DEGENERATIONS OF SURFACE SCROLLS AND THE GROMOV-WITTEN INVARIANTS OF GRASSMANNIANS

IZZET COSKUN

Abstract. We describe an algorithm for computing certain characteristic numbers of rational normal surface scrolls using degenerations. As a corollary we obtain an efficient method for computing the corresponding Gromov-Witten invariants of the Grassmannians of lines.

Contents

1. Introduction

This paper investigates the enumerative geometry of rational normal surface scrolls in $\mathbb{P}^N$ using degenerations. We obtain an effective algorithm for computing certain characteristic numbers of balanced scrolls. Surface scrolls can be interpreted as curves in the Grassmannian of lines $\mathbb{G}(1,N)$. Using our algorithm we calculate the corresponding Gromov-Witten invariants of $\mathbb{G}(1,N)$. We work over the field of complex numbers $\mathbb{C}$.

Motivation. By the *characteristic number problem* we mean the problem of computing the number of varieties in $\mathbb{P}^n$ of a given 'type' (e.g. curves of degree d and genus g) that meet the 'appropriate' number of general linear spaces so that the expected dimension is zero. This problem has attracted a lot of interest since the 19th century (see [Sc], [Kl2]). In the last decade, motivated by the work of string theorists and Kontsevich, there has been significant progress on the problem for curves (see [CH], [V1], [V2] for references).

In comparison the characteristic numbers of higher dimensional varieties are harder to compute, hence have received less attention (see however [VX]). In this paper we start a more systematic study of the characteristic numbers of higher dimensional varieties using degenerations. Here we restrict our attention to rational surface scrolls, although most of the techniques apply with little change to higher dimensional scrolls [C2] and can be modified to apply to Del Pezzo surfaces [C1].

The enumerative geometry of scrolls is also attractive for its connection to the Gromov-Witten theory of $\mathbb{G}(1,N)$. Localization techniques and the associativity relations in the quantum cohomology ring lead to

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recursive algorithms that compute the invariants, but these algorithms are usually inefficient. For example, running FARSTA [Kr], a computer program that computes Gromov-Witten invariants from associativity relations, on Harvard's MECCAH cluster it took over four weeks to determine the cubic invariants of $\mathbb{G}(1,5)$. The algorithm we prove here allows us to compute some of these invariants by hand (§8, 9).

Notation. Let $\overline{M}_{0,n}(\mathbb{G}(1,N),d)$ denote the Kontsevich moduli space of n-pointed genus 0 stable maps to $\mathbb{G}(1,N)$ of Plücker degree d. Let $\mathrm{Hilb}(\mathbb{P}^N, S^d)$ denote the irreducible component of the Hilbert scheme whose general point corresponds to a smooth rational normal surface scroll S of degree d in $\mathbb{P}^N$.

Results. The main results of this paper are the following:

• To calculate the characteristic numbers of scrolls, we specialize the linear spaces meeting the scrolls to a general hyperplane H. We prove that a general, non-degenerate, reducible limit of balanced scrolls incident to the linear spaces consists of the union of two balanced scrolls meeting along a line—provided that the limit of the hyperplane sections in H remains non-degenerate (§6). The precise statements are given in Theorems 6.8 and 6.9.

• By successively breaking the scrolls into lower-degree scrolls, we obtain a recursive algorithm for computing the characteristic numbers of balanced scrolls in $\mathbb{P}^N$ incident to linear spaces of small dimension (§8). Theorem 8.1 summarizes the result.

Example: For instance, the algorithm easily shows that the number of scrolls of degree n in $\mathbb{P}^{n+1}$ containing $n+5$ general points and meeting a general $n-3$ plane is $(n-1)(n-2)$ (§5).

• As a corollary, we obtain an efficient method for computing the corresponding Gromov-Witten invariants of $\mathbb{G}(1,N)$. The proof also yields a method for computing some Gromov-Witten invariants of $\mathbb{F}(0,1;N)$, the partial flag variety of pointed lines in $\mathbb{P}^N$ (§9).

The method. Our method is a degeneration method inspired by [CH] and especially [V2]. The prototypical example answers the question how many lines meet 4 general lines $l_1, \cdots l_4$ in $\mathbb{P}^3$. If l_1 and l_2 lie in a plane P, then the answer is easy to see. Let $q = l_1 \cap l_2$. The two solutions are the line in P through $l_3 \cap P$ and $l_4 \cap P$ and the intersection of the two planes $\overline{ql_3}$ and $\overline{ql_4}$. To answer the original question we can

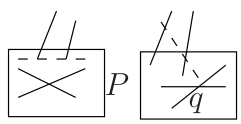

FIGURE 1. Prototypical example of the degeneration method.

specialize two of the lines to a plane. If we know how many of the original solutions approach each of the two special solutions we can answer the problem. Our algorithm carries out this classical idea for rational normal scrolls.

The solution of an enumerative problem by degenerations has two steps. We specialize linear spaces meeting the scrolls to a general hyperplane one at a time. First, we identify the limiting positions of the scrolls.

We prove that non-degenerate limits of scrolls are unions of scrolls where any two "adjacent" components share a common fiber. These limit surfaces occur as images of trees of Hirzebruch surfaces (Proposition 4.1). We describe the trees that occur as limits of scrolls $S_{k,l}$.

Not every tree of scrolls smooths to $S_{k,l}$. The specializations of $S_{k,l}$ contain a connected degree k curve whose components are rational curves in section classes on the scrolls. The existence of a degree k curve with these properties turns out to be sufficient for a union of two scrolls of total degree $k+l$ to smooth to $S_{k,l}$ (Proposition 4.4).

In §6 we carry out a detailed dimension count to identify which unions of scrolls occur as limits under some non-degeneracy assumptions. The dimension calculations are considerably harder for surfaces than for curves because the need to trace both the hyperplane section and the directrix of the surface forces us to work with a non-convex space in the sense of Kontsevich (see [FP]). Nonetheless, under the assumption that the surfaces and their hyperplane sections remain non-degenerate, we prove that balanced scrolls break into unions of balanced scrolls.

Once we determine the limits, we need to determine their multiplicities. We reduce the calculations to the case of curves by constructing a smooth morphism from the space of scrolls to $\overline{M}_{0,n}(\mathbb{P}^N, d)$ and pulling-back the relations between cycles in these spaces [V2] to the space of scrolls (§7). We give many examples to illustrate how the algorithm works in §5.

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2. Preliminaries on Scrolls

This section provides a summary of basic facts about rational scrolls and systems of divisors on them; for more details consult [Bv] Ch. 4, [Fr] Ch. 5 or [GH] §3 Ch. 4.

Rational normal scrolls. Let $k \leq l$ be two non-negative integers with $l > 0$. We will denote a rational normal surface scroll of bidegree k, l by $S_{k,l}$. $S_{k,l}$ is a rational surface of degree $k + l$ in $\mathbb{P}^{k+l+1}$. We now recall its construction.

Fix two rational normal curves of degrees k and l in $\mathbb{P}^{k+l+1}$ with disjoint linear spans. Fix an isomorphism between the curves. $S_{k,l}$ is the surface swept by the lines joining the points corresponding under the isomorphism. The degree k curve is called the *directrix*. If the directrix reduces to a point, we obtain $S_{0,l}$, the cone over a rational normal degree l curve. We will call a scroll *balanced* if $l - k \leq 1$, and *perfectly balanced* if $k = l$. A perfectly balanced scroll has a one-parameter family of directrices.

Rational normal scrolls are non-degenerate surfaces of minimal degree in projective space. Conversely,

Proposition 2.1. *(*[GH] *p.525) Every non-degenerate irreducible surface of degree $m-1$ in $\mathbb{P}^m$ is a rational normal scroll or the Veronese surface in $\mathbb{P}^5$.*

Hirzebruch surfaces. The *Hirzebruch surface* F_r, $r \geq 0$, is the projectivization of the vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r)$ over $\mathbb{P}^1$. In this paper the projectivization of a vector bundle $\mathbb{P}V$ will mean the one-dimensional subspaces of V.

The Picard group of F_r, $r > 0$, is generated by two classes: the class f of a fiber F of the projective bundle and the class e of the unique section E with negative self-intersection. The intersection pairing is given by

$$f^2 = 0, \quad f \cdot e = 1, \quad e^2 = -r.$$

The surface F_0 does not have a section with negative self-intersection; however, the same description holds for its Picard group. The canonical class of F_r is

$$K_{F_r} = -2e - (r+2)f.$$

The automorphism group of F_r. The automorphism group of F_r, for $r > 0$, surjects onto $\mathbb{P}GL_2(\mathbb{C})$. The kernel is the semidirect product of $\mathbb{C}^*$ with $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r))$ where the former acts on the latter by multiplication. Consequently, the dimension of the automorphism group of F_r is $r + 5$. F_0 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$ is the semidirect product of $\mathbb{P}GL_2(\mathbb{C}) \times \mathbb{P}GL_2(\mathbb{C})$ with $\mathbb{Z}/2\mathbb{Z}$, hence has dimension 6.

The following lemma provides the relation between scrolls and Hirzebruch surfaces:

Lemma 2.2. *The scroll $S_{k,l}$ is the image of the Hirzebruch surface F_{l-k} under the complete linear series $|\mathcal{O}_{F_{l-k}}(e + lf)|$. For $k \neq 0, l$, the image of the curve E is the unique rational normal degree k curve on the scroll. The fibers F are mapped to lines. Irreducible curves in the class $e + (l-k)f$ map to rational normal degree l curves with linear span disjoint from the linear span of the image of E.*

Section classes. During degenerations of scrolls it will be essential to determine the limits of their hyperplane sections. When scrolls become reducible, their hyperplane sections remain in section classes.

Definition 2.3. *On a Hirzebruch surface F_r a cohomology class of the form $e + mf$ is called a* **section class**.

Irreducible curves in a section class are sections of the projective bundle. More generally, any curve in a section class consists of a section union some fibers. On $S_{k,l}$ the sections of degree at most $k + l$ are rational normal curves. In particular, the irreducible hyperplane sections are rational normal degree $k + l$ curves. Our description of the cohomology ring of F_r and Lemma 2.2 imply the following lemma:

Lemma 2.4. *A curve of degree d on a scroll $S_{k,l}$ that has intersection multiplicity one with fibers is an element of the linear series $|e + (d - k)f|$ on F_{l-k}.*

Cohomology calculations. Since $\mathcal{O}_E(e + mf) \cong \mathcal{O}_{\mathbb{P}^1}(m - r)$ the long exact sequence associated to the sequence

$$0 \to \mathcal{O}_{F_r}(mf) \to \mathcal{O}_{F_r}(e + mf) \to \mathcal{O}_E(e + mf) \to 0$$

implies that if $m \geq r - 1$, then

$$H^1(F_r, \mathcal{O}_{F_r}(mf)) \to H^1(F_r, \mathcal{O}_{F_r}(e + mf))$$

is surjective; and if $0 \leq m \leq r - 1$, then

$$h^1(\mathcal{O}_{F_r}(e + mf)) = h^1(\mathcal{O}_{F_r}(mf)) + h^1(\mathcal{O}_{\mathbb{P}^1}(m - r)).$$

Since F_r is a rational surface $H^1(F_r, \mathcal{O}_{F_r}) = 0$. Using the exact sequence

$$0 \to \mathcal{O}_{F_r}(mf) \to \mathcal{O}_{F_r}((m + 1)f) \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

we conclude that $H^1(\mathcal{O}_{F_r}(mf)) = 0$ for $m \geq 0$ by induction on m. We, thus, compute the dimensions of all the cohomology groups for the line bundles $\mathcal{O}_{F_r}(e + mf)$, $m \geq 0$.

Lemma 2.5. *The projective dimension of the linear series $|e + mf|$ on $S_{k,l}$, $m \geq 0$ is given by*

$$r(e + mf) = \max(k - l + 2m + 1,\ m). \tag{1}$$

Remark: The preceding discussion proves that when $m < l - k$ the only curves in the section classes $e + mf$ consist of the directrix E union $m - k$ fibers. However, when $m \geq l - k$, the same dimension count implies that there must be irreducible curves in the class $e + mf$.

Lemma 2.6. *The dimension of the locus in the Hilbert scheme whose general point represents a smooth scroll $S_{k,l}$ in $\mathbb{P}^N$ is*

$$(k + l + 2)N + 2k - 4 - \delta_{k,l}.$$

Proof: We can think of the scrolls $S_{k,l}$ as maps from the Hirzebruch surfaces F_{l-k} into projective space $\mathbb{P}^N$. The map is given by $N + 1$ sections of the line bundle $\mathcal{O}_{F_{l-k}}(e + lf)$. This gives $(N + 1)(k + l + 2)$ dimensional choices of sections. After we projectivize and account for the automorphism group of F_{l-k}, which has dimension $l - k + 5 + \delta_{k,l}$, the lemma follows. □

Remark: We defined scrolls $S_{k,l}$ as surfaces in $\mathbb{P}^{k+l+1}$. In case $N < k + l + 1$, Lemma 2.6 provides the dimension of the projections of scrolls to $\mathbb{P}^N$.

Since $\mathbb{P}GL(N+1)$ acts transitively on the non-degenerate scrolls $S_{k,l}$, Kleiman's theorem assures us that if we pick general linear subspaces $\Lambda_i \subset \mathbb{P}^N$ of codimension c_i such that

$$\sum_i (c_i - 2) = (k + l + 2)N + 2k - 4 - \delta_{k,l}$$

then there will be finitely many scrolls $S_{k,l}$ meeting all Λ_i. In the rest of the paper, we address the question of determining this number. Since we will appeal to Kleiman's theorem [Kl1] frequently, we recall it for the reader's convenience.

Theorem 2.7 (Kleiman). *Let G be an integral algebraic group scheme, X an integral algebraic scheme with a transitive G action. Let $f : Y \to X$ and $g : Z \to X$ be two maps of algebraic schemes. For each rational element s of G, denote by sY the X-scheme given by $y \mapsto sf(y)$.*

There exists a dense open subset U of G such that for every rational element in U, the fibered product $(sY) \times_X Z$ is either empty or equidimensional and its dimension is the expected dimension

$$dim(Y) + dim(Z) - dim(X).$$

Furthermore, for a dense open set this fibered product is regular.

Remark: Although we stated the theorem in the language of schemes, Kleiman's proof holds without change for Deligne-Mumford stacks.

3. A Compactification of the Space of Scrolls

In this section we describe a compactification of the space of rational scrolls given by the Kontsevich moduli space of genus 0 stable maps to the Grassmannian.

Scrolls as curves in the Grassmannian. To study the geometry of scrolls it is useful to think of them as rational curves in the Grassmannian $\mathbb{G}(1, N)$ of lines in $\mathbb{P}^N$.

$S_{k,l}$ is a projective bundle over $\mathbb{P}^1$. The fibers of the projection map $\pi : S_{k,l} \to \mathbb{P}^1$ are lines in $\mathbb{P}^N$. Hence, π induces a rational curve of Plücker degree $k+l$ in $\mathbb{G}(1, N)$. More explicitly, consider the incidence correspondence

$$\Phi = \{(p, [L]) : p \in \mathbb{P}^1,\ [L] \in \mathbb{G}(1, N),\ L \subset S_{k,l},\ \pi(L) = p\} \subset \mathbb{P}^1 \times \mathbb{G}(1, N).$$

The image of Φ under the projection of $\mathbb{P}^1 \times \mathbb{G}(1, N)$ to the second factor gives us the required rational curve $C \subset \mathbb{G}(1, N)$.

Conversely, given an irreducible, reduced rational curve C of degree $k + l$ in $\mathbb{G}(1, N)$ we can construct a rational ruled surface of degree $k + l$ in $\mathbb{P}^N$. Consider the projectivization of the tautological bundle T of $\mathbb{G}(1, N)$ over the curve C

$$\Psi = \{([L_c], p) : p \in L_c,\ c \in C\} \subset C \times \mathbb{P}^N \subset \mathbb{G}(1, N) \times \mathbb{P}^N.$$

Projection to the second factor gives a surface S of degree $k + l$ in $\mathbb{P}^N$. If the span of S is $\mathbb{P}^{k+l+1}$, then by Proposition 2.1 the surface is a rational normal scroll. If the span of S is smaller, then S is the projection of a rational normal scroll from a linear subspace of $\mathbb{P}^N$.

Non-degenerate curves. The span of the surface has dimension smaller than $k + l + 1$ if and only if the curve is contained in a $\mathbb{G}(1, r)$ for some $r < k + l + 1$. We will refer to rational curves $C \subset \mathbb{G}(1, N)$ which do not lie in any $\mathbb{G}(1, r)$ for $r < k + l + 1$ as *non-degenerate rational curves in the Grassmannian.*

Non-isomorphic scrolls of the same degree. The automorphism group of $\mathbb{G}(1, N)$ does not act transitively on non-degenerate rational curves. The restriction of the tautological bundle T of $\mathbb{G}(1, N)$ to different curves can have different splitting types. Let $\phi : \mathbb{P}^1 \to C$ be the normalization of C. Consider the vector bundle $V = \phi^* T$ on $\mathbb{P}^1$.

Definition 3.1. *We define the degree k of the summand of minimal degree in the decomposition of $V \to \mathbb{P}^1$ to be the* **directrix degree** *of C.*

The directrix degree distinguishes curves associated to non-isomorphic scrolls. Suppose $C \subset \mathbb{G}(1, N)$ is an irreducible, non-degenerate curve of directrix degree k, then the projectivization of $V \to \mathbb{P}^1$ is isomorphic to F_{l-k}. The reverse construction shows that the curve associated to the scroll $S_{k,l}$ has directrix degree k. We conclude that there is a natural bijection between the set of scrolls $S_{k,l}$ in $\mathbb{P}^N$ and the set of non-degenerate rational curves of degree $k + l$ and directrix degree k in $\mathbb{G}(1, N)$.

$S_{0,1}$ **and** $S_{1,1}$**.** Unlike other scrolls, $\mathbb{P}^2$ and a smooth quadric $Q \subset \mathbb{P}^3$ have more than one scroll structure. $\mathbb{P}^2$ can be given the structure of $S_{0,1}$ in a two parameter family of ways depending on the choice of the vertex point. The quadric surface has two distinct $S_{1,1}$ structures depending on the choice of ruling on the quadric surface. The correspondence between scrolls and rational curves in the Grassmannian differentiates between these scroll structures.

A compactification of the space of scrolls. Using the preceding discussion we can compactify the space of $S_{k,l}$ using the Kontsevich space of stable maps.

Let $\overline{\mathcal{S}} \subset \mathrm{Hilb}(\mathbb{P}^N, \frac{k+l}{2}\, x^2 + \frac{k+l+2}{2}\, x + 1)$ denote the component (with its reduced induced structure) of the Hilbert scheme which parameterizes rational normal scrolls. Let $\mathcal{S} \subset \overline{\mathcal{S}}$ denote the open subscheme whose points represent reduced, irreducible, non-degenerate scrolls. Let $\mathcal{C} \subset \overline{M}_{0,0}(\mathbb{G}(1,N), k+l)$ be the locus in the Kontsevich moduli scheme of stable maps whose points represent closed immersions from an irreducible $\mathbb{P}^1$ to a non-degenerate curve in $\mathbb{G}(1,N)$ of Plücker degree $k+l$. This locus is contained in the automorphism-free locus.

Theorem 3.2. *When $k+l > 2$, there is a natural isomorphism between $\mathcal{S}$ and $\mathcal{C}$ taking the locus of $S_{k,l}$ to maps to curves of directrix degree k.*

Proof: Projection to the second factor from the incidence correspondence Φ induces a morphism from $\mathcal{S}$ to $\mathcal{C}$. We already observed that this morphism is a bijection on points. Since $\mathcal{C}$ is a smooth, quasi-projective variety [FP], Zariski's Main Theorem implies that this morphism is an isomorphism. □

Remark 1. When $k+l \leq 2$, Theorem 3.2 is still valid if instead of the Hilbert scheme we use the space of pointed planes for $k+l=1$ and the space of quadric surfaces with a choice of ruling when $k+l=2$.

