}}_{0,1}(X, e) \to X$$

*is flat for $e \geq 1$.*

Theorem 2.1 in [HRS] proves that Conjecture 1.4 is true for $e = 1$. Let $e_0$ be the minimal integer such that $d < \frac{e_0}{e_0+1}(n+1)$. By Corollary 5.6 of [HRS], Conjecture 1.4 is true for hypersurfaces of degree $d$ in $\mathbb{P}^n$ if and only if it is true for $1 \leq e \leq e_0$. Moreover, by §4 of [HRS], Conjecture 1.4 is implied by the following conjecture.

**Conjecture 1.5.** *Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree $d \leq n-1$. Let $e \geq 2$. Let $p$ be any point on $X$. Let $F_e(p) = ev_e^{-1}(p) \subset \overline{\mathcal{M}}_{0,1}(X, e)$ be the fiber of the evaluation morphism over $p$. Then every irreducible component of $F_e(p)$ contains a map from a reducible curve into $X$.*

The argument in this note extends the results of [HRS] to include the following cases.

**Theorem 1.6.** *Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree $d$ of dimension at least three.*

(1) *If $2d \leq \min(n+4, 2n-2)$, then Conjecture 1.3 holds.*
(2) *If $2d \leq \min(n+3, 2n-3)$, then Conjecture 1.4 holds.*

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## 2. The proof of Theorem 1.1

In this section we carry out the strategy outlined in the introduction to prove Theorem 1.1.

**Lemma 2.1.** *Let $X$ be a smooth cubic hypersurface of dimension at least 3. If the evaluation morphism $ev_1 : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ from pointed lines to $X$ fails to be flat at a point $p \in X$, then the hyperplane section of $X$ tangent at $p$ is an irreducible cone with vertex $p$. Furthermore, the fiber of $ev_1$ over $p$ has dimension $n-3$.*

*Proof.* The fiber of $ev_1$ at $p \in X$ is the space of lines on $X$ containing the point $p$. We can realize the space of lines passing through $p$ as follows. Choose coordinates for $p$ such that $p = [0 : \cdots : 0 : 1]$ and the tangent hyperplane to $X$ at $p$ is given by $x_0 = 0$. Expand the equation of $X$ around $p$ in affine coordinates

$$x_0 + f_2(x_0, \dots, x_{n-1}) + f_3(x_0, \dots, x_{n-1}).$$

We can write the equation of a line passing through $p$ as $[a_0 t, a_1 t, \cdots, a_{n-1} t]$. The line is contained in $X$ if and only if $a_0 = 0$ and $f_2(0, a_1, \dots, a_{n-1}) = f_3(0, a_1, \dots, a_{n-1}) = 0$. Hence, the space of lines contained in $X$ and passing through $p$ can be viewed as the intersection of a quadratic and cubic form in the projectivized tangent space at $p$. Therefore, the morphism $ev_1$ fails to be flat at $p$ if and only if $f_2(0, x_1, \dots, x_{n-1})$ and $f_3(0, x_1, \dots, x_n)$ fail to intersect in a variety of dimension $n-4$. This in turn can happen if and only if either at least one of $f_2(0, x_1, \dots, x_{n-1})$ or $f_3(0, x_1, \dots, x_{n-1})$ is identically zero or $f_2(0, x_1, \dots, x_{n-1})$ and $f_3(0, x_1, \dots, x_{n-1})$ have a common component. If $f_2(0, x_1, \dots, x_{n-1})$ is identically zero, then the hyperplane section $x_0 = 0$ is a cone with vertex at $p$. If $f_3(0, x_1, \dots, x_{n-1})$ is identically zero, then $X$ contains the $(n-2)$-dimensional quadric $x_0 = f_2(x_0, \dots, x_{n-1}) = 0$. Similarly, if $f_2(0, x_1, \dots, x_{n-1})$ and $f_3(0, x_1, \dots, x_{n-1})$ have a common component, then $X$ contains an $(n-2)$-dimensional subvariety of $\mathbb{P}^n$ of degree at most two. The Lefschetz hyperplane theorem (see, for example, Theorem 3.1.17 in [L]) asserts that if $X$ is a smooth, effective ample divisor in a smooth $n$-dimensional projective variety $Y$, then the restriction

$$H^i(Y, \mathbb{Z}) \to H^i(X, \mathbb{Z})$$

is an isomorphism provided $i \leq n-2$. In particular, applying the theorem with $Y = \mathbb{P}^n$ and $X$ the smooth cubic hypersurface, we conclude that the degree of any subvariety of $X$ whose codimension is less than $(n-1)/2$ is divisible by three. Since $X$ is smooth and its dimension is at least three, we conclude that $X$ cannot contain an $n-2$ dimensional variety of degree one or two. Therefore, the morphism $ev_1$ fails to be flat at $p$ if and only if the hyperplane section of $X$ is an irreducible cone with vertex at $p$. Note that in that case the dimension of the space of lines contained in $X$ and containing $p$ is $n-3$. The tangent hyperplane to $X$ at $p$ contains every line contained in $X$ and containing $p$. Since the space of lines in $\mathbb{P}^{n-1}$ containing a fixed point is irreducible of dimension $n-2$, if the dimension were any larger then the whole tangent plane would be contained in the span of these lines. Hence, $X$ would be reducible. This concludes the proof of the lemma. ∎

**Corollary 2.2.** *Let $X$ be a smooth cubic hypersurface of dimension at least three. Then the evaluation morphism*

$$ev_1 : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$$

*is flat in the complement of finitely many points.*

*Proof.* By the previous lemma, $ev_1$ fails to be flat at $p$ if and only if the tangent hyperplane at $p$ is a cone. To prove the corollary, it suffices to show that a smooth hypersurface of degree at least three can have at most finitely many conical hyperplane sections. Suppose there were a one-parameter family $C$ of conical hyperplane sections of $X$. If the degree of $X$ is at least three, then at a point with a conical hyperplane section the second fundamental form vanishes. The second fundamental form is the differential of the Gauss map (which assigns to a point $p$ in $X$ the projective tangent space to $X$ at $p$) (see 1.62 in [GH]). Consequently, the Gauss map is constant along $C$. This contradicts the fact that the Gauss map of a smooth hypersurface in $\mathbb{P}^n$ of degree at least two is finite (see Corollary 3.4.18 in [L]). We conclude that a smooth hypersurface of degree at least three has at most finitely many conical hyperplane sections. This concludes the proof of the corollary. $\square$

**Definition 2.3.** *Let $S = \{p_1, \dots, p_m\}$ denote the finite (possibly empty) set of points $p_i$ where the tangent hyperplane section of the smooth cubic hypersurface $X$ at $p_i$ is a cone with vertex at $p_i$. In the classical literature these points are called Eckardt points.*

**Proposition 2.4.** *Let $X$ be a smooth cubic hypersurface of dimension at least 4. Let $e \geq 1$ denote an integer. The fiber dimension of the evaluation morphism $ev_e : \overline{\mathcal{M}}_{0,1}(X, e) \to X$ is constant, equal to $e(n-2) - 2$, in the complement of the points $p \in S$. The fiber dimension of $ev_e$ over a point $p \in S$ is at most $e(n-2) - 1$. Moreover, if $p \notin S$, then the general point of every irreducible component of $ev_e^{-1}(p)$ parameterizes a map with irreducible domain.*

*Proof.* We will prove this proposition by induction on $e$. Lemma 2.1 proves the base case of the induction. The rest of the argument follows from Mori's bend-and-break lemma and formal properties of the Kontsevich moduli spaces. For applications we state this more generally in Proposition 2.5. Proposition 2.4 follows by taking $d = 3$ in Proposition 2.5 and noting that $\tilde{e} = 1$. $\square$

**Proposition 2.5.** *Let $X$ be a smooth hypersurface of degree $d < n-1$ in $\mathbb{P}^n$. Let $\tilde{e}$ be the minimal integer such that*

$$d \leq \frac{\tilde{e}(n+1)}{\tilde{e}+1}.$$

*Assume that there exists a finite set $S \subset X$ such that for $1 \leq e \leq \tilde{e}$,*

$$\dim(ev_e^{-1}(p)) = e(n-d+1) - 2 \quad \textit{for} \quad p \in X - S \quad \textit{and} \quad \dim(ev_e^{-1}(p)) \leq e(n-d+1) - 1 \quad \textit{for} \quad p \in S.$$