Remark 2. Theorem 3.2 implies that $\mathcal{S}$ is smooth. Note that the Hilbert scheme can be singular along subloci of $\mathcal{S}$. For example, the Hilbert scheme of quartic scrolls is singular along the locus of rational quartic cones—the component corresponding to Veronese surfaces meets the component of scrolls along that locus.

Theorem 3.2 provides us with a compactification of the space of scrolls $S_{k,l}$. For balanced scrolls we can take the Kontsevich moduli space of stable maps. A Zariski-open set of $\overline{M}_{0,0}(\mathbb{G}(1,N), k+l)$ corresponds to maps from an irreducible $\mathbb{P}^1$ to a curve of directrix degree $\lfloor \frac{k+l}{2} \rfloor$. For other scrolls we take the scheme-theoretic closure of the locus of maps from $\mathbb{P}^1$ to $\mathbb{G}(1,N)$ whose image has directrix degree k.

4. Limits of Scrolls in One-Parameter Families

In this section we describe the limits of scrolls and their section classes in one-parameter families.

Let $\mathcal{X} \to B$ denote a flat family of surfaces over a smooth, connected base curve B. We assume that every member of the family except for the central fiber $\mathcal{X}_0 \to b_0 \in B$ is a scroll $S_{k,l}$. To simplify the statements we assume that the surface underlying $\mathcal{X}_0$ (with its reduced induced structure) is still non-degenerate. This assumption can be weakened by considering projections.

Proposition 4.1. *The special fiber $\mathcal{X}_0$ is a connected surface whose irreducible components are scrolls S_{k_i,l_i}. $\mathcal{X}_0$ is the image of a union of Hirzebruch surfaces whose dual graph is a connected tree. The indices k_i, l_i satisfy the constraints:*

$$1.\ \sum_i (k_i + l_i) = k + l \qquad \textit{(degree constraint)}$$

$$2.\ \sum_i k_i \leq k$$

Proof: Since the family $\mathcal{X} \to B$ is flat the central fiber $\mathcal{X}_0$ has to be a connected surface of degree $k+l$. The family $\mathcal{X} \to B$ gives rise to a family of curves $\mathcal{Y} \to B \backslash b_0$ in $\mathbb{G}(1,N)$, hence to a curve in $\overline{\mathcal{M}}_{0,0}(\mathbb{G}(1,N), k+l)$. Since the latter is complete we can extend the family over b_0 by a stable map to $\mathbb{G}(1,N)$ (perhaps after base change). The projectivization of the pull back of the tautological bundle maps to $\mathbb{P}^N$ giving a family that agrees with $\mathcal{X}$ except possibly at $\mathcal{X}_0$. There is a unique scheme structure on the image that makes the family flat. Since over a smooth base curve there is a unique way to complete a family to a flat family ([Ha] III.9.8), this family must coincide with our original family. Hence the underlying surface of $\mathcal{X}_0$ is the image of a tree of Hirzebruch surfaces.

The Veronese surface does not contain any lines. Since every component of $\mathcal{X}_0$ is covered by lines, the components cannot be Veronese surfaces. Moreover, every component of $\mathcal{X}_0$ must span a linear space of dimension one larger than its degree. Otherwise, the total span of their union would have dimension bounded by $k+l$. (If $\mathcal{X}_0$ has j components and the i-th component spans a d_i-dimensional linear space, then the span of X_0 has dimension at most $\sum_{i=1}^{j} d_i - j + 1$.) This would contradict the assumption that $\mathcal{X}_0$ is non-degenerate. By Proposition 2.1 we conclude that every component of $\mathcal{X}_0$ is a scroll. In fact, these scrolls must form a tree except more than 2 components might meet along a fiber. Using the exact sequence that relates the Hilbert polynomials of the union of two varieties to the Hilbert polynomial of the components and their intersection ([Ha] p. 52), we see that a union of scrolls has the same Hilbert polynomial as $S_{k,l}$. Since any embedded components contribute non-trivially to the Hilbert polynomial, we conclude that $\mathcal{X}_0$ does not have any embedded components.

If $k=l$, relation 1 implies inequality 2. Hence, to prove that $\sum k_i \leq k$, we can assume that $k<l$. Then $S_{k,l}$ has a unique degree k rational curve meeting all the fibers or is a cone. The flat limit of the degree k curve is again an arithmetic genus 0 curve of degree k. Since meeting the fibers is a closed condition, the limit curve has to meet all the fibers. The smallest degree curve meeting all the fibers on S_{k_i,l_i} has degree k_i. Hence, $\sum_i k_i \leq k$. $\square$

Proposition 4.1 raises the question of which unions of scrolls can be the limits of $S_{k,l}$. We now describe two standard constructions of families of $S_{k,l}$ breaking into a collection of S_{k_i,l_i}. Using these inductively we can degenerate $S_{k,l}$ to a tree of surfaces with any k_i, l_i satisfying the numerical conditions of Proposition 4.1. However, we cannot smooth all such trees to an $S_{k,l}$.

Example: Cones provide the simplest counterexample. The limit of a family of cones is a union of cones whose vertices coincide. We can take two quadric cones meeting along a line, but whose vertices are distinct. This surface cannot be deformed to an $S_{0,4}$.

Construction 1. For any $k \geq r \geq 0$ there exists a flat family of scrolls $S_{k,l}$ specializing to $S_{k-r,l+r}$. To construct such a family it suffices to exhibit a flat family of vector bundles $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(l)$ degenerating to $\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(k+l-r)$. Since $r<k$ there exists a non-trivial injective bundle map from $\mathcal{O}_{\mathbb{P}^1}(r)$ to $\mathcal{O}_{\mathbb{P}^1}(k)$ giving rise to the extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}(r) \to \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(l) \to \mathcal{O}_{\mathbb{P}^1}(k+l-r) \to 0.$$

This extension gives rise to a family E_t of vector bundles whose general member is $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(l)$, but $E_0 \cong \mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(k+l-r)$. Pick the one-dimensional subspace of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-(k+l-r)) \otimes \mathcal{O}_{\mathbb{P}^1}(r))$ which contains the extension in question. This provides us with a family which gives $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(l)$ when $t \neq 0$ and $\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(l+k-r)$ when $t=0$.

For a more geometric description of a family of $S_{k,l}$ degenerating to $S_{k-1,l+1}$ consider a surface $S_{k,l+1}$. When we project the surface from a point away from the directrix, we obtain $S_{k,l}$. However, when we project the surface from a point on the directrix we obtain $S_{k-1,l+1}$. Now projecting $S_{k,l+1}$ from the points along a curve that meets the directrix in isolated points, we obtain the desired family. This construction easily generalizes to $r>1$.

Construction 2. There exists a family of scrolls $S_{k,l}$ degenerating to the union of S_{k_1,l_1} and S_{k_2,l_2} with $k_1+k_2=k$. We think of scrolls as projectivizations of vector bundles over $\mathbb{P}^1$. We choose a flat family of $\mathbb{P}^1$s with smooth total space over the unit disk whose general member is smooth, but whose central fiber has two components meeting transversely at one point. Given a line bundle $\mathcal{O}_{\mathbb{P}^1}(k)$ on the general fiber there is always a limit line bundle on the special curve. However, the limit does not have to be unique. Limits differ by twists of one of the components of the reducible fiber. We can get any splitting of k on the two components. A similar consideration applies for $\mathcal{O}_{\mathbb{P}^1}(l)$. Taking the desired splitting and projectivizing gives us the desired family of Hirzebruch surfaces.

Remark: Since $\overline{\mathcal{M}}_{0,0}(\mathbb{G}(1,N), k+l)$ is an irreducible, smooth Deligne-Mumford stack, every union of scrolls whose dual graph is a connected tree can be smoothed to an $S_{k,l}$ for some k and l. However, the minimal k depends on the alignment of directrices on the reducible surface (see Example preceding Construction 1). This is the phenomenon we would like to analyze next.

The limits of section classes. Let $\mathcal{X} \to B$ be a flat family of scrolls subject to the hypotheses in the first paragraph of §4. Let $\mathcal{C} \to B$ be a flat family of curves such that $\mathcal{C}_b \subset \mathcal{X}_b$ is a smooth curve in a section class for $b \neq b_0$. We say a curve on $S_{0,l}$ is in a section class if it is the image of a curve in a section class on F_l.

Proposition 4.2. *The limit $\mathcal{C}_{b_0}$ restricts to a curve in a section class on each component of $\mathcal{X}_0$.*

Proof: By Proposition 4.1 the central fiber $\mathcal{X}_0$ is the union of scrolls, so it is meaningful to require the restriction of a curve to a component to be in a section class. Since meeting the fibers is a closed condition, $\mathcal{C}_{b_0}$ meets each fiber. To see that it does not meet the general fiber more than once (away from the cone point of any $S_{0,l}$), consider the one parameter family $\mathcal{Y} \to B$ of rational curves in $\mathbb{G}(1, N)$ corresponding to our family of surfaces. Every component of the central fiber in this family is reduced. Hence, the total space of the family cannot be singular along an entire component. Each curve $\mathcal{C}_b$, for $b \neq b_0$, maps isomorphically to $\mathcal{Y}_b$. By Zariski's Connectedness Theorem the fibers of the map are connected over the smooth locus. The proposition follows. □

Finally, we recall a fact from intersection theory (see [Ful] chapter 12) that will be helpful in determining limits of section classes. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two flat families of curves contained in a flat family of surfaces $\mathcal{X} \to B$ over a smooth base curve. Assume that the general fiber of the family is smooth and that on the general fiber $\mathcal{C}_{1,b}$ and $\mathcal{C}_{2,b}$ are smooth curves that meet transversely at γ points. Let $I_0 \subset \mathcal{X}_0$ denote the set of isolated points of intersection of $\mathcal{C}_{1,0}$ and $\mathcal{C}_{2,0}$ contained in the smooth locus of $\mathcal{X}_0$. Let i_p denote the intersection multiplicity at the point p.

Lemma 4.3. *The intersection multiplicities satisfy the inequality*

$$\gamma \geq \sum_{p \in I_0} i_p(\mathcal{C}_{1,0}, \mathcal{C}_{2,0}).$$

Limits of directrices. Proposition 4.2 allows us to determine the limits of directrices as scrolls degenerate. To ease the exposition first assume that the central fiber of $\mathcal{X} \to B$ has two components S_{k_i,l_i}, $i = 1, 2$. For definiteness let $k_1 + l_2 \leq k_2 + l_1$. We still assume that the limit surface spans $\mathbb{P}^{k+l+1}$. Whenever we refer to the directrix of a perfectly balanced scroll, we mean 'a' directrix.

Proposition 4.4. *The flat limit of the directrices is a connected curve of total degree k whose restriction to each surface is in a section class. Conversely, any connected curve D of degree k, $k \leq \sum(k_i + l_i)/2$, whose restriction to each of the surfaces is in a section class is the limit of the directrices of a one parameter family of scrolls $S_{k,l}$.*

Proof: The first assertion is a restatement of Proposition 4.2. To prove the second assertion we will explicitly construct the desired families of $S_{k,l}$. There are two cases depending on whether D contains a multiple of the line L joining the two surfaces. If D does not contain L, then D must consist of a section (possibly with some fibers) in each surface meeting L at the same point.

When $S_{k,l}$ degenerates to $S_{k-1,l+1}$, the limit of the directrices of $S_{k,l}$ is the union of the directrix of $S_{k-l,l+1}$ and a fiber since these are the only connected degree k curves in a section class on $S_{k-1,l+1}$. Since the projective linear group acts transitively on the fibers of a scroll, using Construction 1 we conclude that the union of the directrix and a fiber on $S_{k-1,l+1}$ can be smoothed to the directrix of an $S_{k,l}$. Inductively, we conclude the analogous statement for $S_{k-r,l+r}$. We can, therefore, assume that D consists of 2 sections meeting L at the same point.

If $k = k_1 + k_2$, then the limit of the directrices must be the union of the directrices of the limit surfaces. A section class on a scroll S_{k_i,l_i} has degree at least k_i. Since the total degree is $k = \sum_i k_i$, the curve must break exactly into degree k_i section classes. A section class of degree k_i on S_{k_i,l_i} is the directrix. Using Construction 2 and the fact that $\mathbb{P}GL(k + l + 2)$ acts transitively on the pairs of surfaces S_{k_i,l_i} that span $\mathbb{P}^{k+l+1}$, meet along a line and whose directrices meet along their common line, we conclude that such pairs can be smoothed to $S_{k,l}$ so that the union of their directrices deforms to the directrix of $S_{k,l}$.

The case $k_1 + k_2 < k$. Since we are assuming that $k_1 + l_2 \leq k_2 + l_1$, D must consist of the directrix in S_{k_1,l_1} and a section of degree $k - k_1$ in S_{k_2,l_2}. Set $j = l + k - k_1 - l_1 - k_2$. Consider a family of $S_{k,l+k-k_1-k_2}$

over a nonsingular curve degenerating to the union of S_{k_1,l_1} and $S_{k-k_1,j}$ as described in Construction 2. By our discussion the directrices must specialize to the union of the directrices. Now choose $k-k_1-k_2$ general sections of the family (possibly after passing to an open subset of the base) that all pass through general points of $S_{k-k_1,j}$. (Observe that the total space of the family we exhibited in Construction 2 is generically smooth on every component of the special fiber, so we can select such sections.) Projecting the family along these sections gives a family of $S_{k,l}$ having the desired numerical properties.

We need to verify that we can get all sections C of degree $k-k_1$ on S_{k_2,l_2} as the projection of the directrix of $S_{k-k_1,j}$ from suitable points on the surface. On S_{k_2,l_2} blow up $k-k_1-k_2$ general points p_i on C. Let Ξ be the sum of the exceptional divisors. The linear series $e+(k-k_1-k_2)f-\Xi$ maps the blow up to projective space as $S_{k-k_1,j}$. This map contracts the fibers passing through p_i and maps C to the directrix. Projecting $S_{k-k_1,j}$ from the points corresponding to the image of the contracted fibers projects $S_{k-k_1,j}$ to S_{k_2,l_2} and the directrix onto C. This concludes the construction.

Now we treat the case when D contains a multiple of the common line L. As in the previous case we can assume that D consists of two sections and L with multiplicity m. D must contain the directrix in S_{k_1,l_1} and a section of degree $k-k_1-m$ in S_{k_2,l_2}. Using the previous case inductively, we can find a family of scrolls $S_{k+m,l+m}$ degenerating to a chain of S_{k_1,l_1}, m quadric surfaces and S_{k_2,l_2} such that their directrices specialize to the directrix of S_{k_1,l_1}, a section of degree $k-k_1-m$ in S_{k_2,l_2} and a chain of conics on the quadric surfaces connecting these two curves. Choose $2m$ general sections which specialize to a pair of points on a fiber on each quadric surface. Projecting the family from those sections gives the desired family. □

Remark: We can inductively extend Proposition 4.4 to the case when the central fiber contains more than two components. The following theorem summarizes the conclusion:

Theorem 4.5. *Suppose a one-parameter family of scrolls $S_{k,l}$ specializes to a tree of scrolls $\bigcup_{i=1}^r S_{k_i,l_i}$. Then the limit of the directrices is a connected curve of degree k whose restriction to each surface is in a section class. Conversely, given any connected curve C of degree $k \leq \sum_i (k_i+l_i)/2$ whose restriction to each component is in a section class, there exists a one-parameter family of $S_{k,l}$ specializing to the reducible surface such that the limit of the directrices is C.*

Limits of other section classes. When a family $\mathcal{X} \to B$ of scrolls $S_{k,l}$ specializes to a union of two scrolls $S_{k_1,l_1} \bigcup S_{k_2,l_2}$, then the flat limit of curves in a section class $e+mf$, $m \geq l-k$, are connected curves of total degree $m+k$ that restrict to section classes. Suppose that the total space of the family is smooth. The curves give a line bundle L over $\mathcal{X}\backslash\mathcal{X}_0$. We can always extend this line bundle to the entire family. However, when the central fiber is reducible, this extension is not unique. Twisting by the components of the central fiber give different extensions.

For concreteness, suppose the line bundle L is the pull-back of $\mathcal{O}_{\mathbb{P}^N}(1)$ to $\mathcal{X}\backslash\mathcal{X}_0$. Let L_0 be the line bundle over $\mathcal{X}_0$ arising from any extension of L. If the restriction of L_0 to each component is effective, then the degree of $L_0|_{S_{k_1,l_1}}$ ranges between k_1 and $k+l-k_2$. One component of the limit curves corresponds to hyperplanes not containing either of the components of the limit scroll. The other limits correspond to hyperplane sections by hyperplanes containing one of the scrolls and tangent to a certain order to the other one along their common line.

Example: When a family of scrolls $S_{2,4}$ specialize to $S_{1,1} \cup S_{1,3}$, a possible limit of the hyperplane sections restricts to a degree 5 curve on $S_{1,1}$ and the directrix on $S_{1,3}$. The dimension of the space of such curves is 8. However, the dimension of the space of hyperplane sections was only 7. Consequently, not all quintics can arise as the limit of hyperplane sections of our family of scrolls. Lemma 4.3 provides the answer. On the smooth surfaces the directrices have intersection number 2 with the hyperplane sections. The limit of the directrices is the union of the directrix D of $S_{1,3}$ and the directrix L on $S_{1,1}$ meeting D. A general quintic meeting D meets L transversely in 3 other points. We conclude that a quintic can be part of a limit of the hyperplane section only if it has contact of order 2 with L at their point of intersection on the common fiber.

The following lemma, which is an immediate consequence of Lemma 4.3 and Proposition 4.2, summarizes the general situation.

Lemma 4.6. *Suppose a one parameter family of scrolls $S_{k,l}$ specializes to a union $\bigcup_{i=1}^{r} S_{k_i,l_i}$. Then the limit of curves in a section class $e+mf$ specialize to connected curves of degree $m+k$. Their restrictions to each surface lie in a section class. Furthermore, the sum of the intersection multiplicities of a limit curve with the limit of the directrices at isolated points of their intersection on the smooth locus of the surface cannot exceed $m+k-l$.*

5. Examples

Example A. Counting cubic scrolls in $\mathbb{P}^4$. Since by Lemma 2.6 there is an 18 dimensional family of cubic scrolls $S_{1,2}$ in $\mathbb{P}^4$, there are finitely many containing $9-n$ general points and meeting $2n$ general lines.

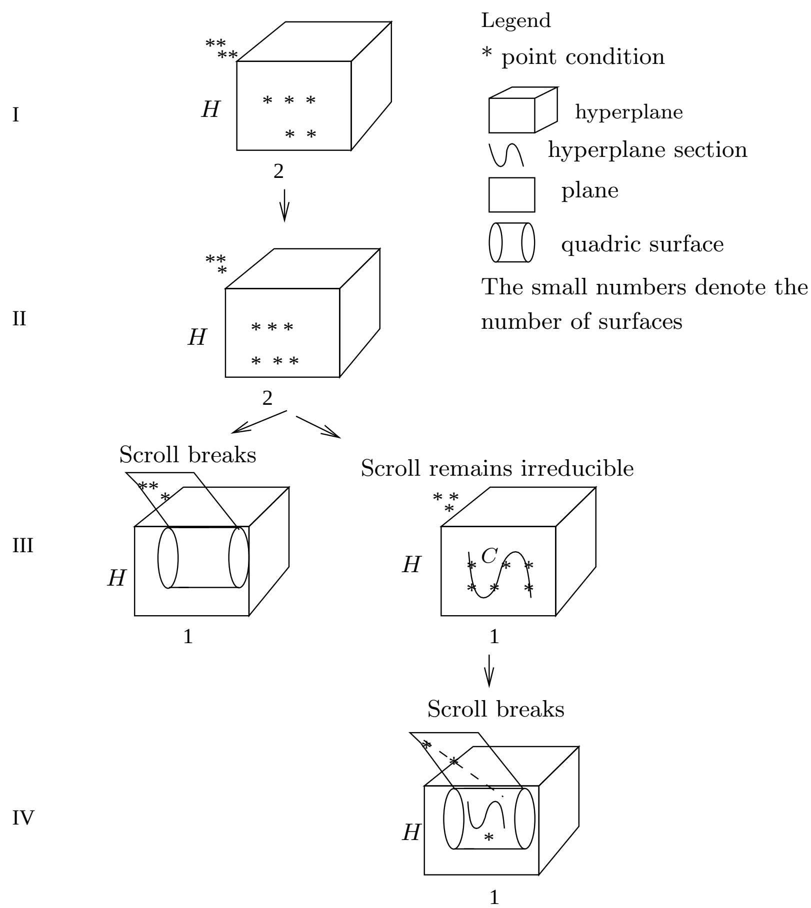

Figure 2. Example A1. Cubic scrolls containing 9 general points in $\mathbb{P}^4$.

A1. Cubic scrolls containing 9 general points. We specialize the nine points one by one to a fixed hyperplane H in $\mathbb{P}^4$ (see Figure 2). We can take H to be the span of four of the points.

Step I. Specialize a fifth point p_5 to a general point of H. There are no reducible scrolls at this stage containing all the points. Any reducible scroll would be the union of a quadric and a plane meeting along a line. Since no six of the points lie in a $\mathbb{P}^3$, the quadric could contain at most 5 of the points. However, the remaining 4 points do not lie on a plane.

Step II. Specialize a sixth point p_6 to H. Now there is a reducible solution: the plane P spanned by the 3 points not in H and the unique quadric in H containing $H \cap P$ and the six points in H. However, at this stage there might still be irreducible solutions. Their hyperplane section in H must be the unique twisted cubic C that contains the 6 points in H.

Step III. Specialize a seventh point p_7 to a general point of H. Bezout's theorem forces the scrolls to break into a union of a quadric surface and a plane. The quadric Q must contain the twisted cubic C and p_7. Since the plane and the quadric meet in a line, Q must also contain the point of intersection q of H with the line spanned by the points not contained in H. This determines the quadric uniquely. The plane is also determined because it must contain the line in the quadric through q which meets C only once: recall by Lemma 4.2 the curve C is in a section class.

Later we will check that both of the solutions occur with multiplicity 1. This will prove that there are 2 cubic scrolls containing 9 points in $\mathbb{P}^4$.

A2. Cubic scrolls in $\mathbb{P}^4$ containing 6 points and meeting 6 lines. We carry out the degenerations required to see that there are 1140 cubic scrolls in $\mathbb{P}^4$ containing 6 points and meeting 6 lines (see Figure 3).

Step I. We specialize 5 points and a line l_1 to a fixed hyperplane H. This is the first stage where reducible solutions exist. There can be a quadric Q contained in H and a plane P not contained in H meeting Q in a line. There are 4 possibilities: of the 5 lines not contained in H, 4, 3, 2, or 1 of them can meet Q and 1, 2, 3, or 4 remaining lines, respectively, can meet P.

If 4 of the lines meet Q, then Q is determined uniquely. P can be any of the 4 planes meeting the remaining line, containing the remaining point and meeting Q in a line. Since l_1 meets Q in 2 points, each of the solutions count twice for the choice of point. Finally, we have a factor of 5 for the choice of which 4 lines among the 5 meet the quadric Q. We conclude that there are 40 such surfaces. The analysis of the other three cases is similar.

At this stage some scrolls can remain irreducible. Their hyperplane section C in H is then a twisted cubic containing the 5 points and meeting l_1.

Step II. We specialize another line l_2 to H. There are new reducible solutions.

Case i. There can be a plane P in H and a quadric Q meeting it along a line. P must be one of the 10 planes spanned by 3 of the points in H. Finally, Q must contain the other two points in H, meet P in a line, and meet the lines and the point p not contained in H. Further specialization shows that there are 6 such quadrics. Hence we get 60 solutions.

Case ii. There can be a quadric Q in H and a plane P not contained in H meeting Q in a line. There are four possibilities: 1, 2, 3 or 4 of the lines can meet P. Let us analyze the case when 3 lines meet Q. The limit hyperplane section C contained in Q must meet 5 points and l_1. Among the the quadrics containing 8 points only a finite number contains such a twisted cubic. To determine the number we specialize the conditions on the cubic curve.

Take a general plane Π and specialize 3 of the points and l_1 to Π. By Bezout's theorem C has to break into a conic and a line. The line can be any of the three lines containing two of the points in Π or it can be the line joining the two points not contained in Π. Once we require Q to contain any of the lines, it is uniquely determined. When there is a line l in Π, the limit of C meets l_1 only in $l \cap l_1$. When the line l is not contained in Π, the limit of C has a conic in Π which must meet l_1 in 2 points, hence we get a multiplicity of 2. The other cases are analogous.

By Lemma 4.2 the curve C must be in a section class in Q, hence should meet the fibers only once. When counting quadric surfaces, one has to be careful to distinguish between the rulings.

Step III. Finally, there can still be irreducible scrolls. In that case their hyperplane section must be one of the 5 twisted cubics in H containing 5 points and meeting l_1 and l_2 (see §2.3 of [V2]). The analysis is similar to the previous cases.

Example B. Counting quadric surfaces in $\mathbb{P}^4$. The degeneration method allows us to count different types of scrolls. We illustrate this by counting quadric surfaces and quadric cones in $\mathbb{P}^4$ (see Figure 4).

B1. Quadric surfaces in $\mathbb{P}^4$ containing 3 points and meeting 7 lines. Step I. Specialize the three points p_1, p_2, p_3 and a line l_1 to the hyperplane H. At this stage there is a unique quadric contained in

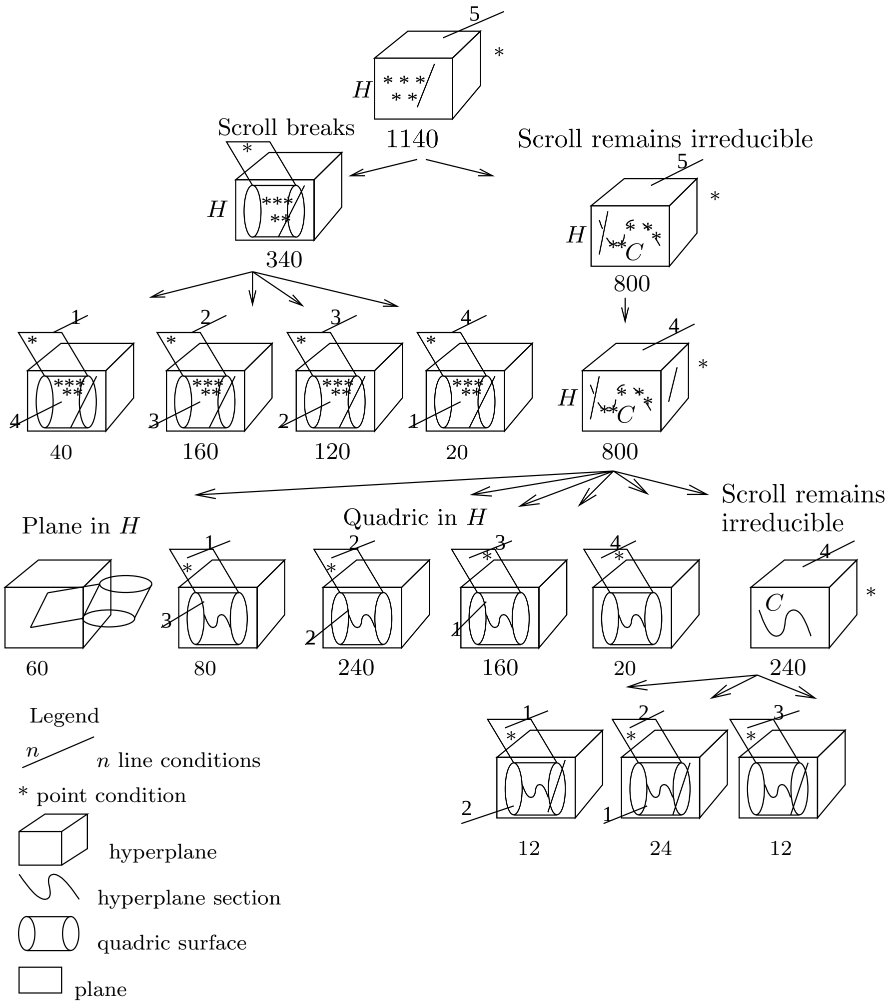

FIGURE 3. Example A2. Cubic scrolls in $\mathbb{P}^4$ containing 6 points and meeting 6 lines.

H satisfying all the incidence conditions. It counts with multiplicity 2 for the choice of intersection point with l_1.

If a quadric is not contained in H, its hyperplane section must lie in the plane Π spanned by the points p_1, p_2, p_3 and must meet l_1 at $l_1 \cap \Pi$.

Step II. We specialize l_2. The quadric can lie in H. If not, the hyperplane section must be the unique conic containing p_i and $l_j \cap \Pi$. Specializing a third line l_3 forces the quadrics to either become reducible or to lie in H. We obtain a total of 9 quadric surfaces containing 3 points and meeting 7 lines.

B2. Quadric cones in $\mathbb{P}^4$ containing 3 points and meeting 6 lines. We compare the case of quadric cones to the case of quadric surfaces.

Step I. Specialize the points p_1, p_2, p_3 and the line l_1 to H. The cone can lie in H. There are 4 quadric cones in $\mathbb{P}^3$ containing 8 general points. Each solution counts with multiplicity 2 for the choice of intersection point with l_1.

Step II. If a cone does not lie in H, its hyperplane section in H must lie in the plane Π spanned by p_i, so it must meet p_i and $q = l_1 \cap \Pi$. We specialize another line l_2 to H. The cone can lie in H. Again there are 4 solutions each counted with multiplicity 2.

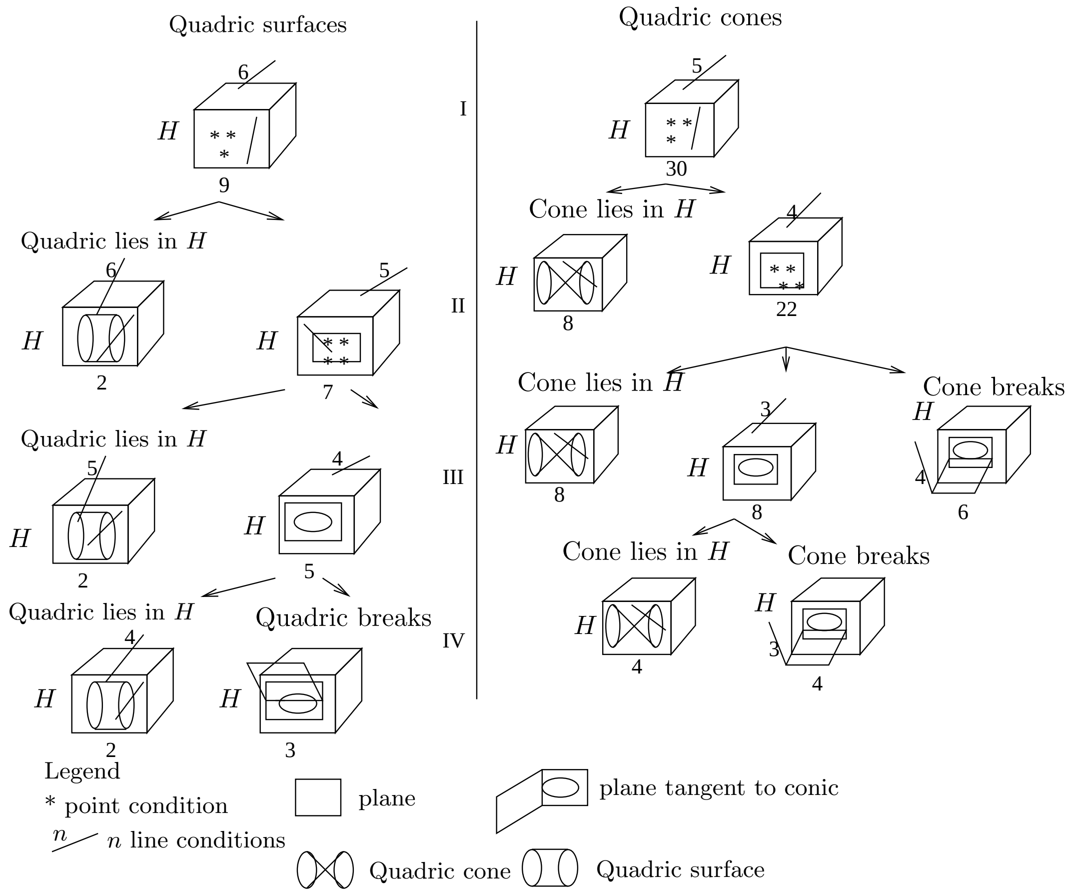

FIGURE 4. Example B. Counting quadric surfaces containing 3 points and meeting 7 lines and quadric cones containing 3 points and meeting 6 lines.

There can also be reducible solutions: the union of Π and one of the three planes that meet Π in a line and meet the three lines not contained in H. This case is delicate. The limit of the hyperplane sections is a conic containing p_i and q. The two planes are images of Hirzebruch surfaces F_1 whose directrices are contracted. The image conic is in the class $e + 2f$ on the F_1 it lies in. It also meets the directrix of the other F_1. Hence, the limit of the hyperplane section has to contain p_i and q and be tangent to the line common to the planes at the limit of the vertices of the nearby cones. There are two conics containing 4 points and tangent to a line in $\mathbb{P}^2$. The two points of tangency give us the possible limiting positions of the vertices of our original family of cones. So each of the pairs of planes can be the limit of cones in two ways depending on the choice of the vertex point. We thus get a count of 6.

Step III. If the cone is neither reducible nor contained in H, then the hyperplane section in H is determined. Specializing a third line l_3 forces the cones to break or to lie in H. The calculations are analogous to the previous case. We obtain a total of 30 quadric cones containing 3 points and meeting 6 lines.

Example C. Counting scrolls of degree n in $\mathbb{P}^{n+1}$ containing $n+5$ points and meeting an $n-3$ plane. We give a final example to illustrate the types of recursive formulae one might hope to obtain from our method of counting. Observe that by Lemma 2.6 there will be finitely many scrolls $S_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n+1}{2} \rfloor}$ in $\mathbb{P}^{n+1}$ containing $n+5$ points and meeting an $n-3$ plane P. Let us denote this number by $S(n)$.

Step I. We specialize the points to a hyperplane H. The first reducible solution occurs when $n+3$ of the points lie in H: a plane not contained in H and a scroll of degree $n-1$ in H meeting the plane in a

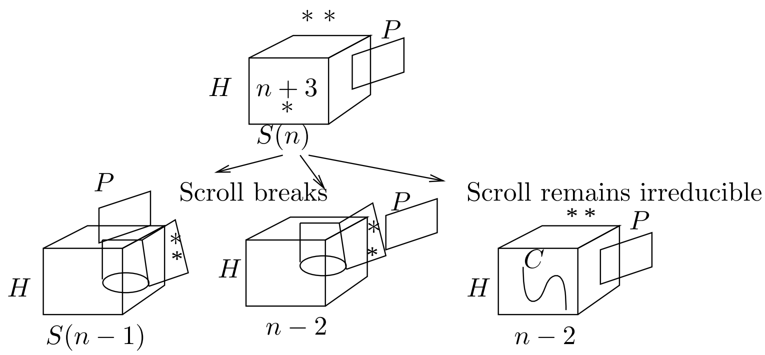

Figure 5. Counting degree n scrolls in $\mathbb{P}^{n+1}$ containing $n+5$ points and meeting an $n-3$ plane.

line. The degree $n-1$ scroll has to meet q, the point of intersection of H with the span of the two points not contained in H. If the scroll meets P, then we reduce to the same problem in degree one less, so the number is $S(n-1)$.

If the plane meets P, then the scroll must contain a line l in $P' = H \cap P$ containing q. We have to count scrolls of degree $n-1$ in $\mathbb{P}^n$ containing a line through a point in a $\mathbb{P}^{n-2}$ and containing an additional $n+3$ points in general position. A further specialization shows that there are $n-2$ such scrolls.

If the scrolls remain irreducible, then their hyperplane section C in H is the unique rational normal curve containing all the points in H. When we specialize the $n-3$ plane P to H the scroll breaks into a degree $n-1$ scroll union a plane. We are reduced to counting degree $n-1$ scrolls containing C, a point and meeting P. It is easy to see that there are $n-2$ such scrolls by first breaking C into a degree $n-1$ curve union a line and then specializing P into the linear space spanned by the degree $n-1$ curve. Solving the recursion we conclude that there are $(n-1)(n-2)$ degree n scrolls in $\mathbb{P}^{n+1}$ containing $n+5$ points and meeting an $n-3$ plane.

We will now justify the calculations made above by making the necessary dimension counts and multiplicity calculations. In a table at the end of the paper we will provide some other characteristic numbers of surfaces.

6. Degenerations Set Theoretically

In this section we describe the set theoretical limits of surface degenerations under the assumption that the surfaces and their successive hyperplane sections remain non-degenerate. The calculation in this section will be purely set theoretic. We collect our notation here for the reader's convenience.

The table of notation
H, Π hyperplanes in $\mathbb{P}^n$
$\{\Delta^i_{a_i}\}_{i=1}^I$, $\{\Sigma^i_{a'_i}\}_{i=1}^{I'}$ $\{\tilde{\Lambda}^i_{\tilde{b}_i} \subset \Lambda^i_{b_i}\}_{i=1}^Y$ general linear subspaces of $\mathbb{P}^n$
$\{\Gamma^j_{c_j}\}_{j=1}^{J_H}$ general linear subspaces of H
$\{\Omega^j_{d_j}\}_{j=1}^{J_\Pi}$ general linear subspaces of Π
$\Delta(0), \dots, \Delta(M)$ a partition of the collection $\{\Delta^i_{a_i}\}_{i=1}^I$ into $M+1$ parts
MS_H, MS_Π spaces of maps from a Hirzebruch surface
M MS_H or MS_Π
D_H divisor of MS_H defined by requiring a point to map to H
D_Π divisor of MS_Π defined by requiring a point to map to Π
D directrix or the limit of the directrix
C hyperplane section of the surface in H or its limit
q_i marked points on the surface
q'_i, p_j marked points of C
λ_i marked fiber lines on the surface
X_H, X_Π, W_H, W_Π divisors in $\overline{MS}_H$ and $\overline{MS}_\Pi$

Notation: Let H and Π be general hyperplanes in $\mathbb{P}^N$. In our algorithm we will specialize a linear space meeting either a surface or a curve on a surface to a hyperplane. We will refer to the hyperplane by H when we specialize a condition on a surface and by Π when we specialize a condition on a curve.

The dimension of a linear space will be indicated by a subscript. When the dimensions are fixed in a discussion, we will omit them from the notation. Let $\{\Delta^i_{a_i}\}_{i=1}^I$ and $\{\Sigma^i_{a'_i}\}_{i=1}^{I'}$ be two collections of general linear subspaces of $\mathbb{P}^n$. These linear subspaces will impose conditions that we have not yet specialized to H or Π. We will require our surfaces to meet the first collection of linear spaces and our marked curve to meet the latter. Let $\{\tilde{\Lambda}^i_{\tilde{b}_i} \subset \Lambda^i_{b_i}\}_{i=1}^Y$ be a collection of pairs of general linear subspaces of $\mathbb{P}^N$. We will have marked fiber lines on our surfaces that are required to lie in $\Lambda^i_{b_i}$ and meet $\tilde{\Lambda}^i_{\tilde{b}_i}$.

Let $\{\Gamma^j_{c_j}\}_{j=1}^{J_H}$ and $\{\Omega^j_{d_j}\}_{j=1}^{J_\Pi}$ be collections of general linear subspaces of H and Π, respectively. These will be the linear spaces that we have already specialized to H and Π, respectively.

When the surface becomes reducible, each irreducible component satisfies some of the constraints. $\Delta(0), \cdots, \Delta(M)$ will denote a partition of the collection $\{\Delta^i_{a_i}\}_{i=1}^I$ into $M+1$ subcollections. The i-th irreducible component of the surface will satisfy the constraints $\Delta(i)$. We will use analogous notation for the other collections of linear spaces and for the points and the fibers.