*Then for every $e \geq 1$,*

$$\dim(ev_e^{-1}(p)) = e(n-d+1) - 2 \quad \textit{for} \quad p \in X - S \quad \textit{and} \quad \dim(ev_e^{-1}(p)) \leq e(n-d+1) - 1 \quad \textit{for} \quad p \in S.$$

*Furthermore, if $p \in X - S$, then the general map in $ev_e^{-1}(p)$ has irreducible domain.*

*Proof.* The proof is by induction on $e$. The proposition holds for $e \leq \tilde{e}$ by assumption. By basic deformation theory (see Lemma 4.2 in [HRS]), the dimension of every irreducible component of $\overline{\mathcal{M}}_{0,m}(X, e)$ is at least $e(n-d+1) + n + m - 4$. Consider the evaluation morphism $ev_e \times ev_e : \overline{\mathcal{M}}_{0,2}(X, e) \to X \times X$. By the theorem on the dimension of fibers, every irreducible component of a fiber of this morphism has dimension at least

$$\dim(\overline{\mathcal{M}}_{0,2}(X, e)) - \dim(X \times X) \geq (e-1)(n+1) - ed + 1.$$

If $n \geq d$ and $e > \tilde{e}$, this dimension is at least one. By Mori's bend-and-break lemma (Lemma 1.9 in [KM]), every complete curve in a fiber of $ev_e \times ev_e$ contains maps from a reducible domain or a multiple cover of a smaller degree curve.

Let $p \notin S$. Suppose by induction that for $e < e_0$, the dimension of every irreducible component of $ev_e^{-1}(p) \subset \overline{\mathcal{M}}_{0,1}(X, e)$ is $e(n-d+1) - 2$. We check that multiple covers of an irreducible curve containing $p$ cannot form an irreducible component of $ev_{e_0}^{-1}(p)$. By the induction hypothesis, we may assume that the dimension of every irreducible component of the space of curves of degree $e'$ containing $p$ is $e'(n-d+1) - 2$. The dimension of the space of $e_0/e'$-sheeted covers of such curves is $e'(n-d+1)-2+2(e_1/e')-2$. Consider the difference in dimensions

$$e_0(n-d+1)-2-[e'(n-d+1)-2+2(e_0/e')-2]=(e_0-e')(n-d+1-\frac{2}{e'})>0.$$

Since $n>d+1$, this difference is strictly positive. Hence, multiple covers cannot form a component of $ev_{e_0}^{-1}(p)$. Moreover, multiple covers of irreducible curves have codimension at least two in the fibers except when $n=d+2$, $e_0=2$ and $e'=1$. Similarly, if $p \in S$, then the dimension of the locus of multiple covers in $ev_{e_0}^{-1}(p)$ is at most $e'(n-d+1)-1+2(e_0/e')-2$. Note that this is still bounded above by $e_0(n-d+1)-2$ when $n>d+1$.

Next we show that the locus of maps with reducible domains do not form an irreducible component of $ev_e^{-1}(p)$ when $p \notin S$. Let $p \notin S$. Let $R(p)$ be a locus of maps in $ev_e^{-1}(p)$ consisting of maps with reducible domains. We would like to give an upper bound on the dimension of $R(p)$. Suppose we know the Proposition for all degrees $e<e_0$. Let $C$ be the domain of a map parameterized by $R(p)$. There are three possibilities.

(1) A node of $C$ may map to the point $p$. Let $D$ be a maximal connected subset of $C$ contracted to the point $p$ by the stable map. Let $C_1,\dots,C_u$ be the closures of the connected components of $C-D$. Suppose the map has degree $e_i$ on $C_i$ for $1\le i\le u$. Then by the induction hypothesis the dimension of $R(p)$ is bounded above by

$$\sum_{i=1}^{u}(e_i(n-d+1)-2)+u-2,$$

where the term $u-2$ accounts for the dimension of the moduli of the marked point and the points of attachment of $D$ with $C_1,\dots,C_u$. Since $\sum e_i=e_0$, we see that

$$e_0(n-d+1)-u-2\ge \dim(R(p)).$$

Since $u\ge 2$ by assumption, we see that such a locus has codimension at least two in $ev_{e_0}^{-1}(p)$. We may now assume that the nodes of $C$ do not map to $p$.