Spaces of maps. We now define a sequence of spaces of maps from Hirzebruch surfaces to $\mathbb{P}^N$. These will correspond to scrolls with two marked curves C and D, marked fibers λ_i and marked points p_j, q_i, q'_i. C will denote the hyperplane section in H or a limit of it. D will be the directrix or a limit of it. The scrolls will meet the linear spaces Δ^i along q_i. The curve C will meet the linear constraints along p_j and q'_i. In addition the fiber λ_i will be required to lie in the linear space Λ^i and meet the linear space $\tilde{\Lambda}^i$ (see Figure 6).

Definition of MS_H. Let MS_H be the set of maps π, up to isomorphism, from a Hirzebruch surface F_{l-k}, with two marked sections (C, D), Y marked fibers λ_i and $I + J_H$ marked points $\{q_i\}_{i=1}^I$ and $\{p_j\}_{j=1}^{J_H}$, to $\mathbb{P}^N$ whose image is an $S_{k,l}$ not contained in H such that

1. $\pi(C) = \pi(S) \cap H$.
2. D is the directrix (or if $k = l$, a choice of directrix).
3. The images of the marked fibers λ_i are contained in the linear spaces Λ^i and meet the linear spaces $\tilde{\Lambda}^i$.
4. q_i are marked points on the surface and p_j are marked points on the curve C. We assume that they are distinct points whose images lie in the linear spaces Δ^i and Γ^j, respectively.

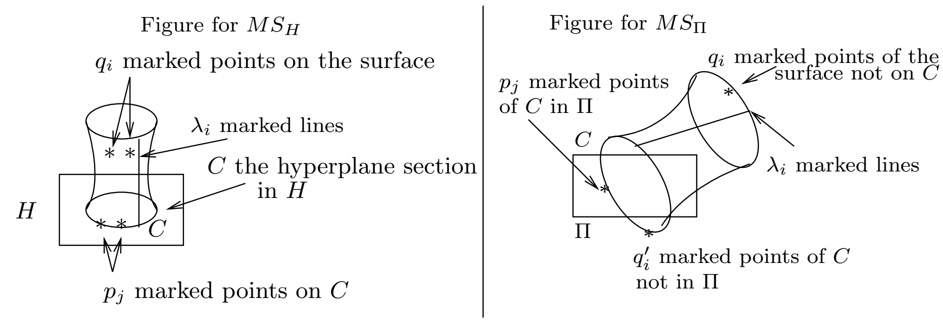

Figure 6. Definitions of MS_H and MS_Π.

In what follows we will use

$$MS_H(\mathbb{P}^N;\ k,l;\ C(k+l),\ D;\ Y,\ I,\ J_H,\ \pi : \pi^{-1}(H) = C;$$
$$(\pi(\lambda_i) \subset \Lambda^i, \pi(\lambda_i) \cap \tilde{\Lambda}^i \neq \emptyset)_{i=1}^Y, (\pi(q_i) \in \Delta^i)_{i=1}^I, (\pi(p_j) \in \pi(C) \cap \Gamma^j)_{j=1}^{J_H})$$

or any subset of this notation when the rest of the data is clear from the discussion.

Definition of MS_Π. Similarly let MS_Π be the set of maps π, up to isomorphism, from a Hirzebruch surface F_{l-k}, with two marked sections (C, D), Y marked fibers λ_i and $I + I' + J_\Pi$ marked points $\{q_i\}_{i=1}^I$, $\{q_i'\}_{i=1}^{I'}$ and $\{p_j\}_{j=1}^{J_\Pi}$, to $\mathbb{P}^N$ whose image is an $S_{k,l}$ such that

1. $\pi(C)$ has degree d.
2. D is the directrix (or if $k = l$, a choice of directrix).
3. The images of the marked fibers λ_i are contained in the linear spaces Λ^i and meet the linear spaces $\tilde{\Lambda}^i$.
4. q_i are marked points on the surface, q_i' are marked points on the curve C whose images are not contained in Π and p_j are marked points on the curve C that map to Π. We assume that they are distinct points whose images lie in the linear spaces Δ^i and Σ^i and Ω^j, respectively.

In what follows we will use

$$MS_\Pi(\mathbb{P}^N;\ k,l;\ C(d), D; Y, I, I', J_\Pi; \pi : (\pi(\lambda_i) \subset \Lambda^i, \pi(\lambda_i) \cap \tilde{\Lambda}^i \neq \emptyset)_{i=1}^Y,$$
$$(\pi(q_i) \in \Delta^i)_{i=1}^I, (\pi(q_i') \in \pi(C) \cap \Sigma^i)_{i=1}^{I'}, (\pi(p_j) \in \pi(C) \cap \Omega^j)_{j=1}^{J_\Pi})$$

or any subset of this notation when the rest of the data is constant during a discussion.

We can compactify both MS_H and MS_Π in a manner analogous to §3. For concreteness we explain the construction for MS_H. The case of MS_Π is identical. Let $\mathbb{F}(0,0,1;N)$ denote the variety of two-pointed lines in $\mathbb{P}^N$. A curve in $\mathbb{F}(0,0,1;N)$ is determined by three degrees, the degrees (d_0, d_1) of the two projections γ_0, γ_1 to $\mathbb{P}^N$ and the degree d_2 of the projection γ_2 to $\mathbb{G}(1,N)$. Given a map π in MS from a surface S with sections C and D, we get a stable map to $\mathbb{F}(0,0,1;N)$ by sending (S, C, D, π) to the map σ from C given by $\sigma(p) = (\pi(p), \pi(D \cap F_p), [\pi(F_p)])$ where F_p is the fiber through p. We have to enhance this correspondence to mark the points on the surface. To do that we simply take the I-th fold fiber product of the universal family $\mathcal{U}$ over the stack $\overline{\mathcal{M}}_{0,J_H+Y}(\mathbb{F}(0,0,1;N),(d_0,d_1,d_2))$. We will denote the closure of MS_H and MS_Π in these stacks as $\overline{\mathcal{MS}}_H$ and $\overline{\mathcal{MS}}_\Pi$, respectively. When we do not want to distinguish between them we will use the notation $\overline{\mathcal{M}}$.

Definition of the Divisors $\mathcal{D}_H$**,** $\mathcal{D}_\Pi$**.** In $\overline{\mathcal{MS}}_H$ requiring the stable map to map p_I into H defines a Cartier divisor. We will denote this divisor by $\mathcal{D}_H$. Similarly, $\overline{\mathcal{MS}}_\Pi$ has a Cartier divisor $\mathcal{D}_\Pi$ defined by requiring the image of $q_{I'}'$ to lie in Π. Colloquially, the surfaces in the divisors are the surfaces we see after we specialize one of the points to a hyperplane.

Definitions of X_H and X_Π. Let

$$X_H(\mathbb{P}^N; (k_i, l_i; C(d_i), D(e_i), \lambda(i), q(i), p(i) \in C(i))_{i=0,1}; \pi)$$

be the set of maps, up to isomorphism, from a pair of Hirzebruch surfaces $F_{l_i - k_i}$ meeting transversely along a fiber such that

1. $S_0 := \pi(F_{l_0-k_0}) \subset H$ is a scroll S_{k_0,l_0} contained in H.
2. $S_1 := \pi(F_{l_1-k_1})$ is a scroll S_{k_1,l_1} not contained in H and which meets H transversely along the line that joins it to S_0 (and possibly elsewhere).
3. $C(d_i)$ is a section of degree d_i and $D(e_i)$ is a section of degree e_i on S_i such that $C(d_0) \cup C(d_1)$ and $D(e_0) \cup D(e_1)$ form a connected curve.
4. $\lambda(i)$, $q(i)$ and $p(i)$ is a partition of the marked fibers and points to the two components. They satisfy the same incidence and containment relations as in the definition of MS_H.

The definition of X_Π is analogous, but with H replaced by Π and the names of the linear spaces and points modified as in the definition of M_Π. Denote the corresponding stacks in $\overline{\mathcal{MS}}_H$ and $\overline{\mathcal{MS}}_\Pi$ by $\mathcal{X}_H$ and $\mathcal{X}_\Pi$, respectively.

Definitions of W_H and W_Π. Let

$$W_H(\mathbb{P}^N; (k_i, l_i; C(d_i), D(e_i), \lambda(i), q(i), p(i) \in C(d_i))_{i=0}^M; \pi)$$

be the set of maps, up to isomorphism, from the union of $M+1$ Hirzebruch surfaces to $\mathbb{P}^N$ such that

1. All the components S_i for $i > 0$ are attached along distinct fibers to a central component S_0,
2. $\pi(S_0)$ is a cone not contained in H,
3. $C(d_0)$ and $D(e_0)$ both contain the directrix of S_0,
4. The other surfaces all map to H ($\pi(S_i) \subset H$ for $i > 0$),
5. The fibers $\lambda(i)$ and the marked points $q(i)$ and $p(i)$ are distributed according to a partition and satisfy the same incidence and containment relations as in the definition of MS_H.

The definition of W_Π is analogous, but with H replaced by Π and the names of the linear spaces and points modified as in the definition of MS_Π. We will denote the corresponding stacks by $\mathcal{W}_H$ and $\mathcal{W}_\Pi$.

Dimension counts. With this preparation we can begin the dimension counts.

Proposition 6.1. *Let A be a reduced, irreducible substack of $\overline{\mathcal{M}}$ and let p be any of the labeled points. Then there exists a Zariski-open subset U of the dual projective space $\mathbb{P}^{N*}$ such that for all hyperplanes $[H] \in U$ the intersection $A \cap \{\pi(p) \in H\}$ is either empty or reduced of dimension* $\dim A - 1$.

Proof: $\mathbb{P}^N$ is a homogeneous space under the action of the group $\mathbb{P}GL(N+1)$. The proposition follows from Theorem 2.7. In the notation of the theorem take $f : H \to \mathbb{P}^N$ to be the immersion of a hyperplane. Let $g : A \to \mathbb{P}^N$ be the evaluation morphism at p. □

Proposition 6.2. *Let A be a reduced, irreducible substack of $\overline{\mathcal{M}}$ and let λ be any of the marked fibers. Then there exists a Zariski-open subset U of the dual projective space $\mathbb{P}^{N*}$ such that for all hyperplanes $[H] \in U$ the intersection $A \cap \{\pi(\lambda) \in H\}$ is either empty or reduced of dimension* $\dim A - 2$.

Proof: Consider the action of $\mathbb{P}GL(N+1)$ on the Grassmannian $\mathbb{G}(1,N)$. Let $f : \Sigma_{1,1}(H) \to \mathbb{G}(1,N)$ be the immersion of the Schubert cycle of lines contained in H. Let $g : A \to \mathbb{G}(1,N)$ be the evaluation morphism. The proposition follows from Theorem 2.7 and the fact that the cycle $\Sigma_{1,1}(H)$ has codimension 2 in the Grassmannian. □

Remark. In the notation of the Proposition 6.1 or 6.2, we can further deduce that if B is a proper, closed substack of A then every component of $B \cap \{\pi(p) \in H\}$ is a proper closed substack of a component of $A \cap \{\pi(p) \in H\}$ by using the dimension statement in Proposition 6.1 or 6.2 for every component of B.

A convention: Since perfectly balanced scrolls have a one-parameter family of directrices, they behave differently for dimension calculations. In order to have a more uniform treatment, we will declare that a *perfectly balanced scroll does not have any directrices.* We will treat *a directrix of a perfectly balanced scroll like any other section.*

The dimension of the building blocks. We now compute the dimension of the locus of maps, up to isomorphism, from an irreducible Hirzebruch surface F_{l-k} with two marked irreducible sections C and D and $m+n$ marked fibers λ_i to $\mathbb{P}^N$ such that

(i) the image of F_{l-k} is a scroll $S_{k,l}$ that has contact of order m_i along fibers λ_i, $1 \leq i \leq m$, with a fixed hyperplane H,

(ii) the marked curves C, D have degrees d and e, respectively. In case these curves are distinct we assume that they have contact of order n_i with each other along distinct fibers $\lambda_1, \dots, \lambda_k$, $\lambda_{m+1}, \dots, \lambda_{m+n}$ $(k \leq m)$.

Proposition 6.3. *If $\sum_{i=1}^m m_i > k$, then this dimension is*

$$N(k+l+2) + k - l - 5 - \delta_{k,l} + m + \alpha, \text{ where}$$

(i) $\alpha = 0$ if C and D lie in H;

(ii) $\alpha = 2d - k - l + 1$ if C and D coincide, but do not lie in H;

(iii) $\alpha = 2e - k - l + 1 + n - \sum_{i=1}^{n+m} n_i$ if C lies in H, but D does not;

(iv) $\alpha = 2d + 2e - 2k - 2l + 2 + n - \sum_{i=1}^{n+m} n_i$ if C and D are distinct and do not lie in H.

If $\sum_{i=1}^m m_i \leq k$, then the dimension is

$$N(k+l+2) + 2k - 4 - \delta_{k,l} - 2\sum_{i=1}^{m} m_i + m + \alpha, \text{ where}$$

(i) $\alpha = 0$ if C and D both coincide with the directrix or are contained in H;

(ii) $\alpha = n - \sum_{i=1}^{m+n} n_i$ if one of C or D coincides with the directrix and the other lies in H;

(iii) $\alpha = 2d - k - l + 1$ if C and D coincide but are distinct from the directrix and do not lie in H;

(iv) $\alpha = 2d - k - l + 1 + n - \sum_{i=1}^{n+m} n_i$ if D coincides with the directrix or lies in H and C is a section distinct from them;

(v) $\alpha = 2d + 2e - 2k - 2l + 2 + n - \sum_{i=1}^{n+m} n_i$ if C and D do not coincide; are distinct from the directrix and do not lie in H.

Proof: The map from F_{l-k} is given by $N+1$ sections $s_0, \cdots, s_N$ in the class $\mathcal{O}_{F_{l-k}}(e+lf)$. We assume s_0 corresponds to the hyperplane H. In case the sections C and D do not coincide, we choose the additional n lines along which C and D meet. Since there is a one parameter family of fibers this gives us a choice of n dimensions. Finally, we mark two sections C and D on the surface which have the required incidence with each other along the chosen fibers. We need to know the dimension of pairs of sections with specified incidence along specified fibers. The following Lemma gives this dimension.

Lemma 6.4. *On F_{l-k} the projective dimension of pairs of distinct irreducible sections (s_1, s_2) in the classes $e + m_1 f, e + m_2 f$, $m_1 \leq m_2$, respectively, having contact of order n_i, $\sum_i n_i \leq m_1 + m_2 + k - l$, with each other along specified fibers is*

$$2m_1 + 2m_2 + 2k - 2l + 2 - \sum_i n_i.$$

unless $m_1 = 0$ and $k \neq l$. In the latter case the dimension is

$$2m_2 + k - l + 1 - \sum_i n_i.$$

Proof: Set $u = m_1 + m_2 + k - l$ and $t = 2m_2 + k - l + 1$. We can pick the section in the class $e + m_1 f$ arbitrarily. By Lemma 2.5 the dimension of the space of these sections is $2m_1 + k - l + 1$, unless $m_1 = 0$ and $k \neq 0$. In the latter case the dimension is 0. We can embed the surface F_{l-k} by the linear system $|\mathcal{O}_{F_{l-k}}(e + m_2 f)|$. In this embedding the first chosen section is a rational normal curve of degree u. Choosing a curve in the second section class with certain tangency conditions at specified points is equivalent to choosing a hyperplane in $\mathbb{P}^t$ with the required tangency conditions to the rational normal curve along the points of its intersection with the specified fibers. Since the curves are distinct and irreducible the hyperplane will not contain the first curve. Hence, its intersection with the span of the first curve will be a hyperplane of the span.

A rational normal curve is embedded by a complete linear system on $\mathbb{P}^1$. Any two positive divisors of degree d on $\mathbb{P}^1$ are linearly equivalent. A hyperplane having the specified order of contact is given by $n_1p_1 + \cdots + n_rp_r + D$ where D is any sum of $m_1 + k_1 - \sum_i n_i$ points. Such hyperplanes form a $u - \sum_i n_i$ dimensional linear subspace of $(\mathbb{P}^u)^*$. For each such linear space the hyperplanes of $\mathbb{P}^t$ that contain it form an irreducible linear space of dimension $m_2 - m_1 + 1$ in $(\mathbb{P}^t)^*$. The lemma follows. □

Using Lemma 6.4 we can complete the proof of Proposition 6.3. Note that if $\sum_{i=1}^m m_i > k$, then the directrix of the surface has to be contained in H and we must have $\sum_{i=1}^m m_i = l$. In this case the dimension for the choice of s_0 is $m+1$. By Lemma 2.5 the dimension for the choice of each of the other N section classes is $k+l+2$. Finally, we have to choose the sections C and D.

- If they both coincide with the directrix, there is nothing to choose.
- If they coincide, but are in a different section class, by Lemma 2.5 we should add $2d-k-l+1$.
- If C coincides with the directrix, then we have nothing further to choose. If D does not coincide with the directrix, then by Lemma 6.4 the choice for its dimension is $2e - k - l + 1 - \sum_{i=1}^{n+m} n_i$.
- If C and D are distinct section classes different from the directrix, Lemma 6.4 provides us the dimension.

If $\sum_{i=1}^m m_i \leq k$, then the hyperplane H does not contain the directrix. The class in H residual to the lines must be $e+(l-\sum_{i=1}^m m_i)f$. Hence, by Lemma 2.5 the dimension of s_0 is $k+l-2\sum_{i=1}^m m_i+m+1$. The rest of the calculation is analogous to the previous case. We must remember to projectivize and subtract the dimension of the automorphism group of $S_{k,l}$. The proposition follows. □

Gluing scrolls. We now prove that gluing scrolls along fibers to form a tree of scrolls imposes the expected number of conditions. When we glue, we will have to identify a fiber of one surface with a fiber of the other one and match the sections in the different surfaces along the fibers.

Let $X \subset \mathbb{P}^N$ be a projective variety. Let $F_X(s)$ be an irreducible, reduced subscheme of its Fano scheme of s-dimensional linear spaces. Let $(p_i)_{i=1}^q$ denote $q \leq s+1$ points in general linear position. Let $H(X)$ denote the $\mathbb{P}GL(N+1)$ orbit of $[X]$ in the Hilbert scheme. For any $[Z] \in H(X)$, let $F_Z(s)$ be the translation of $F_X(s)$ in the Grassmannian by the element that takes X to Z.

Lemma 6.5. *Let $X_1, X_2 \subset \mathbb{P}^N$ be two projective varieties. In the incidence correspondence*

$$\begin{aligned} I := \{(Z_1,\ \lambda_1,\ (p_j^1)_{j=1}^q,\ Z_2,\ \lambda_2,\ (p_j^2)_{j=1}^q) : Z_i \in H(X_i), \lambda_i \in F_{Z_i}(s), p_j^i \in \lambda_i\} \\ \subset H(X_1) \times \mathbb{G}(s,N) \times (\mathbb{P}^N)^q \times H(X_2) \times \mathbb{G}(s,N) \times (\mathbb{P}^N)^q \end{aligned}$$

the locus J defined by $\lambda_1 = \lambda_2$ and $p_j^1 = p_j^2$ for all $1 \leq j \leq q$ has codimension $(s+1)(N-s)+sq$.

Proof: J is contained in the locus where $\lambda_1 = \lambda_2$. Restricted to this locus J can be seen as the projection that forgets the points on λ_2. Since the choice of q points in $\mathbb{P}^s$ has dimension sq, the fibers of this projection are all irreducible of dimension sq. Hence, to conclude the lemma it suffices to settle the case $q = 0$. Consider the projection from I to the product of the Grassmannians $\mathbb{G}(s,N) \times \mathbb{G}(s,N)$. By assumption all fibers are projectively equivalent under the diagonal action of $\mathbb{P}GL(N+1)$, so they all have the same dimension. We conclude that the codimension of J, which is the pull back of the diagonal, is equal to the codimension of the diagonal in $\mathbb{G}(s,N) \times \mathbb{G}(s,N)$. □

Forming trees. Let T be a connected tree with v vertices numbered from 1 to v. Suppose e_1 of its edges are labeled 1, e_2 of its edges are labeled 2 and the rest are labeled 0. Let ν_i be the valence of the i-th vertex.

Let V be a v-tuple of scrolls where on the i-th one there are ν_i marked lines with two distinct points on each. Let W be the subscheme of V corresponding to the v-tuples that form the tree T. That is if vertex i is adjacent to vertex j, then a marked line of the scroll in the i-th position coincides with a marked line of the scroll in the j-th position and if the edge between the vertices is labeled by 1 or 2, one or both of the points, respectively, coincide (see Figure 7).

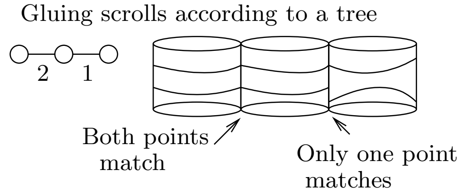

FIGURE 7. An example of gluing scrolls according to a tree.

Proposition 6.6. *The codimension of W in V is*

$$(2N-2)(v-1)+e_1+2e_2.$$

Proof: When there are only two surfaces, the proposition is a special case of Lemma 6.5. To prove the proposition in general induct on the number of vertices by removing a root and using Lemma 6.5 again. A similar argument applies when the lines joining a surface to two adjacent surfaces coincide. □

Main Enumerative Theorems: We now state and prove the main enumerative theorems.

For an open subset of $\overline{\mathcal{M}}$, the space of surfaces, the image of the maps have the same Hilbert polynomial. Hence, we obtain a rational morphism to the Hilbert scheme. The subloci in $\mathcal{X}$ and $\mathcal{W}$ whose general points correspond to maps with non-degenerate image also admit rational maps to the same Hilbert scheme.

Definition 6.7. *A divisor of $\overline{\mathcal{M}}$ whose general point corresponds to a map with non-degenerate image is called* **enumeratively relevant** *if its image in the Hilbert scheme has codimension one in the image of $\overline{\mathcal{M}}$.*

To determine the characteristic numbers of scrolls we count the number of reduced points in the locus of the Hilbert scheme corresponding to the type of scroll we are interested in. Viewed from this perspective only enumeratively relevant divisors of $\overline{\mathcal{M}}$ contribute to the enumerative calculations. Contracted components can add moduli to maps from trees of Hirzebruch surfaces without changing the number of moduli of the image surfaces. In order to eliminate these extra moduli we define the modified tree.

The modified tree. Let π be a map from a tree $\tilde{T}$ of Hirzebruch surfaces. We will refer to vertices corresponding to surfaces that are contracted by π as *contracted vertices* and to subtrees consisting entirely of contracted vertices as *contracted subtrees*. The *modified tree* T is defined to be the same as $\tilde{T}$ if the map π does not contract any surfaces. If π contracts some surfaces, we modify $\tilde{T}$ as follows:

• If a maximal, connected, contracted subtree abuts only one non-contracted vertex, we simply remove it.

• If a maximal, connected, contracted subtree abuts exactly two non-contracted vertices, we remove the subtree and join the two vertices by an edge.

• If a maximal, connected contracted subtree abuts three or more non-contracted vertices, then we remove the contracted tree and join the adjacent vertices to a junction. We mark the junction by c, the number of non-contracted vertices adjacent to the contracted tree. The reader can think of the junction as a fiber common to more than two surfaces in the image of π (Figure 8).

We call a subtree of a modified tree connected, if every vertex has an edge to some other vertex in the subtree or connects to a junction that another vertex in the subtree is connected to. The image of a connected subtree is connected in codimension 1.

Although Theorems 6.8 and 6.9 initially look complicated, they state that when we carry out the degenerations, only very few types of behavior occur (provided we require the surface and the limit of its hyperplane sections in H have maximal possible span). For example, when we specialize the linear space Δ^I to lie in the hyperplane H, then Theorem 6.8 states that the limits of balanced scrolls $S_{k,l}$ meeting Δ^I is one of the following. If it is possible for a rational normal curve of degree $k+l$ to meet all the linear

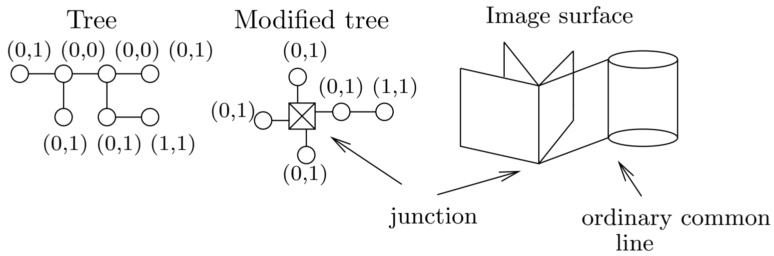

FIGURE 8. The modified tree.

spaces contained in H, then the surface might remain irreducible and balanced and meet Δ^I along its hyperplane section C in H. The entire surface might lie in H. In this case the surface remains irreducible and balanced and must meet the linear spaces not contained in H along their intersections with H. Finally, the surface may become reducible. In this case there are two components both of which are balanced scrolls. Exactly one of them is contained in H. The constraints are distributed among the two components with the exception that Δ^I meets the component of the surface in H. In case the original scrolls were perfectly balanced, the two components do not have to be perfectly balanced. In case the original scrolls were not balanced, then there can be a few other possibilities.

Recall that the divisor $\mathcal{D}_H$ ($\mathcal{D}_\Pi$) in $\overline{\mathcal{MS}}_H$ ($\overline{\mathcal{MS}}_\Pi$) is defined by requiring the marked point q_I ($q'_{I'}$) to lie in H (Π, respectively). The following theorems list the enumeratively relevant components of $\mathcal{D}_H$ and $\mathcal{D}_\Pi$ that satisfy our non-degeneracy assumption.

Theorem 6.8. *Every enumeratively relevant component of the divisor*

$$\mathcal{D}_H \subset \overline{\mathcal{MS}}_H(N; k, l, C(k+l), D, Y, I, J_H)$$

whose general member corresponds to a map where the set theoretic images of the surface and C have maximal span is one of the following:

1. $\overline{\mathcal{MS}}_H(N; k, l, C(k+l), D, Y, I-1, J_H+1)$, the locus of maps whose general point corresponds to maps from F_{l-k} into $\mathbb{P}^N$ with image $S_{k,l}$ satisfying the same conditions as the original surfaces except that the hyperplane section C in H contains the point q_I. This locus is non-empty provided that the constraints that the linear spaces impose on the rational normal curves of degree $k+l$ do not exceed the dimension of the curves.

2. $\overline{\mathcal{MS}}_H(N-1; k, l, C(k+l), D, Y, I, J_H)$, the locus of maps whose general point corresponds to maps from F_{l-k} into H with image $S_{k,l}$ subject to the same constraints as the original surfaces. Since the images lie in H, the surfaces must meet the linear constraints Δ^i, $1 \leq i \leq I-1$, and $\tilde{\Lambda}^i, \Lambda^i$ along their intersections with H. This locus is not enumeratively relevant unless $J_H \geq k+l+1$.

3. $\mathcal{X}(\mathbb{P}^N; (k_i, l_i; C(d_i), D(e_i), \lambda(i), q(i), p(i))_{i=0,1}; \pi)$, where $q_I \in q(0)$, $d_0 = k_0 + l_0 + 1$, $d_1 = k_1 + l_1 - 1$, $e_i = k_i$, $k = k_0 + k_1$ and $J_H(0) \geq k_0 + l_0 + 2$. This is the locus of maps from unions of two Hirzebruch surfaces $F_{l_0 - k_0}$ and $F_{l_1 - k_1}$ into $\mathbb{P}^N$ where the image of the first one lies in H. The points and fibers are distributed among the two components except q_I is required to lie in the surface in H. The directrix specializes to the union of the directrices of the two surfaces. Such a component is enumeratively relevant when $J_H(0) \geq k_0 + l_0 + 2$. In case the surface outside H is a plane, we must in fact have $J_H(0) \geq k_0 + l_0 + 3$.

4. $\mathcal{W}(\mathbb{P}^N; (k_i, l_i; C(d_i), D(k_i), \lambda(i), q(i), p(i))_{i=0}^M; \pi)$, where $k_0 = 0$, $d_0 = k_0 + l_0 - M$, $d_i = k_i + l_i + 1$ for $i > 0$, $q_I \in q(i)$ for some $i > 0$ and $J_H(i) \geq k_i + l_i + 2$ for $i > 0$. This is the locus of maps from a union of $M+1$ Hirzebruch surfaces where M surfaces are attached to a central one. The image of the central surface is a cone and lies outside H. All the other surfaces map to H. Both C and D contain the cone point of the central surface. q_I lies on a surface in H. This component is enumeratively relevant when $J_H(i) \geq k_i + l_i + 2$ for $i > 0$.

In addition, if $k = l$ *there can be*

5. $\mathcal{X}(\mathbb{P}^N; (k_0, k_0+1; C(2k_0+2), D(k_0+1), \lambda(0), p(0), q(0)), (k-k_0-1, k-k_0;$ $C(2k-2k_0-1), D(k-k_0), \lambda(1), p(1), q(1)) : q_I \in q(0), J_H(0) \geq 2k_0+3))$.
6. $\mathcal{X}(\mathbb{P}^N; (k_0, k_0+1; C(2k_0+2), D(k_0), \lambda(0), p(0), q(0)), (k-k_0-1, k-k_0;$ $C(2k-2k_0-1), D(k-k_0+1), \lambda(1), p(1), q(1)) : q_I \in q(0), J_H(0) \geq 2k_0+3)$,

where these are the divisors arising when a perfectly balanced scroll breaks into a union of two balanced scrolls. They are enumeratively relevant when $J_H(0) \geq 2k_0+3$.

Theorem 6.9. *Let* $k+l \leq d \leq k+l+1$. *Every enumeratively relevant component of*

$$\mathcal{D}_\Pi \subset \overline{\mathcal{MS}}_\Pi(\mathbb{P}^N; k, l, C(d), D, Y, I, I', J_\Pi)$$

where the curve C *and the limit divisor cut out on* C *by* Π *remain non-degenerate is of the form*

1. $\overline{\mathcal{MS}}_\Pi(\mathbb{P}^N; k, l, C(d), D, Y, I, I'-1, J_\Pi+1 : \Omega^{J_\Pi+1} = \Sigma^{I'})$*, the locus of maps satisfying the same conditions as* $\overline{\mathcal{MS}}_\Pi$ *except that the curve* C *meets* Π *in the additional marked point* $q'_{I'}$.

2. $\overline{\mathcal{MS}}_\Pi(\mathbb{P}^N; k, l, C(d), D, Y, I, I'-1, J_\Pi : q'_{I'} = p_j)$*, the locus of maps satisfying the same conditions as* $\overline{\mathcal{MS}}_\Pi$ *except that the two marked points* $q'_{I'}$ *and* p_j *map to the same point.* C *meets* $\Sigma^{I'} \cap \Omega^j$ *along this point provided this intersection is non-empty.*

3. $\overline{\mathcal{MS}}_\Pi(\mathbb{P}^N; k, l, C(d), D, Y, I, 0, I'+J_\Pi : \pi(C) \subset \Pi, J_\Pi = d \leq k+l)$*, the locus of maps where* C *specializes to lie in* Π*. Since* C *lies in* Π*, it meets all the conditions along their intersections with* Π*. Otherwise, the surface satisfies the same conditions. In order for this locus to be enumeratively relevant* $J_\Pi = d$*. Moreover,* $d \leq k+l$ *since the surface does not lie in* Π.

4. $\overline{\mathcal{MS}}_\Pi(\mathbb{P}^{N-1}; k, l, C(d), D, Y, I, 0, I'+J_\Pi : \pi(S) \subset \Pi, J_\Pi = d)$*, the locus where the entire surface specializes to lie in* Π*. The conditions on the surfaces and curves remain the same, but since the surface lies in* Π *the surface and the curves must meet the conditions along their intersections with* Π*. This locus is enumeratively relevant when* $J_\Pi = d$.

5. $\overline{\mathcal{MS}}_\Pi(\mathbb{P}^N; k, l, C(d), D, Y, I, I'-1, J_\Pi+1 : C = \tilde{C} \cup F,\ \pi(\tilde{C}) \subset \Pi, J_\Pi = d, q'_{I'} \in \tilde{C})$*, the locus where* C *breaks into a section* $\tilde{C}$ *of one lower degree union a fiber and the section* $\tilde{C}$ *lies in* Π*. The surface remains irreducible. The point* $q'_{I'}$ *lies in* $\tilde{C}$*. This locus is enumeratively relevant when* $J_\Pi = d$

6. $\overline{\mathcal{MS}}_\Pi(\mathbb{P}^N; k, l, C(d), D, Y, I, I'-1, J_\Pi+1 : C = \tilde{C} \cup F,\ \pi(F) \subset \Pi, J_\Pi \geq 2, p_{j_1}, p_{j_2}, q'_{I'} \in F)$*, the locus where* C *breaks into a section* $\tilde{C}$ *of one lower degree union a fiber and the fiber lies in* Π*. The point* $q'_{I'}$ *lies in the fiber in* Π*. This locus is enumeratively relevant when* $J_\Pi \geq 2$.

In case the degree of C *is* l*, then* C *can break into a union of the directrix and* $l-k$ *fibers. Either some subset of the fibers or some subset of the fibers and the directrix can lie in* Π*. The following two cases summarize these loci:*

7. $\overline{\mathcal{MS}}_\Pi(\mathbb{P}^N; k, l, C(l), D, Y, I, I'-1, J_\Pi+1 : C = D \cup F_1 \cdots \cup F_{l-k},$ $q'_{I'} \in \pi(D \cup \mathbb{F}_1 \cdots \cup F_r) \subset \Pi,\ J_\Pi \geq k+r+1)$
8. $\overline{\mathcal{MS}}_\Pi(\mathbb{P}^N; k, l, C(l), D, Y, I, I'-1, J_\Pi+1 : k > 0,\ C = D \cup F_1 \cdots \cup F_{l-k},$ $q'_{I'} \in \pi(\mathbb{F}_1 \cup \cdots \cup F_r) \subset \Pi,\ J_\Pi \geq 2r)$

9. $\mathcal{X}(\mathbb{P}^N; (k_i, l_i; C(d_i), D(e_i), \lambda(i), q(i), q'(i), p(i))_{i=0,1}; \pi)$*, where* $q'_{I'} \in q'(0)$, $d_0 = k_0+l_0+1$, $d_1 = k_1+l_1-1$, $e_i = k_i$, $k = k_0+k_1$ *and* $J_\Pi(0) \geq k_0+l_0+2$*. This is the locus of maps from unions of two Hirzebruch surfaces* $F_{l_0-k_0}$ *and* $F_{l_1-k_1}$ *into* $\mathbb{P}^N$ *where the image of the first one lies in* Π*. The points and fibers are distributed among the two components except* $q'_{I'}$ *is required to lie in the surface in* Π*. The directrix specializes to the union of the directrices of the two surfaces. Such a component is enumeratively relevant when* $J_\Pi(0) \geq k_0+l_0+2$*. In case the surface outside* Π *is a plane, we must in fact have* $J_\Pi(0) \geq k_0+l_0+3$.

In addition if $k = l$, we can have the two types of loci where the perfectly balanced scroll breaks into a union of two balanced scrolls.

10. $\mathcal{X}(\mathbb{P}^N; (k_0, k_0+1; C(2k_0+2), D(k_0+1), \lambda(0), q(0), q'(0), p(0)), (k-k_0-1, k-k_0;$
$C(d-2k_0-2), D(k-k_0), \lambda(1), q(1), q'(1), p(1)) : q'_{I'} \in q'(0), J_H(0) \geq 2k_0+3)$.
11. $\mathcal{X}(\mathbb{P}^N; (k_0, k_0+1; C(2k_0+2), D(k_0), \lambda(0), q(0), q'(0), p(0)), (k-k_0-1, k-k_0;$
$C(d-2k_0-2), D(k-k_0+1), \lambda(1), q(1), q'(1), p(1)) : q'_{I'} \in q'(0), J_H(0) \geq 2k_0+3)$.

We will prove both theorems by dimension counts. The arguments become intricate because we need to account for competing phenomena. For example, in a union of scrolls requiring three surfaces to meet along the same line costs dimension. On a scroll requiring a curve to contain a line also costs dimension. However, if a curve contains the common line of three surfaces, then that voids the matching conditions between pieces of the curve on different components of the surface. We will prove that the gain is always less than the cost.

Proof: By repeatedly using Propositions 6.1 and 6.2 and the remark following them, it suffices to prove Theorem 6.8 when $I = 1, Y = 0$ and Theorem 6.9 when $I = Y = 0, I' = 1$. We can assume that the linear spaces Γ^j contained in H are all equal to H and the linear spaces Ω^j contained in Π are all equal to Π. We can also set $\Delta^I = \Sigma^{I'} = \mathbb{P}^N$. The general case follows by adding marked points and marked fibers and requiring them to lie in the intersection of general hyperplanes.

Proof of Theorem 6.8: By Proposition 6.3 the dimension of $\overline{\mathcal{MS}}_H$ is

$$N(k+l+2) + 2k - 4 + (J_H + 2).$$

Let U be an irreducible component of $\mathcal{D}_H$. Since we are interested only in the enumeratively relevant components, we can work with the modified tree T.

I. No component lies in H. In this case, the hyperplane section C must contain the point q_I. Hence the last term decreases by 1 to $J_H + 1$. Since a reducible surface or a more unbalanced surface that is the limit of scrolls $S_{k,l}$ has dimension at most $N(k+l+2) + 2k - 5$, the only divisor where the map π does not map a component of the surface to H is given by the first possibility in the theorem.

II. The entire surface lies in H. In this case the dimension of the surface with a choice of hyperplane section and directrix is at most $N(k+l+2) + 2k - 5$ with equality if and only if the surface is an $S_{k,l}$. Since the additional term we add is still $J_H + 2$ we conclude that the only divisors have the form described by the second possibility in the theorem. Furthermore, suppose $J_H < k + l + 1$. Then there is a positive dimensional family of limits of the hyperplane section C that pass through the chosen J_H points. Such a divisor is not enumeratively relevant.

III. From now on we can assume that π maps at least one component of the surface into H and maps at least one component to a surface not contained in H.

Using the non-degeneracy assumption to simplify T. Since we are assuming that the image surface spans $\mathbb{P}^{k+l+1}$, each of the non-contracted components map to rational normal scrolls. Moreover, the span of any subsurface connected in codimension 1 of degree d must be $\mathbb{P}^{d+1}$. Since the limit curve C is non-degenerate each of its non-contracted components maps to a rational normal curve. We conclude that the restriction of C to any subsurface of degree d connected in codimension 1 has degree at most $d + 1$ since such a surface can span at most $\mathbb{P}^{d+1}$.

Claim: *The surfaces outside H form a connected tree and meet H with multiplicity 1 along their lines contained in H.*

Let T_i be a maximal, connected subtree of the modified tree T where all the vertices correspond to surfaces mapped into H. At most one surface not contained in H can be connected to T_i. Otherwise, the restriction of C would have degree at least 2 more than the total degree of the surfaces in T_i contradicting the observation in the previous paragraph. As a corollary we conclude that the surfaces not contained in H form a single connected tree. Again by the non-degeneracy of C the surfaces outside have multiplicity 1 along any line they meet H including the lines where they are connected to surfaces inside. Moreover, a line joining two surfaces not contained in H is not contained in H.

Except for these restrictions the image surface can have many components. Three or more surfaces can meet along junctions. The limits of the curves C and D have to be connected, but they can be reducible. C and D can contain fibers. Some of these fibers can be common to two or more surfaces. In some components C and D can have common sections. In some components one or both of them can contain the directrix of that component. Moreover, D can contain some of the fibers with higher multiplicity. We need to introduce notation to account for all these possibilities.

Notation: We assume that π is a map whose image forms a modified tree T of scrolls with v vertices and σ junctions marked $c_1, \dots, c_\sigma$. We assume that c_i scrolls contain the i-th junction as a common line. We assume that T has

• t maximal connected subtrees $T_1, \dots, T_t$ with $v_1, \dots, v_t$ vertices that correspond to surfaces whose image under π lies in H and

• one connected subtree T_0 with v_0 vertices that correspond to surfaces that do not map to H. We must have that $v = \sum_{i=0}^t v_i$. Geometrically, the limit surface contains v irreducible components. Among these components v_0 of them are not in H. The rest lie in the hyperplane H.