(2) A node of $C$ may map to a point $p_i$ contained in the set $S$. Let $D$ be a maximal connected subset of $C$ contracted to the point $p_i$ by the stable map. Let $C_1,\dots,C_u$ be the closures of the connected components of $C-D$. Suppose that the map has degree $e_i$ on $C_i$ for $1\le i\le u$. Let the inverse image of $p$ be contained in $C_1$. Then by the induction hypothesis the dimension of $R(p)$ is bounded above by

$$e_1(n-d+1)-2+\sum_{i=2}^{u}(e_i(n-d+1)-1)+\max(0,u-3),$$

where the term $\max(0,u-3)$ accounts for the dimension of the moduli of the points of attachment of $D$ with $C_1,\dots,C_u$. If $u=2$, then we obtain the inequality

$$e_0(n-d+1)-3\ge \dim(R(p)).$$

If $u>2$, then the inequality becomes

$$e_0(n-d+1)-u-1+u-3=e_0(n-d+1)-4\ge \dim(R(p)).$$

We see that such loci have codimension at least one in $ev_{e_0}^{-1}(p)$. We may now assume that the nodes of $C$ do not map to points in $\{p\}\cup S$.

(3) A node of $C$ may map to a point $q\notin\{p\}\cup S$. Let $D$ be a maximal connected subset of $C$ contracted to the point $q$ by the stable map. Let $C_1,\dots,C_u$ be the closures of the connected components of $C-D$. Suppose that the map has degree $e_i$ on $C_i$ for $1\le i\le u$. Let the inverse image of $p$ be contained in $C_1$. Then by the induction hypothesis, the dimension of $R(p)$ is bounded by

$$\sum_{i=1}^{u}(e_i(n-d+1)-2)+\max(0,u-3)+1,$$

where the term $\max(0,u-3)+1$ accounts for the dimension of the moduli of the points of attachment of $D$ with $C_1,\dots,C_u$. When $u=2$, we see that

$$e_0(n-d+1)-3\ge \dim(R(p)).$$

When $u>2$, then

$$e_0(n-d+1)-u-2\ge \dim(R(p)).$$

We see that such loci have codimension at least one in $ev_{e_0}^{-1}(p)$.

We conclude that maps consisting of multiple covers of smaller degree curves and maps with reducible domains cannot form a component of $ev_e^{-1}(p)$ when $p \notin S$ since the dimension of such loci is strictly less than the lower bound on the dimension of the irreducible components of $ev_e^{-1}(p)$. Since by Mori's bend-and-break lemma, every irreducible component of $ev_e^{-1}(p)$ contains maps with reducible domains or maps multiple to their image in codimension one, we conclude that the dimension of every irreducible component of $ev_e^{-1}(p)$ for $p \notin S$ is $e(n-d+1)-2$.

Next, we repeat the analysis assuming that $p \in S$. As in the previous case, there are three possibilities. The point $p$ may be the image of a node. If the point $p$ is not the image of a node, another point in $S$ may be the image of a node. Finally, the images of the nodes of the domain curve may be disjoint from $S$.

(1) A node of $C$ may map to a point $p$. Let $D$ be a maximal connected subset of $C$ contracted to $p$ by the stable map. Let $C_1, \dots, C_u$ be the closures of the connected components of $C - D$. Suppose that the map has degree $e_i$ on $C_i$ for $1 \leq i \leq u$. Then by the induction hypothesis, the dimension of $R(p)$ is bounded above by
$$\sum_{i=1}^{u}(e_i(n-d+1)-1)+u-2,$$
where the term $u-2$ accounts for the dimension of the moduli of the points of attachment of $D$ with $C_1, \dots, C_u$ and the marked point mapping to $p$. We obtain the inequality
$$e_0(n-d+1)-2 \geq \dim(R(p)).$$
We may now assume that the nodes of $C$ do not map to the point $p$.

(2) A node of $C$ may map to a point $p_i$ in $S - \{p\}$. Let $D$ be a maximal connected subset of $C$ contracted to $p_i$ by the stable map. Let $C_1, \dots, C_u$ be the closures of the connected components of $C - D$. Suppose that the map has degree $e_i$ on $C_i$ for $1 \leq i \leq u$ and that $p$ is contained in the image of $C_1$. Then by the induction hypothesis, the dimension of $R(p)$ is bounded above by
$$e_1(n-d+1)-1+\sum_{i=2}^{u}(e_i(n-d+1)-1)+\max(0,u-3),$$
where the term $\max(0, u-3)$ accounts for the moduli of the points of attachments of $C_1, \dots, C_u$ and $D$. We thus get the bound
$$e_0(n-d+1)-2 \geq \dim(R(p)).$$
We may now assume that the nodes of $C$ do not map to points of $S$.