We assume that the i-th vertex in the modified tree T is marked by two integers (k_i, l_i) to denote that the surface corresponding to that vertex is S_{k_i,l_i}. We have $\sum(k_i + l_i) = k + l$ (degree constraint).

Each vertex also has the information regarding the two limit curves C, D. On each component S the restriction of the curves C and D to S will consist of a section (which we will denote by $\mathrm{s}(C_{|S})$ and $\mathrm{s}(D_{|S})$, respectively) union fibers. If on the j-th surface C does not contain the directrix E_S of the surface, we assume that it consists of a section $\mathrm{s}(C_{|S})$ of degree d_j union fibers. If on the j-th surface D does not contain the directrix E_S, we assume that it consists of a section $\mathrm{s}(D_{|S})$ of degree e_j union fibers. If on the j-th surface $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S})$, then $d_j = e_j$. We further assume that

• of the σ junctions in the tree, exactly σ_1 of them are common to both the curves C and D and that D contains the i-th among them with multiplicity f_i,

• the curves C and D both contain exactly μ_1 of the ordinary common lines, fiber lines along which exactly two surfaces meet, and that D contains the i-th one with multiplicity g_i,

• D contains μ_2 additional ordinary common lines and σ_2 additional junctions with multiplicities f_i and g_i, respectively, and

• C contains μ_3 additional ordinary common lines and σ_3 junctions.

• On the j-th surface apart from the junctions and the ordinary common lines, C and D contain exactly ζ_j common fibers, C contains ξ_j additional fibers and D contains ψ_j additional fibers. We allow D to contain the common fibers with multiplicity ρ_{ji} and the additional fibers with multiplicity ϕ_{ji}.

• Finally, on surfaces where $\mathrm{s}(C_{|S}) \neq \mathrm{s}(D_{|S})$, the sections $\mathrm{s}(C_{|S})$ and $\mathrm{s}(D_{|S})$ have contact of order m_{ji} with each other along junctions and ordinary common lines and contact of order n_{ji} along other fibers common to both C and D.

In summary, C contains $\sigma_1 + \sigma_3$ junctions and $\mu_1 + \mu_3$ ordinary common lines. On the j-th surface S, C has a section part $\mathrm{s}(C_{|S})$ (of degree d_j if $\mathrm{s}(C_{|S}) \neq E_S$) and $\zeta_j + \xi_j$ fibers. D contains $\sigma_1 + \sigma_2$ junctions and $\mu_1 + \mu_2$ ordinary common lines with multiplicities f_i and g_i. On the j-th surface S, D contains a section part $\mathrm{s}(D_{|S})$ (of degree e_j if $\mathrm{s}(D_{|S}) \neq E_S$) and $\zeta_j + \psi_j$ fibers with multiplicities ρ_{ji} and ϕ_{ji}.

The dimension of the building blocks. Using Proposition 6.3 we compute the dimension of the data of the surfaces S_{k_i,l_i} together with the curves C and D on them prior to the gluing conditions.

The contribution of the surfaces that lie in H: We have to choose the surfaces, the section parts $\mathrm{s}(C_{|S})$ and $\mathrm{s}(D_{|S})$ of the limits of C and D on the surfaces and the ζ, ξ, ψ fiber lines that C and D contain. We also need to ensure that $\mathrm{s}(C_{|S})$ and $\mathrm{s}(D_{|S})$ have the specified order of contact with each other. The contributions are as follows:

• If both C and D contain the directrix E_S of the surface,

$$\sum_{\substack{E_S=\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})\\ S\subset H}} N(k_j+l_j+2)+k_j-l_j-6-\delta_{k_j,l_j}+\zeta_j+\xi_j+\psi_j$$

• If C contains the directrix E_S of the surface, but D does not,

$$\sum_{\substack{\mathrm{s}(C_{|S})=E_S\neq\mathrm{s}(D_{|S})\\ S\subset H}} N(k_j+l_j+2)+k_j-l_j-6-\delta_{k_j,l_j}+2e_j-k_j-l_j+1$$
$$+\zeta_j+\xi_j+\psi_j-\sum_i(m_{ji}+n_{ji})$$

• If D contains E_S, but C does not,

$$\sum_{\substack{\mathrm{s}(D_{|S})=E_S\neq\mathrm{s}(C_{|S})\\ S\subset H}} N(k_j+l_j+2)+k_j-l_j-6-\delta_{k_j,l_j}+2d_j-k_j-l_j+1$$
$$+\zeta_j+\xi_j+\psi_j-\sum_i(m_{ji}+n_{ji})$$

• If neither C nor D contains E_S, but $\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})$,

$$\sum_{\substack{\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})\neq E_S\\ S\subset H}} N(k_j+l_j+2)+k_j-l_j-6-\delta_{k_j,l_j}+2d_j-k_j-l_j+1$$
$$+\zeta_j+\xi_j+\psi_j$$

• Finally, if $\mathrm{s}(C_{|S})\neq\mathrm{s}(D_{|S})$ and they are different from the directrix E_S,

$$\sum_{\substack{\mathrm{s}(C_{|S})\neq\mathrm{s}(D_{|S})\neq E_S\\ S\subset H}} N(k_j+l_j+2)+k_j-l_j-6-\delta_{k_j,l_j}+2(d_j+e_j-k_j-l_j+1)$$
$$+\zeta_j+\xi_j+\psi_j-\sum_i(m_{ji}+n_{ji})$$

where the index j runs over the designated types of surfaces.

The contribution of surfaces not contained in H. These calculations are similar except that the data of C is determined by the intersection of the surface with H. The contributions are given as follows:

• If $\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})=E_s$,

$$\sum_{\substack{E_S=\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})\\ S\not\subset H}} N(k_j+l_j+2)+k_j-l_j-6-\delta_{k_j,l_j}+l_j+1+\psi_j$$

• If $\mathrm{s}(C_{|S})=E_s\neq\mathrm{s}(D_{|S})$,

$$\sum_{\substack{\mathrm{s}(C_{|S})=E_S\neq\mathrm{s}(D_{|S})\\ S\not\subset H}} N(k_j+l_j+2)+k_j-l_j-6-\delta_{k_j,l_j}+2e_j-k_j+2$$
$$+\psi_j-\sum_i(m_{ji}+n_{ji})$$

• If $\mathrm{s}(D_{|S})=E_s\neq\mathrm{s}(C_{|S})$,

$$\sum_{\substack{\mathrm{s}(D_{|S})=E_S\neq\mathrm{s}(C_{|S})\\ S\not\subset H}} N(k_j+l_j+2)+k_j-l_j-6-\delta_{k_j,l_j}+d_j+2$$
$$+\psi_j-\sum_i(m_{ji}+n_{ji})$$

• If $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) \neq E_s$,

$$\sum_{\substack{E_S=\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})\\ S\not\subset H}} N(k_j+l_j+2)+k_j-l_j-6-\delta_{k_j,l_j}+d_j+2+\psi_j$$

• If $\mathrm{s}(C_{|S}) \neq \mathrm{s}(D_{|S})$ and both are different from E_S,

$$\sum_{\substack{\mathrm{s}(C_{|S})\neq\mathrm{s}(D_{|S})\neq E_S\\ S\not\subset H}} N(k_j+l_j+2)+k_j-l_j-6-\delta_{k_j,l_j}+d_j+2e_j-k_j-l_j+3$$
$$+\psi_j-\sum_i(m_{ji}+n_{ji})$$

To find the total contribution we add these 10 formulae.

Gluing the pieces. So far we have the dimension of v surfaces S_{k_j,l_j}, all but v_0 of them lying in H and the choices of two curves C and D on them. We now have to glue the surfaces along fiber lines ensuring that C and D are connected.

When a fiber joining two surfaces is contained in a curve, then the union of the curves in these two surfaces is automatically connected. When the fiber is not part of the curve, we normally have two sections to match, except when the sections already meet the fiber at the same point. The curves can meet the fiber at the same point if their section parts coincide or if their contact order m at that fiber is greater than zero. We now introduce some more notation to record whether C and D meet lines common to two surfaces at the same point or not.

More Notation. Recall from above that the surface has v irreducible components and σ junctions, where c_i surfaces contain the i-th junction line. Since there must be $v-1$ fibers where the surfaces are joined, we conclude that the image surface has $v-1-\sum_{i=1}^{\sigma}(c_i-1)$ ordinary common lines.

Recall that C and D both contain exactly μ_1 of these ordinary common lines. D contains an additional μ_2 of them and C contains an additional μ_3 of them. We now need to specify the types of surfaces that meet along ordinary common lines that are neither contained in C nor D. Suppose among these ordinary common lines in κ_1 of them two surfaces for which $\mathrm{s}(C_{|S}) \neq \mathrm{s}(D_{|S})$ meet and $\mathrm{s}(C_{|S})$ intersects $\mathrm{s}(D_{|S})$ along the fiber (i.e. $m_{ji} > 0$). In κ_2 ordinary common lines two surfaces with distinct sections meet and these sections do not intersect (i.e. $m_{ji} = 0$). In κ_3 of them a surface for which $\mathrm{s}(C_{|S}) \neq \mathrm{s}(D_{|S})$ meets a surface for which $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S})$. Finally, in κ_4 of them two surfaces for which $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S})$ meet. Observe that we exhausted the ordinary common lines:

$$v-1-\sum_{i=1}^{\sigma}(c_i-1)=\sum_{i=1}^{3}\mu_i+\sum_{i=1}^{4}\kappa_i$$

Finally, we have to specify the matching conditions on the curves C and D along the σ junctions. Recall that above we required C and D to both contain σ_1 junctions, D to contain an additional σ_2 junctions and C to contain an additional σ_3 junctions. Suppose in the next σ_4 junctions the sections in each of the surfaces meet the junction at the same point. Finally, in the last σ_5 junctions surfaces with distinct sections ($\mathrm{s}(C_{|S}) \neq \mathrm{s}(D_{|S})$) meet the junction at two distinct points. Note that $\sum_{i=1}^{5}\sigma_i = \sigma$.

Choosing the lines to glue along. On each surface we have to choose the lines along which we will glue it to other surfaces. For gluing two surfaces not contained in H or two surfaces in H this gives us a choice of 2 for each connection except at the junctions where this over counts by $c_i - 2$. For gluing a surface not in H and a surface in H we have only a choice of 1 dimension because the surface not contained in H has only finitely many lines in H.

Combining the gluing conditions. When we glue the sections if in one surface the sections coincide, then on the other surface we must have $m > 0$ or the two sections must coincide, hence we only have 1 point to match. In case the sections do not match on either of the two surfaces, we have two points to match, unless $m > 0$ on one, hence both of them. By Lemma 6.5 and Proposition 6.6 we conclude that we have to subtract the following gluing conditions from our running dimension count.

• We subtract $\sum_{i=1}^{t}(2N-6)(v_i-1)$ for gluing the surfaces in H to each other,
• We subtract $(2N-4)(v_0-1)$ for gluing the surfaces outside H to each other,
• We subtract $(2N-5)t$ for gluing the t trees of surfaces in H to the surfaces outside,
• We subtract $\sum_{i=1}^{\sigma}(c_i-2)$ to account for the fact that σ of the common lines are junctions where at the i-th one c_i surfaces meet,
• Finally, we subtract

$$\mu_2+\mu_3+\kappa_1+2\kappa_2+\kappa_3+\kappa_4+\sum_{i=\sigma_1+1}^{\sigma-\sigma_5}(1-c_i)+\sum_{i=\sigma-\sigma_5+1}^{\sigma}2(1-c_i)$$

to ensure that the curves C and D are connected.

When we add the contribution of the surfaces and subtract all the gluing conditions we obtain the dimension of surfaces of the specified type. We can simplify the end result by using the following relations:

• The curve D is the limit of directrices, so it has degree k. The sum of the degrees of the section parts $\mathrm{s}(D_{|S})$ and the number of fiber lines counted with multiplicity must add to k.

$$k=\sum_{\mathrm{s}(D_{|S})=E_S}k_j+\sum_{\mathrm{s}(D_{|S})\neq E_S}e_j+\sum_{i,j}(\rho_{ji}+\phi_{ji})+\sum_{i=1}^{\sigma_1+\sigma_2}f_i+\sum_{i=1}^{\mu_1+\mu_2}g_i$$

• Similarly, using the fact that the curve C has degree $k+l$, we obtain

$$k+l=\sum_{\mathrm{s}(C_{|S})=E_S}k_j+\sum_{\mathrm{s}(C_{|S})\neq E_S}d_j+\sum_j(\zeta_j+\xi_j)+\mu_1+\mu_3+\sigma_1+\sigma_3.$$

• We know that the total degree of the surfaces is $k+l$. We can express this as the sum of the degrees of the surfaces in the hyperplane H and the sum of the degrees of the hyperplane sections in H of the surfaces outside H. We can express the latter as the t lines joining the t trees of surfaces in H to the surfaces outside H and the data of C away from these common lines.

$$k+l=\sum_{S\subset H}(k_j+l_j)+t+\sum_{S\not\subset H}(\zeta_j+\xi_j)+\sum_{\substack{\mathrm{s}(C_{|S})=E_S\\ S\not\subset H}}k_j+\sum_{\substack{\mathrm{s}(C_{|S})\neq E_S\\ S\not\subset H}}d_j$$

• Finally, we use Lemma 4.3 to conclude that the C and D cannot have total intersection multiplicity larger than k at isolated points on the smooth locus of the surface. If we assume that the total intersection multiplicity at the smooth points is $k-w$ for some nonnegative w, we obtain the relation:

$$\begin{aligned}k-w=&\sum_{\mathrm{s}(C_{|S})=E_S\neq\mathrm{s}(D_{|S})}(e_j+\xi_j+\phi_j-l_j)+\sum_{\mathrm{s}(D_{|S})=E_S\neq\mathrm{s}(C_{|S})}(d_j+\xi_j+\phi_j-l_j)\\&+\sum_{\mathrm{s}(C_{|S})\neq\mathrm{s}(D_{|S})\neq E_S}(e_j+\xi_j+\phi_j+d_j-k_j-l_j)-\sum_j\left(\sum_i m_{ji}+\sum_i n_{ji}\right)\end{aligned}$$

Final formula. Adding the contributions of the ten types of surfaces, subtracting the gluing conditions and simplifying using these four relations, we obtain the following formula for the dimension of the surfaces under consideration:

$$N(k+l+2)+2k-3-v_0-w+\sum_{\substack{\mathrm{s}(C_{|S})\neq E_S\\ \mathrm{s}(D_{|S})\neq E_S}}(k_j-l_j)-\sum_j \delta_{k_j,l_j}-\sum_{\substack{\mathrm{s}(C_{|S})=E_S\\ =\mathrm{s}(D_{|S})}}1$$

$$+\sum_{\substack{\mathrm{s}(C_{|S})\neq E_S\\ \neq \mathrm{s}(D_{|S})}}1-\sum_{\mathrm{s}(C_{|S})\neq \mathrm{s}(D_{|S})}(\xi_j+\sum_i \phi_{ji})+\sum_j \psi_j-\sum_j\sum_i(\rho_{ji}+\phi_{ji})-\sum_{i=1}^{\sigma_1+\sigma_2}f_i$$

$$-\sum_{i=1}^{\mu_1+\mu_2}g_i-\mu_3+\sigma-\sigma_1-\sigma_3-\sum_{i=\sigma_1+1}^{\sigma-\sigma_5}(c_i-1)-\sum_{i=\sigma-\sigma_5+1}^{\sigma}2(c_i-1)-\kappa_2$$

Interpreting the formula. Before choosing the points p_j that lie in H and the point q_I which we just specialized to H, the dimension of the space of maps $\overline{\mathcal{MS}}_H$ is $N(k+l+2)+2k-4$. To determine the components of the divisor D_H, we need to see when the above formula is one less than this dimension.

Since there is at least one surface outside H, we have $v_0 \geq 1$, so

$$N(k+l+2)+2k-3-v_0 \leq N(k+l+2)+2k-4.$$

None of the other terms contribute positively to the sum. To see this observe the following:

• Since $k_j-l_j-\delta_{k_j,l_j} \leq -1$, the surfaces where the section parts of C and D and the directrix are all distinct ($\mathrm{s}(C_{|S}) \neq \mathrm{s}(D_{|S}) \neq E_S$) contribute less than or equal to 0 to the sum with equality if and only if the surface is balanced.

• Note since ϕ_{ji} is the multiplicity with which D contains the fiber ψ_j, we have $\psi_j-\sum_i \phi_{ji} \leq 0$.

• Since c_i, the number of surfaces that meet at the i-th junction, is at least three, the junctions between 1 and $\sigma-\sigma_5$ contribute -1 or less to the sum. The remaining σ_5 junctions contribute -2 to the sum.

• All the remaining terms, if they are non-zero, they are strictly negative.

This analysis allows us to conclude the following about codimension one loci:

• There are no junctions where the curves C and D meet the fiber at distinct points, i.e. $\sigma_5=0$.

• There can be at most one surface where $\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})$ since the surfaces on which the section parts of C and D coincide each contribute -1 or less to the sum.

• If the surface contains a subsurface where $\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})$, then C and D cannot contain any fibers (since each fiber contributes at least an additional -1).

Above we expressed w, the defect from having the limits of the k intersections of C and D along the smooth points of the surface, as a sum over surfaces where $\mathrm{s}(C_{|S}) \neq \mathrm{s}(D_{|S})$. We can also express w using the equations above as follows:

$$w=\sum_{\substack{\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})\\ \neq E_S}}(2d_j-k_j-l_j)+\sum_{\substack{\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})\\ =E_S}}(k_j-l_j)+\mu_1+\mu_3+\sigma_1+\sigma_3$$

$$+\sum_j \zeta_j+\sum_{i=1}^{\sigma_1+\sigma_2}f_i+\sum_{i=1}^{\mu_1+\mu_2}g_i+\sum_{\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})}(\xi_j+\phi_{ji})+\sum_{i,j}(m_{ji}+n_{ji})$$

• It follows that in a codimension 1 locus there are no surfaces where $\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})$ and $\mathrm{s}(C_{|S}) \neq E_S$. Since in that case $w \geq 1$ bringing the dimension to at least two less than the dimension of $\overline{\mathcal{MS}}_H$.

Suppose there are no surfaces where $\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})$. Then in all the surfaces $\mathrm{s}(C_{|S}) \neq \mathrm{s}(D_{|S})$. Any ordinary common line or junction contained in the curves C or D contributes less than or equal to -2 to the dimension. Hence, C and D cannot contain any junctions or ordinary common lines. Each time we glue two of these surfaces either $\kappa_2=1$ (i.e. C and D meet the ordinary common line at distinct points) or $m \geq 1$ for both of the surfaces (i.e. C and D meet the ordinary common line at the same point on both components). We conclude that there must be exactly two surfaces and the curves C and D must meet the common fiber at distinct points. Since C and D cannot contain fibers (this would contribute a further -1

to the dimension), their restriction to each component must be a section. If the surface in H is S_{k_0,l_0} and the surface not contained in H is S_{k_1,l_1}, then C restricted to S_{k_0,l_0} has degree k_0+l_0+1 and C restricted to S_{k_1,l_1} has degree k_1+l_1-1.

If for both surfaces $\mathrm{s}(D_{|S})$ is the directrix, then $k_0+k_1=k$. If for one of the surfaces $\mathrm{s}(C_{|S})=E_S$, but $\mathrm{s}(D_{|S})\neq E_S$, then that surface is not contained in H. Since the directrix of the surface (which by assumption is equal to $\mathrm{s}(C_{|S})$) has degree one less than its degree the surface must be a plane. (Recall our convention that perfectly balanced scrolls do not have any directrices. That is why we do not include the quadric surface.) The degree of D is at least 1 and it is required to pass through a point. Hence the choice of D contributes a one-parameter family to our dimension. Such a component cannot be enumeratively relevant unless $k=l$. In the latter case a perfectly balanced scroll breaks into a plane and another balanced scroll.

Finally, if there is a surface where both $\mathrm{s}(C_{|S})$ and $\mathrm{s}(D_{|S})$ are different from the directrix, than that surface must be balanced. Either the scroll S is perfectly balanced and $\mathrm{s}(D_{|S})$ is a minimal degree section and $k=k_0+k_1$ or the choice of $\mathrm{s}(D_{|S})$ contributes at least 1 to the dimension. It contributes exactly one when the surface is balanced and not perfectly balanced and the degree of $\mathrm{s}(D_{|S})$ is one larger than the degree of the directrix. Such a case can be enumeratively relevant only if $k=l$. In that case we must have $k=k_0+k_1+1$ and a perfectly balanced scroll breaks into a union of two balanced scrolls.

Suppose that there is a surface S where $\mathbf{s}(C_{|S})=\mathbf{s}(D_{|S})=E_S$. We already observed that S is the unique such surface; there are no junctions and the curves C,D do not contain any ordinary common lines or lines on the surfaces of other types. Furthermore, we must have $w=0$. Hence, the equation for w simplifies to

$$0=k_S-l_S+\xi_S+\sum_i m_{ji}.$$

The degree of C restricted to S is $k_S+\xi_S$. Since any surface emanating from S can account for at most one degree by our non-degeneracy assumption, we conclude that there must be at least $l_S-\xi_S$ surfaces adjacent to S. Since the other surfaces all have $\mathrm{s}(C_{|S})\neq \mathrm{s}(D_{|S})$, $m\geq 1$ for each of them. Since $w=0$, we conclude that there are exactly $l_S-\xi_S$ surfaces, all adjacent to S. The curves C and D meet each other simply along the lines joining the other surfaces to S. S is not contained in H and all the other surfaces are in H. Moreover, $k_S=0$, so S is a cone.

For the surfaces in H the degree of C restricted to them is one more than their degree, so C cannot restrict to their directrix. If D restricts to their directrix in all the surfaces in H, then $k=\sum k_i$. The surfaces in H on which D does not restrict to the directrix must be balanced. On any surface which is perfectly balanced D restricts to a section of minimal degree (otherwise, the locus is not enumeratively relevant). If all of the surfaces are perfectly balanced, then we must have $k=\sum k_i$. If one of the surfaces is balanced, but not perfectly balanced, then for this locus to be enumeratively relevant, we must have $k=l$ and the degree of D restricted to this surface must be one more than the degree of the directrix. There can be at most one such surface. In this case S must be a plane and there are only two components to the limit surface.

Finally, we have to add the choices for the J_H points and the point q_I. We must have q_I in a surface inside, otherwise we lose 1 dimension by restricting it to the hyperplane section. Finally, to be enumeratively relevant C should not move in a linear system. The number of conditions that are needed to ensure this have been noted in the theorem. This completes the proof. □

Proof of Theorem 6.9. The proof is very similar to the previous one. As discussed above, it suffices to consider the case $I=Y=0,\ I'=1,\ \Omega^j=\Pi$ for all j and $\Sigma^{I'}=\mathbb{P}^N$. In this case the dimension of $\overline{\mathcal{MS}}_\Pi$ is

$$(N+1)(k+l+2)+2d-2l-4.$$

Let U be a component of the divisor $\mathcal{D}_\Pi$ defined by requiring $q'_{I'}\in\Pi$. Since the degree of C is d, the number of marked points of C in Π is bounded by d, i.e. $J_\Pi\leq d$.

Case 1: The surface remains irreducible. If the curve C also remains irreducible and is not contained in Π, then $q'_{I'}\in C\cap\Pi$. Either $q'_{I'}$ becomes one of the unmarked points of the intersection $C\cap\Pi$ or it coincides with one of the previously marked ones. These loci are divisors only if the surface remains $S_{k,l}$. The curve C cannot remain irreducible if the cardinality of $C \cap \Pi$ exceeds the degree d of C. These divisors are listed as the first two divisors in the theorem.

If the curve remains irreducible, but lies in Π, then either $d \leq k+l$ or the surface also lies in Π. If the curve lies in Π but the surface does not, then $d = k+l$. For the locus to be a divisor the surface must remain $S_{k,l}$, the curve must remain irreducible and $J_\Pi = d$. Only when $J_\Pi = d$, the loss in dimension due to the choice of C exceeds the gain in dimension due to the choice of the points p_j and $q'_{I'}$ by only one. Consequently, the surface or the curve cannot become more special. The case when the surface lies in Π is similar.

If the curve becomes reducible, then it must be the union of a section of degree d' together with $d-d'$ fibers. We denote this curve as $\tilde{C} \cup F_1 \cup \cdots \cup F_{d-d'}$. If the section $\tilde{C}$ is contained in Π, then the locus is a divisor when $d' = d-1$ and $J_\Pi = d$ and the remaining fiber is not contained in Π or $d = l$ and the directrix and r of the fibers are contained in Π and $J_\Pi = k+r+1$.

If only fiber components of the limit curve C are contained in Π, then either there must be a unique fiber and two points among the J_Π must specialize to it or $d = l$ and the curve C must break to a union of the directrix and fibers—r of which are contained in Π. In the latter case $2r$ of the points in J_Π must specialize pairwise to the lines. These statements can be checked by a dimension count.

Case 2: The surface becomes reducible. We will perform a calculation analogous to the proof of the previous theorem to find the divisors. We work with the modified tree T.

Using the non-degeneracy assumption to simplify T. Since the hyperplane section of C in Π is non-degenerate, the components of C not in Π meet Π transversely. Each tree of curves in Π can meet at most one curve not in Π. In particular, the curves not contained in Π form a connected tree. The surfaces not contained in Π have contact of order one with Π along their lines contained in Π except possibly when the surface is a cone and the section part of C reduces to the vertex. However, in the latter case forcing the cone to have higher order of contact with Π strictly lowers the dimension, so we will ignore this case.

Three or more curves. Unfortunately, in addition to the cases we considered in the previous proof, now for the surfaces outside Π we have to distinguish whether $\mathrm{s}(C_{|S})$ and $\mathrm{s}(D_{|S})$ lie in Π or remain outside Π. Fortunately, due to our non-degeneracy assumptions we do not need to record the incidence data of C with the hyperplane. If we remove the non-degeneracy assumption, it is easy to construct situations where the directrix has high order of contact with the hyperplane and the curve has high order of contact with both the hyperplane and the directrix at different points. It is a very hard problem to determine the limits in this generality. This is the main obstruction for carrying out our algorithm in greater generality.

Notation: We preserve our notation from the previous proof (with the caveat that we have to replace the hyperplane H with the hyperplane Π everywhere). For the surfaces not contained in Π, we will need to keep some additional information. If none of $\mathrm{s}(C_{|S})$, $\mathrm{s}(D_{|S})$ or E_S lies in Π, we will denote by x_j the number of lines the surface S has in Π. In the surfaces outside Π, among the fiber lines ζ_j, ξ_j and ψ_j we will assume $\tilde{\zeta}_j, \tilde{\xi}_j$ and $\tilde{\psi}_j$ of them are not contained in Π and the rest are contained in Π.

Building blocks: Using Proposition 6.3 we can calculate the dimension that the choice of surfaces and curves contribute prior to gluing.

• If the surface S lies in Π, then the contributions are exactly as in the case $S \subset H$ of the previous theorem. We can summarize them in the following equation, where the summation runs over the surfaces contained in Π.

$$\sum_{S \subset \Pi} (N(k_j + l_j + 2) - 6 + k_j - l_j - \delta_{k_j,l_j} + \sum_{\mathrm{s}(C_{|S}) = E_S \neq \mathrm{s}(D_{|S})} (2e_j - k_j - l_j + 1)$$
$$+ \sum_{\mathrm{s}(D_{|S}) = E_S \neq \mathrm{s}(C_{|S})} (2d_j - k_j - l_j + 1) + \sum_{\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) \neq E_S} (2d_j - k_j - l_j + 1)$$
$$+ \sum_{\mathrm{s}(C_{|S}) \neq E_S \neq \mathrm{s}(D_{|S})} 2(d_j + e_j - k_j - l_j + 1) + \zeta_j + \xi_j + \psi_j - \sum_i (m_{ji} + n_{ji}))$$

• If $S \not\subset \Pi$, then the contributions are similar to the case when $S \not\subset H$ of the previous theorem. However, in this case we have more possibilities depending on whether the restrictions of C and D contain the section in $S \cap \Pi$ or contain fiber lines in Π. We list these contributions as follows:

- If $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) = E_S$, but they are not contained in Π

$$\sum_{\substack{\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})=E_S \\ \mathrm{s}(C_{|S})\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+k_j+l_j-x_j+2+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j$$

- If $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) \neq E_S$, and neither $\mathrm{s}(C_{|S})$ nor E_S is contained in Π

$$\sum_{\substack{\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})=E_S \\ \mathrm{s}(C_{|S}),\mathrm{s}(D_{|S}),E_S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+2d_j-x_j+3+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j$$

- If $\mathrm{s}(D_{|S}) = E_S \neq \mathrm{s}(C_{|S})$, and neither $\mathrm{s}(C_{|S})$ nor E_S is contained in Π

$$\sum_{\substack{\mathrm{s}(D_{|S})=E_S\neq\mathrm{s}(C_{|S}) \\ \mathrm{s}(C_{|S}),\mathrm{s}(D_{|S}),E_S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+2d_j-x_j+3$$
$$+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j-\sum_i(m_{ji}+n_{ji})$$

- If $\mathrm{s}(C_{|S}) = E_S \neq \mathrm{s}(D_{|S})$ and none are contained in Π

$$\sum_{\substack{\mathrm{s}(C_{|S})=E_S\neq\mathrm{s}(D_{|S}) \\ \mathrm{s}(C_{|S}),\mathrm{s}(D_{|S}),E_S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+2e_j-x_j+3$$
$$+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j-\sum_i(m_{ji}+n_{ji})$$

- If $\mathrm{s}(C_{|S})$, $\mathrm{s}(D_{|S})$ and E_S are all distinct and none are contained in Π

$$\sum_{\substack{\mathrm{s}(D_{|S})=E_S\neq\mathrm{s}(C_{|S}) \\ \mathrm{s}(C_{|S}),\mathrm{s}(D_{|S}),E_S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+2d_j+2e_j-k_j-l_j-x_j+4$$
$$+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j-\sum_i(m_{ji}+n_{ji})$$

- If $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) = E_S$ and they lie in Π

$$\sum_{\substack{\mathrm{s}(D_{|S})=\mathrm{s}(C_{|S})=E_S \\ E_S\subset\Pi,S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+l_j+1+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j$$

- If $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) \neq E_S$ and $\mathrm{s}(C_{|S})$ lies in Π

$$\sum_{\substack{\mathrm{s}(D_{|S})=\mathrm{s}(C_{|S})\neq E_S \\ \mathrm{s}(C_{|S})\subset\Pi,S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+d_j+2+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j$$

- If $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) \neq E_S$ and E_S lies in Π

$$\sum_{\substack{\mathrm{s}(D_{|S})=\mathrm{s}(C_{|S})\neq E_S \\ E_S\subset\Pi,S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+2d_j-k_j+2+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j$$

- If $\mathrm{s}(D_{|S}) = E_S \neq \mathrm{s}(C_{|S})$ and $\mathrm{s}(C_{|S})$ lies in Π

$$\sum_{\substack{\mathrm{s}(D_{|S})=E_S\neq\mathrm{s}(C_{|S})\\ \mathrm{s}(C_{|S})\subset\Pi, S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+d_j+2$$
$$+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j-\sum_i(m_{ji}+n_{ji})$$

- If $\mathrm{s}(D_{|S}) = E_S \neq \mathrm{s}(C_{|S})$ and E_S lies in Π

$$\sum_{\substack{\mathrm{s}(D_{|S})=E_S\neq\mathrm{s}(C_{|S})\\ E_S\subset\Pi, S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+2d_j-k_j+2$$
$$+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j-\sum_i(m_{ji}+n_{ji})$$

- If $\mathrm{s}(C_{|S}) = E_S \neq \mathrm{s}(D_{|S})$ and $\mathrm{s}(D_{|S})$ lies in Π

$$\sum_{\substack{\mathrm{s}(C_{|S})=E_S\neq\mathrm{s}(D_{|S})\\ \mathrm{s}(D_{|S})\subset\Pi, S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+e_j+2$$
$$+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j-\sum_i(m_{ji}+n_{ji})$$

- If $\mathrm{s}(C_{|S}) = E_S \neq \mathrm{s}(D_{|S})$ and E_S lies in Π

$$\sum_{\substack{\mathrm{s}(C_{|S})=E_S\neq\mathrm{s}(D_{|S})\\ E_S\subset\Pi, S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+2e_j-k_j+2$$
$$+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j-\sum_i(m_{ji}+n_{ji})$$

- If $\mathrm{s}(C_{|S})$, $\mathrm{s}(D_{|S})$ and E_S are all distinct and $\mathrm{s}(C_{|S})$ is contained in Π

$$\sum_{\substack{\mathrm{s}(C_{|S})\neq E_S\neq\mathrm{s}(D_{|S})\\ \mathrm{s}(C_{|S})\subset\Pi, S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+d_j+2e_j-k_j-l_j+3$$
$$+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j-\sum_i(m_{ji}+n_{ji})$$

- If $\mathrm{s}(C_{|S})$, $\mathrm{s}(D_{|S})$ and E_S are all distinct and $\mathrm{s}(D_{|S})$ is contained in Π

$$\sum_{\substack{\mathrm{s}(C_{|S})\neq E_S\neq\mathrm{s}(D_{|S})\\ \mathrm{s}(C_{|S})\subset\Pi, S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+2d_j+e_j-k_j-l_j+3$$
$$+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j-\sum_i(m_{ji}+n_{ji})$$

- Finally, if $\mathrm{s}(C_{|S})$, $\mathrm{s}(D_{|S})$ and E_S are all distinct and E_S is contained in Π

$$\sum_{\substack{\mathrm{s}(C_{|S})\neq E_S\neq\mathrm{s}(D_{|S})\\ \mathrm{s}(C_{|S})\subset\Pi, S\not\subset\Pi}} N(k_j+l_j+2)-6+k_j-l_j-\delta_{k_j,l_j}+2d_j+2e_j-2k_j-l_j+3$$
$$+\tilde{\zeta}_j+\tilde{\xi}_j+\tilde{\psi}_j-\sum_i(m_{ji}+n_{ji})$$

Final Formula. The gluing conditions are exactly as in the proof of the previous theorem. We retain the notation we used there. Finally, we use the facts that the degree of C is d, the degree of D is k and that their intersection multiplicities at isolated points along the smooth points of the surface has to be $d-l-w$ for some positive w. These are expressed as in the previous theorem. Adding the contributions of each type of surface, subtracting the gluing conditions and simplifying using these facts, we obtain

$$
\begin{aligned}
&(N+1)(k+l+2)+2d-2l-5-v_0-w-t-\sum_j \delta_{k_j,l_j}-\sum_{\substack{E_S\subset\Pi\\ S\not\subset\Pi}} k_j-\sum_{\substack{\mathrm{s}(C_{|S})\subset\Pi\\ S\not\subset\Pi}} d_j\\
&-\sum_{\substack{\mathrm{s}(D_{|S})\neq\mathrm{s}(C_{|S}),\mathrm{s}(D_{|S})\neq E_S\\ \mathrm{s}(D_{|S})\subset\Pi, S\not\subset\Pi}} e_j-\sum_{S\subset\Pi}(k_j+l_j)+\sum_{\substack{\mathrm{s}(C_{|S})\neq E_S\\ \mathrm{s}(D_{|S})\neq E_S}}(k_j-l_j)+\sum_{S\subset\Pi}\psi_j\\
&+\sum_{S\not\subset\Pi}\tilde{\psi}_j-\sum_i\phi_{ji}-\sum_1^{\sigma_1+\sigma_2} f_i-\sum_1^{\mu_1+\mu_2} g_i-\sum_{j,i}\rho_{ji}-\mu_3-\sum_{\mathrm{s}(C_{|S})\neq\mathrm{s}(D_{|S})}(\xi_j+\sum_i\phi_{ji})\\
&+\sigma-\sigma_1-\sigma_3-\sum_{\sigma_1+1}^{\sigma-\sigma_5}(c_i-1)-\sum_{\sigma-\sigma_5+1}^{\sigma}2(c_i-1)-\kappa_2-\sum_{\mathrm{s}(C_{|S})=\mathrm{s}(D_{|S})=E_S}1\\
&+\sum_{\substack{S\not\subset\Pi\\ \mathrm{s}(C_{|S}),\mathrm{s}(D_{|S}),E_S\not\subset\Pi}}(1-x_j)\;+\sum_{\mathrm{s}(C_{|S})\neq\mathrm{s}(D_{|S})\neq E_S}1\;-\sum_{S\not\subset\Pi}((\zeta_j-\tilde{\zeta}_j)+(\xi_j-\tilde{\xi}_j)+(\psi_j-\tilde{\psi}_j))
\end{aligned}
$$

Interpretation. This expression gives us the dimension of the tree of scrolls before we choose the J_Π points that are the marked points of $C\cap\Pi$. Since we are assuming that the points remain non-degenerate, for each connected tree of curves contained in $C\cap\Pi$ we can choose one more point than the total degree of the curves in that tree. Once we choose the points we compare the above expression with $(N+1)(k+l+2)+2d-2l-4$, the dimension of $\overline{\mathcal{MS}}_\Pi$.

We need to control the contribution of the choice of points to the dimension. If $C_{|S}$ lies outside Π, then we have only finitely many choices for the points on that curve. However, when $C_{|S}$ lies in Π, then we have a one-parameter family of choices for the points on that part of the curve.

The restriction of C to a tree of surfaces in Π forms a connected tree. Hence, each tree of surfaces in Π contributes at most one tree of curves. The total degree of C restricted to the trees of surfaces in Π is at most $\sum_{S\subset\Pi}(k_j+l_j)+t$. The degree of the components of C that are on surfaces outside Π, but lie in Π can be obtained by summing the following: We need to add d_j as j runs over the surfaces $S\not\subset\Pi, \mathrm{s}(C_{|S})\subset\Pi$ for the contribution of the section parts. We add $(\zeta_j-\tilde{\zeta}_j)+(\xi_j-\tilde{\xi}_j)$ taken over $S\not\subset\Pi$ to account for the fibers that are in Π. Finally, we need to add the number of junctions and ordinary common lines that are part of C and lie in Π. These terms all appear in the final formula with negative signs. We conclude that the choice of points for each such tree contributes at most 1 to the dimension with equality if and only if in each tree the maximum possible is attained. The extremal case occurs when the t trees of curves on surfaces contained in Π and the curves $\mathrm{s}(C_{|S})\subset\Pi$ on the surfaces $S\not\subset\Pi$ all remain disjoint. It suffices to consider the extremal case.

Recall that v_0 denotes the number of surfaces not contained in Π. For a surface $S\not\subset\Pi$ if $\mathrm{s}(C_{|S})\subset\Pi$, then the contribution of that surface to v_0 cancels the contribution of the tree of curves lying on that surface. We can therefore concentrate only on trees of curves lying on surfaces in Π. These trees need to be connected by surfaces not contained in Π. Let us call surfaces that connect pieces of C on surfaces in Π connecting surfaces. A connecting surface cannot have the section part of C in Π since otherwise the trees it connects in Π would already form one tree in Π. If for a connecting surface $\mathrm{s}(D_{|S})\subset\Pi$, then the terms $-e_j$ summed over these surfaces contribute an amount more than the number of trees that surface connects. If for a connecting surface $E_S\subset\Pi$, the number of trees that surface connects can be at most one more than k_j. This follows from our non-degeneracy assumption. For such surfaces the contributions $-k_j$ and $-v_0$ annul the contribution of all the trees that the surface connects. Finally, if the surface S has no special sections $\mathrm{s}(C_{|S})$, $\mathrm{s}(D_{|S})$ or E_S in Π, then it must contain a line in Π for every tree of curves it connects. Hence the contributions to $-v_0$ and $1-x_j$ for such surfaces annul the contributions of the trees it connects.