(3) If a node of $C$ maps to a point $q \notin S$, let $D$ be a maximal connected subset of $C$ contracted to $q$ by the stable map. Let $C_1, \dots, C_u$ be the closures of the connected components of $C - D$. Suppose that the map has degree $e_i$ on $C_i$ for $1 \leq i \leq u$ and that $p$ is contained in the image of $C_1$. Then by the induction hypothesis, the dimension of $R(p)$ is bounded above by
$$e_1(n-d+1)-1+\sum_{i=2}^{u}(e_i(n-d+1)-2)+\max(0,u-3)+1.$$
Hence
$$e_0(n-d+1)-2 \geq \dim(R(p)).$$

By Mori's bend-and-break lemma, $ev_{e_0}^{-1}(p)$ either consists entirely of maps with reducible domains or contains the union of multiple covers and maps with reducible domains as a codimension one subset. Since we bounded the dimension of the latter from above by $e_0(n-d+1)-2$, we conclude that the dimension of every irreducible component of $ev_{e_0}^{-1}(p)$ for $p \in S$ is at most $e_0(n-d+1)-1$. □

**Corollary 2.6.** *Let $X$ be a smooth hypersurface of degree $d < n-1$ in $\mathbb{P}^n$. Suppose that $X$ satisfies the hypotheses of Proposition 2.5. Then the dimension of every irreducible component of $\overline{\mathcal{M}}_{0,0}(X,e)$ is equal to*
$$\dim(\overline{\mathcal{M}}_{0,0}(X,e)) = e(n-d+1)+n-4.$$

*Furthermore, the restriction of the evaluation morphism to each irreducible component of $\overline{\mathcal{M}}_{0,1}(X,e)$ is dominant.*

*Proof.* By the theorem on the dimension of fibers and Proposition 2.5, the dimension of every irreducible component of $\overline{\mathcal{M}}_{0,1}(X,e)$ is at most $e(n-d+1)+n-3$. Since the dimension is at least $e(n-d+1)+n-3$, we conclude that equality must hold and that the restriction of the evaluation morphism to each irreducible component must be dominant. Consider the forgetful morphism

$$\pi : \overline{\mathcal{M}}_{0,1}(X,e) \to \overline{\mathcal{M}}_{0,0}(X,e).$$

Since the dimension of the fibers of $\pi$ is equal to one, we conclude that

$$\dim(\overline{\mathcal{M}}_{0,0}(X,e)) = e(n-d+1)+n-4.$$

□

In particular, setting $d=3$, we obtain the following corollary.

**Corollary 2.7.** *Let $X$ be a smooth cubic hypersurface of dimension at least 4. Then the dimension of every irreducible component of $\overline{\mathcal{M}}_{0,0}(X,e)$ is equal to*

$$\dim(\overline{\mathcal{M}}_{0,0}(X,e)) = e(n-2)+n-4.$$

*Furthermore, the restriction of the evaluation morphism to each irreducible component of $\overline{\mathcal{M}}_{0,1}(X,e)$ is dominant.*

We are now ready to prove Theorems 1.1 and 1.6.

*Proof of Theorem 1.1.* Let $X$ be a smooth cubic hypersurface in $\mathbb{P}^n$ of dimension at least 4. In Corollary 2.7, we showed that

$$\dim(\overline{\mathcal{M}}_{0,0}(X,e)) = e(n-2)+n-4.$$

To prove Theorem 1.1 there remains to check that $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible. The argument mimics the one given in [S] or [HRS]. For the convenience of the reader we recall the argument.

It is well-known that $\overline{\mathcal{M}}_{0,0}(X,1)$ is smooth and irreducible. The Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(X,1)$ is isomorphic to the Fano variety of lines on $X$. The Zariski tangent space to the Fano variety at a point $L$ is given by $H^0(L, N_{L/X})$. Considering the exact sequence

$$0 \to N_{L/X} \to N_{L/\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(3)_{|X} \to 0,$$

we conclude that $N_{L/X}$ is a vector bundle of rank $n-2$ and degree $n-4$ on $\mathbb{P}^1$ that admits an injective map to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1}$. Therefore, the only possible splitting types of $N_{L/X}$ are

$$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-4} \ \text{ or } \ \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-3}.$$

In either case, $h^0(L, N_{L/X}) = 2n-6$. Since the dimension of the Zariski tangent space is equal to the dimension of $\overline{\mathcal{M}}_{0,0}(X,1)$ at every point, we conclude that $\overline{\mathcal{M}}_{0,0}(X,1)$ is smooth. $\overline{\mathcal{M}}_{0,1}(X,1)$ is connected because the evaluation morphism maps it to $X$ with connected fibers. Hence, $\overline{\mathcal{M}}_{0,0}(X,1)$ is irreducible. Furthermore, since lines cover $X$, the normal bundle of a line containing a general point in $X$ has the form

$$N_{L/X} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-4}$$

by Theorem 3.11 (or Exercise 4.4.2) of [K]. In particular, the space of lines on $X$ containing a fixed general point $p$ is smooth. The rest of the argument is formal, so we carry it out in greater generality.

Let $X$ be a smooth hypersurface of degree $d$ satisfying the hypotheses of Proposition 2.5. By Corollary 2.6, the map

$$ev_e : \overline{\mathcal{M}}_{0,1}(X,e) \to X$$

restricted to every irreducible component of $\overline{\mathcal{M}}_{0,1}(X,e)$ is dominant. Therefore, if the general fiber of $ev_e$ is irreducible, then $\overline{\mathcal{M}}_{0,1}(X,e)$ and consequently $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible. Suppose that for $e \le \tilde{e}$, the general fiber of the evaluation morphism is irreducible. By the discussion in the previous paragraph, this assumption holds for every smooth cubic hypersurface of dimension at least four. We will prove the irreducibility of the general fiber of $ev_e$ by induction on $e$. Assume that the general fiber of $ev_e$ is irreducible for $e < e_0$. It follows that at a general point $p \in X$ the intersection of the fiber of the evaluation morphism with the boundary divisor $\Delta_{(e,\{1\}),(e_0-e,\emptyset)}$ is irreducible. The boundary divisor is the image of the gluing morphism

$$\overline{\mathcal{M}}_{0,2}(X,e) \times_X \overline{\mathcal{M}}_{0,1}(X,e_0-e) \to \Delta_{(e,\{1\}),(e_0-e,\emptyset)}.$$

By Proposition 2.5, the general map in $ev_e^{-1}(p)$ has irreducible domain. Consequently, by the induction hypothesis, the inverse image via the forgetful morphism $\pi$ of the fiber of the evaluation morphism, $\pi^{-1}(ev_e^{-1}(p))$, is irreducible in $\overline{\mathcal{M}}_{0,2}(X,e)$. By induction, the general fiber of the first projection

$$\pi_1 : \overline{\mathcal{M}}_{0,2}(X,e) \times_X \overline{\mathcal{M}}_{0,1}(X,e_0-e) \to \overline{\mathcal{M}}_{0,2}(X,e)$$

is irreducible. Using the dimension estimates in Proposition 2.5, we conclude the irreducibility of the intersection of $ev_{e_0}^{-1}(p)$ with the boundary divisor $\Delta_{(e,\{1\}),(e_0-e,\emptyset)}$. Note that for every $1 \leq e < e_0$, this boundary divisor contains all the maps whose domain is a chain of $e_0$ rational curves $C_1 \cup \cdots \cup C_{e_0}$ and whose image is a chain of $e_0$ lines where the image of the first curve in the chain contains $p$. We will refer to such maps as linear chains starting at $p$. We are now ready to conclude the proof.

Let $M$ be an irreducible component of $ev_{e_0}^{-1}(p) \subset \overline{\mathcal{M}}_{0,1}(X,e_0)$ for a general point $p \in X$. The general member of $M$ is a map from an irreducible domain. By Mori's bend-and-break lemma, $M$ contains maps from reducible curves in codimension one. Hence, $M$ must contain the entire intersection of at least one of the boundary components with $ev_{e_0}^{-1}(p)$. In particular, every irreducible component $M$ of $ev_{e_0}^{-1}(p)$ contains all the linear chains starting at $p$. Let $C$ be a general linear chain starting at $p$. Since $H^1(C, N_{C/X}(-p)) = 0$, we conclude that the fiber of $ev_{e_0}^{-1}(p)$ is smooth at such a point. It follows that the fiber of $ev_{e_0}^{-1}(p)$ is irreducible. Hence, $\overline{\mathcal{M}}_{0,0}(X,e_0)$ is irreducible. This concludes the proof of the theorem. □

*Proof of Theorem 1.6.* Let $X$ be a general hypersurface in $\mathbb{P}^n$ of dimension at least three and degree $d \leq n-1$. Then by Theorem V.4.3 of [K], we may assume that $\overline{\mathcal{M}}_{0,0}(X,1)$ is irreducible of dimension $2n-3-d$. Moreover, by Theorem 2.1 of [HRS], we may also assume that the evaluation morphism $ev_1 : \overline{\mathcal{M}}_{0,0}(X,1) \to X$ is flat. First, suppose $2d \leq \min(n+3, 2n-3)$. Then the threshold value $\tilde{e}$ in Proposition 2.5 is less than or equal to two. Since the hypotheses of Proposition 2.5 are satisfied for $e=1$, we only need to check them for $e=2$. Suppose there exists a component of the fiber of the evaluation morphism $ev_2^{-1}(p)$ of dimension larger than $2(n-d+1)-2 \geq n-4$. Since double covers of lines and reducible conics have the expected dimension, we may assume that the general point represents an irreducible conic. Note that the maps parameterized by this component cannot cover $X$. Otherwise, by Theorem 3.11 of [K], the general member in this family would be free. Since $H^1(C, N_{C/X}(-p)) = 0$ for free curves, the dimension of $ev_2^{-1}(p)$ would be $2(n-d+1)-2$ contradicting that they form a larger dimensional component. Hence, the images of these maps cover at most a subvariety of dimension $n-2$. If $2d \leq n+3$, then $2(n-d+1)-2 \geq n-3$. Hence, any component of $ev_2^{-1}(p)$ with larger than expected dimension would have dimension at least $n-2$. By Mori's bend-and-break lemma, reducible conics or double covers of lines would form a divisor in that irreducible component. Since the evaluation morphism for lines is flat, by the calculations in the proof of Proposition 2.5, the dimension of either loci is $2(n-d+1)-3$. This is a contradiction since this locus would have codimension at least two. We conclude that when $2d \leq \min(n+3, 2n-3)$, if the evaluation morphism is flat in degree one, then it is also flat in degree two. Note that by Theorem 23.1 in [M] the flatness of the evaluation morphism is equivalent to the fibers all having the same dimension. Proposition 2.5 then implies that the evaluation morphism $ev_e$ is flat for all degrees $e$. This concludes the proof of Theorem 1.6 (2). If $2d = n+4$, then suppose there exists a point $p$ such that $ev_2^{-1}(p)$ has a component of dimension at least $n-2$ or a one-parameter-family of points with $\dim(ev_2^{-1}(p)) \geq n-3$. Then by Mori's bend-and-break lemma, that component must contain reducible conics or double covers of lines in codimension one. However, since the evaluation morphism is flat in degree one, we again obtain a contradiction. We conclude that the hypotheses of Proposition 2.5 are satisfied. By Corollary 2.6 the evaluation morphism is dominant on each component for $e=2$. It follows by Theorem 3.11 in [K] that the general fiber of the evaluation morphism is smooth. It is easy to see that the fiber is connected, hence it is irreducible. The irreducibility of $\overline{\mathcal{M}}_{0,0}(X,e)$ follows by induction on the degree $e$ by the argument given in the proof of Theorem 1.1. Since the locus of smooth rational curves of degree $e$ in the Hilbert scheme may be embedded in $\overline{\mathcal{M}}_{0,0}(X,e)$ as a dense open subset, we conclude that $R_e$ is irreducible. This concludes the proof of Theorem 1.6 except in the three cases $(d,n) = (3,4), (4,5), (5,6)$ when $2d \leq n+4$ and $d = n-1$.

When $(d,n) = (3,4), (4,5), (5,6)$, the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(X,e)$ is reducible. We will show that it has two irreducible components. The locus $N_e$ of degree $e$ covers of lines (for $e > 1$) forms a second component also of dimension $2e+n-4$. $N_e$ is irreducible (since it maps to the Fano variety of lines which is irreducible and the fibers are all isomorphic to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, e)$). Moreover, the normal bundle of a cover of a free line is globally generated, hence the Kontsevich moduli space is smooth of dimension $2e + n - 4$ at a general point of $N_e$. We conclude that it forms a separate component. When applying Mori's bend-and-break argument to the locus of irreducible conics in $ev_2^{-1}(p)$, we may assume that the conics become reducible by choosing the second point not to lie on one of the finitely many lines containing $p$. Hence, the argument given in the previous paragraph still shows that the evaluation morphism $ev_2$ has constant fiber dimension when $(d,n) = (3,4)$ or $(4,5)$. Similarly, when $(d,n) = (5,6)$, the evaluation morphism $ev_2$ has constant fiber dimension in the complement of a finite set $S$ and the fiber dimension increases by at most one over the points in $S$. Hence, all the hypotheses of Proposition 2.5 with the exception of the requirement that $d < n-1$ are satisfied. The only place where the assumption $d < n-1$ is used in the proof of Proposition 2.5 is in bounding the dimension of multiple covers of irreducible curves. Suppose by induction that for $e < e_0$, $\dim(ev_e^{-1}(p)) = 2e-2$ if $p \notin S$ and $\dim(ev_e^{-1}(p)) = 2e-1$ if $p \in S$. If $p \notin S$, then the quantity

$$(e_0 - e')(2 - \frac{2}{e'})$$

occurring in the proof of Proposition 2.5 is zero if and only if $e' = 1$. Furthermore, this quantity is at least two if $e' > 1$. If $p \in S$ and $e' > 1$, then

$$(e_0 - e')(2 - \frac{2}{e'}) - 1$$

is at least one. We conclude that the locus of degree $e_0/e'$ covers of irreducible curves of degree $e' > 1$ has codimension one or more in $ev_{e_0}^{-1}(p)$. For $e_0 > 2$, Mori's bend-and-break argument applies for maps with irreducible domains. Furthermore, since there are finitely many lines on $X$ containing $p$, we may assume that the limiting map is not a degree $e_0$ cover of a line by requiring our second point not to lie on a line in $X$ containing $p$. The argument in Proposition 2.5 is then valid without any change. We conclude that for every degree $e$, the fibers of $ev_e$ have constant dimension $2e-2$ over $p \notin S$ and dimension at most $2e-1$ over points $p \in S$. It follows that every irreducible component of $\overline{\mathcal{M}}_{0,0}(X,e)$ has dimension $2e+n-4$ and that the evaluation morphism restricted to each irreducible component of $\overline{\mathcal{M}}_{0,1}(X,e)$ is dominant. It is well-known that the locus of smooth rational curves embeds as a non-empty Zariski open set in $\overline{\mathcal{M}}_{0,0}(X,e)$. To conclude that $R_e$ is irreducible, it suffices to show that $\overline{\mathcal{M}}_{0,0}(X,e)$ has two irreducible components. Since $N_e$ does not intersect the locus of smooth rational curves of degree $e$, it follows that the locus of smooth rational curves must be a Zariski open set in the other irreducible component. Therefore, $R_e$ is irreducible. Let $p$ be a general point on $X$. By running the bend-and-break argument in the proof of Theorem 1.1 with two points $p, q \in X$ such that there are no lines in $X$ containing $p$ and $q$ and where the finitely many lines through $p$ and $q$ are free, we can assume that every irreducible component of $ev_e^{-1}(p)$ with $e \geq 2$ not contained in $N_e$ contains generically one-to-one maps to $X$ with reducible domain. Hence, by induction every irreducible component of $ev_e^{-1}(p)$ not contained in $N_e$ contains the chains of lines starting at $p$. Since $ev_e^{-1}(p)$ is smooth at a general chain of lines starting at $p$, we conclude that the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(X,e)$ for $(d,n) = (3,4), (4,5), (5,6)$ and $e \geq 2$ has two irreducible components. Therefore, $R_e$ is irreducible. This concludes the proof of Theorem 1.6. □

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