Note that if there is a surface S not in Π which contains no special sections $\mathrm{s}(C_{|S})$, $\mathrm{s}(D_{|S})$ or E_S in Π and has no lines in Π, the contributions of v_0 and $1-x_j$ exactly cancel each other out. We conclude that the contributions from the choice of J_Π points at most cancel the terms

$$\begin{aligned} &-v_0 - t - \sum_{\substack{E_S \subset \Pi \\ S \not\subset \Pi}} k_j - \sum_{\substack{\mathrm{s}(C_{|S}) \subset \Pi \\ S \not\subset \Pi}} d_j - \sum_{\substack{\mathrm{s}(D_{|S}) \neq \mathrm{s}(C_{|S}), \mathrm{s}(D_{|S}) \neq E_S \\ \mathrm{s}(D_{|S}) \subset \Pi, S \not\subset \Pi}} e_j - \sum_{S \subset \Pi} (k_j + l_j) \\ &-\sigma_1 - \sigma_3 + \sum_{\substack{S \not\subset \Pi \\ \mathrm{s}(C_{|S}), \mathrm{s}(D_{|S}), E_S \not\subset \Pi}} (1 - x_j) - \sum_{S \not\subset \Pi} ((\zeta_j - \tilde{\zeta}_j) + (\xi_j - \tilde{\xi}_j)) \end{aligned}$$

However, we still need to add one to the dimension for the choice of $q'_{I'}$.

Moreover, since $k_j - l_j - \delta_{k_j,l_j} \leq -1$, a component where $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) \neq E_S$ contributes -1 or less and a component where $\mathrm{s}(C_{|S})$, $\mathrm{s}(D_{|S})$ and E_S are all distinct contributes 0 or less to the sum with equality if and only if that component is balanced. A component of type $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) = E_S$ contributes -1. We conclude that there can be at most one surface S where $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S})$.

To continue our analysis we express w as in the proof of the previous theorem. By the arguments given there we conclude the following:

• A general point in a codimension one locus does not contain any surface components where $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) \neq E_S$.

• If the surface does not contain a subsurface where $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) = E_S$, there can be at most 2 components. The curves C and D do not contain any fibers and the degree of the curve C in the surface in Π is one more than the sum of the degrees of the surfaces in Π. The enumeratively relevant codimension one loci have $k_0 + k_1 = k$, unless $k = l$ and $k_0 + k_1 = k - 1$.

• A subsurface where $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) = E_S$ is not contained in Π. Suppose a surface contains a subsurface where $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) = E_S$. There cannot be any junctions and the curves C and D cannot contain fiber lines in this case. All the other surfaces must have $\mathrm{s}(C_{|S}) \neq \mathrm{s}(D_{|S})$. By an argument similar to the one given in the proof of the previous theorem, we see that there cannot be a component of type $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) = E_S$ in case the curve has degree $k + l + 1$. In case the degree of the curve is $k + l$ there can be components of type $\mathrm{s}(C_{|S}) = \mathrm{s}(D_{|S}) = E_S$ only in the case described in the previous theorem. However, then all the curves inside are connected and we conclude that there can be at most one other component. □

7. Multiplicity Calculations

In this section we carry out the multiplicity calculations needed for enumerative computations involving balanced scrolls.

Under the hypotheses of Theorems 6.8 and 6.9 when we specialize a linear space to H or Π, balanced scrolls incident to the linear space break into at most two balanced scrolls. By Proposition 4.4 the limit of the directrices is uniquely determined by the surfaces. For multiplicity calculations it is more convenient to reformulate the degeneration problem in the space of scrolls where we only mark the hyperplane section.

Let $\overline{\mathcal{M}}_H(\mathbb{P}^N; k, l; C, \{\lambda_i\}_{i=1}^Y, \{q_i\}_{i=1}^I, \{p_j\}_{j=1}^{J_H})$, or $\overline{\mathcal{M}}_H$ for short, be defined like the corresponding space $\overline{\mathcal{MS}}_H$ in §6 except that now do not mark the directrix. More explicitly, an open set in $\overline{\mathcal{M}}_H$ corresponds to maps from a Hirzebruch surface into $\mathbb{P}^N$ as a scroll $S_{k,l}$, where the marked curve C maps to the hyperplane section in H and the marked fibers and points are required to lie in various linear spaces. We compactify the space as in §3. In a similar fashion define the space $\overline{\mathcal{M}}_\Pi$ and the loci $\mathcal{X}'_H$ and $\mathcal{X}'_\Pi$ corresponding to $\overline{\mathcal{MS}}_\Pi$ and $\mathcal{X}_H$ and $\mathcal{X}_\Pi$, where again the only difference is that we do not mark the directrix. The spaces $\overline{\mathcal{M}}_H$ and $\overline{\mathcal{M}}_\Pi$ have natural Cartier divisors D'_H and D'_Π defined by requiring the point p_I and $q'_{I'}$ to lie in H and Π, respectively.

Since the limits of directrices are determined uniquely by the surfaces, Theorems 6.8 and 6.9 describe the enumeratively relevant components of D'_H and D'_Π subject to the non-degeneracy assumptions. We would like to compute the multiplicity of the Cartier divisor along each of the Weil divisors appearing in the list.

Lemma 7.1. *Let $S_{k,l}$ be a non-singular scroll in $\mathbb{P}^N$. Let $\nu := \nu_{S_{k,l}/\mathbb{P}^N}$ denote its normal bundle in $\mathbb{P}^N$. Suppose D is a divisor in a section class $e + mf$ for $m \leq l+1$, then*

(1) *$H^i(S_{k,l}, \nu) = 0$, for $i \geq 1$.*
(2) *$H^i(S_{k,l}, \nu \otimes \mathcal{O}_{S_{k,l}}(-D)) = 0$ for $i \geq 1$.*

Proof: The line bundles $\mathcal{O}_{S_{k,l}}$, $\mathcal{O}_{S_{k,l}}(1)$, and $\mathcal{O}_{S_{k,l}}(1) \otimes \mathcal{O}_{S_{k,l}}(-D)$ have no higher cohomology and by Serre duality $h^2(S_{k,l}, \mathcal{O}_{S_{k,l}}(-D)) = 0$. Consequently, the Euler sequence for $\mathbb{P}^N$

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to \oplus_1^{N+1} \mathcal{O}_{\mathbb{P}^N}(1) \to T_{\mathbb{P}^N} \to 0$$

implies that $H^i(S_{k,l}, T_{\mathbb{P}^N} \otimes \mathcal{O}_{S_{k,l}}) = 0$ and $H^i(S_{k,l}, T_{\mathbb{P}^N} \otimes \mathcal{O}_{S_{k,l}}(-D)) = 0$ for $i \geq 1$. The standard exact sequence

$$0 \to T_{S_{k,l}} \to T_{\mathbb{P}^N} \otimes \mathcal{O}_{S_{k,l}} \to \nu \to 0$$

implies that

$$h^i(S_{k,l}, \nu) = h^{i+1}(S_{k,l}, T_{S_{k,l}}) \text{ and } h^i(S_{k,l}, \nu(-D)) = h^{i+1}(S_{k,l}, T_{S_{k,l}}(-D)).$$

When $i = 2$, the right hand sides immediately vanish. When $i = 1$, they also vanish by Serre duality since $T^*_{S_{k,l}} \otimes \mathcal{O}_{S_{k,l}}(K)$ and $T^*_{S_{k,l}} \otimes \mathcal{O}_{S_{k,l}}(K+D)$ have no global sections. $\square$

Theorem 7.2. *When $l - k \leq 1$, the components*

$$1.\overline{\mathcal{M}}_H(N; k, l, C, Y, I-1, J_H+1)$$
$$2.\overline{\mathcal{M}}_H(N-1; k, l, C, Y, I, J_H)$$
$$3.\mathcal{X}'(\mathbb{P}^N; (k_0, l_0, C(k_0+l_0+1)), (k_1, l_1, C(k+l-k_0-l_0-1))),$$

satisfying the constraints listed in 1,2,3,5 and 6 of Theorem 6.8, occur with multiplicity one in

$$D'_H \subset \overline{\mathcal{M}}(N; k, l, C, Y, I, J_H).$$

Proof: By Propositions 6.1 and 6.2 it suffices to restrict to the case $Y = 0$, $I = 1$, $\Gamma^j = H$ for all j. We can then recover the general case by slicing with general hyperplanes.

To determine the multiplicity for the first locus we can assume that Δ^I has dimension $N-3$. Consider the family obtained by rotating Δ^I into H. Let $\Delta^I(t)$ denote this family. We assume that $\Delta^I(0) \subset H$. Consider the family

$$\{(t, S_{k,l}, p_1, \cdots, p_{J_H}) : S \cap \Delta(t) \neq \emptyset\}$$

of balanced scrolls $S_{k,l}$ which meet $\Delta(t)$ and have J_H marked points in H. The multiplicity under consideration is the same as the multiplicity of the divisor $t = 0$ in this family. The family admits a rational map to the space of rational normal curves with J_H marked points by sending each marked surface to the hyperplane section in H. This map is defined over the locus under discussion and is smooth over that locus by Lemma 7.1.

Here we are using the fact that to show that a morphism of stacks $\mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ is smooth and $\mathcal{A}$ is equidimensional and smooth it suffices to show that the Zariski tangent space to the fiber is of dimension $\dim \mathcal{A} - \dim \mathcal{B}$.

The divisor whose multiplicity we are trying to determine occurs as a component of the pull-back of the divisor of rational curves in $H = \mathbb{P}^{N-1}$ that meet an $N-3$ dimensional linear space. Since the latter divisor is reduced and the morphism is smooth we conclude that the multiplicity is one in this case.

To determine the multiplicity of the other loci it is more convenient to look at the case when $\Delta^I = \mathbb{P}^N$. In that case the marking of the point q_I gives the universal surface over the loci in question. To compute the multiplicity we can forget the marking of q_I. Note that in all the loci described the hyperplane section C is uniquely determined by the surface and the marked points. We get a map from the space of surfaces to the space of rational curves by sending the map from the surface to the map from C to H. This is a smooth morphism as in the previous case by Lemma 7.1 and the standard normal bundle sequence relating the normal bundle of the reducible surface to that of the union. When C is reducible, the divisor in question is a component of the pull-back of a boundary divisor of $\overline{M}_{0,J_H}(\mathbb{P}^{N-1}, k+l)$ under a smooth morphism, hence reduced.

Finally, we remark that in case 2 when the surface breaks into a plane union a balanced scroll, one can see that the multiplicity is one by direct computation using the determinantal representation of scrolls. □

Theorem 7.3. *When $l-k \leq 1$, the components $\overline{\mathcal{M}}_\Pi$ and $\mathcal{X}'_\Pi$ of*

$$D'_\Pi \subset \overline{\mathcal{M}}_\Pi(N; k, l, C, Y, I, I', J_\Pi)$$

listed in $1-11$ of Theorem 6.9 occur with multiplicity one in D'_Π.

Proof: It suffices to consider the case $Y = I = 0$, $I' = 1$. We reduce this case to Theorem 6.2 in [V2]. $\overline{\mathcal{M}}_\Pi(\mathbb{P}^N, k, l, C, 0, 0, 1, J_\Pi)$ admits a morphism ι to $\overline{\mathcal{M}}_{0,J_\Pi+1}(\mathbb{P}^N, d)$ which sends (S, C, π) to $\pi : C \to \mathbb{P}^N$ and stabilizes. In light of §3, we can interpret this morphism—at least in an open set containing the loci we are interested in—as the morphism induced between the corresponding Kontsevich spaces by the projection from $\mathbb{F}(0,1;N)$ to $\mathbb{P}^N$.

We claim that ι is smooth at all the loci covered by the theorem. Both the image and the domain of ι are equidimensional and smooth along the loci we are interested in. It suffices to check that the fiber is smooth. We need to compute the dimension of the Zariski tangent space to the fiber. The Zariski tangent space to the fiber at a point (S, C), where S is a Hirzebruch surface and C is a curve in a section class that lie in one of the loci covered by the theorem, is given by $H^0(S, \nu_S(-C))$. If the surface is smooth, then we can conclude that the morphism is smooth by Lemma 7.1. If $S = S_1 \cup S_2$ has two components meeting transversely along their common line L, the claim follows from Lemma 7.1 and the standard exact sequence relating the normal bundle of S to the normal bundles of S_1 and S_2.

D'_Π is the pull-back of the divisor D_H in [V2] by ι. Since ι is smooth our theorem follows from Theorem 6.2 in [V2]. □

Finally, to conclude the multiplicity calculations we recall Proposition 2.18 in [V3]. This proposition asserts that if to the data of a zero dimensional locus $\mathcal{M}$, we add a linear space of dimension $N-2$ that meets the surface or a linear space of dimension $N-1$ that meets a marked curve, then the degree of the stack multiplies by the degree of the surface or the degree of the marked curve, respectively.

8. A Simple Enumerative Consequence

In this section we describe an application of Theorems 6.8 and 6.9 to the characteristic numbers of balanced scrolls. We impose enough point conditions to ensure that we can satisfy the non-degeneracy assumptions of the theorems. When counting scrolls of degree $k+l$ in $\mathbb{P}^{k+l+1}$, the hypotheses of Theorem 8.1 require that at least $k+l+4$ of the linear spaces are points. It is possible to strengthen the theorem at the expense of complicating the algorithm.

Theorem 8.1. *Suppose $0 \leq l-k \leq 1$. Let $\{\Delta^i_{a_i}\}_{i=1}^I$ be a set of linear spaces of dimension $a_i < N-2$ in $\mathbb{P}^N$ in general linear position such that*

1. $\sum_{i=1}^{I} N - 2 - a_i \;=\; (k+l+2)\,N + 2k - 4 - \delta_{k,l}$ *(expected dimension 0)*
2. $a_I \leq N-k-l-1$
3. $a_i = 0$ *for* $1 \leq i \leq k+l+1$ *(enough points)*
4. $a_{k+l+2+j} \leq N-k-l-1,$ *for* $0 \leq j \leq a_I + 1$

Then there exists an algorithm which computes the number of scrolls $S_{k,l}$ meeting $\{\Delta^i_{a_i}\}_{i=1}^I$.

Proof: We now describe the algorithm and prove that it terminates without stepping outside the bounds of our non-degeneracy assumptions. We begin with the case $N = k + l + 1$ and reduce the more general case to it later.

Step 1. Specialize the Δ^i, except for Δ^I, one by one to general linear spaces of a general hyperplane H in order of increasing dimension until a reducible solution appears.

In our case the first $k + l + 3$ linear spaces and Δ^I are points. We can take H to be the span of the first $k + l + 1$ points. We claim that after we specialize Δ^{k+l+2} to H the scrolls incident to all Δ^i are still irreducible balanced scrolls.

The hyperplane section in H has to meet the first $k + l + 1$ points, so it is non-degenerate. Similarly, since the scroll needs to meet Δ^I, it spans $\mathbb{P}^{k+l+1}$. By Theorem 6.8 if there is a reducible scroll, then the component of the scroll in H meets Δ^{k+l+2} and contains $d + 2$ of the first $k + l + 1$ points, where d is the degree of the scroll. Since this is impossible the claim follows. We repeat step 1 by specializing Δ^{k+l+3} to H.

Theorem 6.8 still applies. In this case there are 2 possibilities.

Case i. Some scrolls can remain irreducible. Then their hyperplane section in H is the unique rational normal curve containing the $k + l + 3$ points in H. Repeat Step 1 by specializing Δ^{k+l+4}. Theorem 6.8 still applies and this case can no longer occur. Proceed to the next possibility.

Case ii. Some scrolls can become reducible. By Theorem 6.8 the only reducible scrolls can be a balanced scroll of degree $k + l - 1$ in H and a plane outside.

• If we arrived at Case ii after passing through Case i first, this is clear since otherwise the limit hyperplane section would be reducible. In this case the degree $k + l - 1$ scroll contains the rational normal curve and meets Δ^{k+l+4}. The plane contains Δ^I. The rest of the conditions are distributed among the two. We need to consider each way of partitioning the other conditions in such a way that they do not impose more conditions on either of the components than they can satisfy. (If we do not satisfy the last clause, the algorithm will give 0.)

• If we are in Case ii right after having specialized Δ^{k+l+3}, then the component in H needs to meet Δ^{k+l+3} and contain at least $d+2$ of the first $k+l+2$ points if its degree is d. The only possibility is $d = k+l-1$. The scroll of degree $k+l-1$ contains the $k+l+3$ points in H. The plane contains Δ^I. We consider each partition of the rest of the conditions. In either case proceed to step 2.

Crucial Point: Proposition 4.4 implies that if a balanced scroll breaks into a union of two balanced scrolls the gluing conditions on the directrices are automatically satisfied. Therefore, we reduced the problem of counting scrolls of degree $k+l$ to counting pairs (Plane, Scroll of degree $k+l-1$) meeting along a line and in addition satisfying the conditions described in Case ii. Using Step 2, which we now describe, we further reduce the problem to counting degree $k + l - 1$ scrolls.

Note that the scroll of degree $k+l-1$ needs to contain a line of the plane outside H. The plane in turn contains a point and meets some linear spaces.

Step 2. Use Schubert calculus to re-express the conditions on the plane as multiples of Schubert cycles.

After Step 2, the plane is required to contain a point (Δ^I), meet a linear space Λ_1 in a line, and lie in a linear space Λ_2. In turn the common line between the two scrolls is required to meet the linear space $\Lambda_1 \cap H$ and lie in $\Lambda_2 \cap H$. We have reduced the problem to counting degree $k + l - 1$ balanced scrolls in $\mathbb{P}^{k+l}$ satisfying the conditions in Case ii) and containing a fiber lying in a linear space and meeting another linear space.

• If we arrived at this stage without going through Case i, we are done by induction. The steps so far only used Theorem 6.8 which allows for our new condition without changing the conclusions. In addition the scroll contains $k + l + 3$ points. We can go back to Step 1 and run the process from the beginning.

• If we arrived at this stage after passing through Case i, we have to count scrolls of degree $k + l - 1$ in $\mathbb{P}^{k+l}$ containing a rational normal curve C of degree $k + l$, meeting some linear spaces and containing a fiber which lies in a linear space and meets some other linear space. Proceed to step 3.

Step 3. Specialize a linear space meeting the rational normal curve of degree $k+l$ to a general hyperplane Π in order of increasing dimension, but always keeping a point outside Π, until the curve or the surface becomes reducible.

In our case Step 3 amount to breaking C into a rational normal curve of degree $k+l-1$ union a general line. Theorem 6.9 applies since the surface still spans $\mathbb{P}^{k+l}$ and the limit of the hyperplane section of the curve in Π is non-degenerate. We conclude that the surface cannot break after this degeneration. If Δ^{k+l+4} had codimension more than 2, specialize it to Π after which the surface has to necessarily break into a plane containing l union a degree $k+l-1$ surface. Go back and repeat Steps 2 and 3. If Δ^{k+l+4} has codimension 2, specialize a different linear space again in order of increasing dimension. Go back to Step 2. We have reduced the problem to a problem of one degree lower in $\mathbb{P}^{k+l-1}$.

Inductively, we reduce the problem to a problem of counting planes with conditions of meeting a linear space, containing lines in a linear space or containing a conic. Finally, Theorems 7.2 and 7.3 dictate the multiplicities with which each case occurs. This concludes the description of the algorithm when $N=k+l+1$.

When $N>k+l+1$, the algorithm is almost identical and quickly reduces to the case $N=k+l+1$. Start by specializing Δ^i for $i\leq k+l+2$ to a general hyperplane. By the assumption 3, the first $k+l+1$ span P, a $\mathbb{P}^{k+l}$. The hyperplane section of the scrolls in H have to lie in P. After we specialize Δ^{k+l+2}, there are two possibilities.

1. If the scroll lies in H, we are done by induction.

2. If the scroll does not lie in H, Δ^{k+l+2} meets P in a point. At this stage there cannot be any reducible scrolls by the argument given above. Specialize Δ^{k+l+3} to H.

• If the scroll lies in H, we are again done by induction.

• If the scroll becomes reducible, proceed to Step 2 in the above process since there must be a scroll of degree $k+l-1$ in P.

• If the scroll remains irreducible and outside H, its hyperplane section in P is determined. After the next degeneration proceed with Step 2 in the above process. This concludes the proof. □

Remark 1. Although to satisfy the non-degeneracy assumptions the algorithm dictates an order of specialization, the enumerative numbers are independent of the order. By specializing the conditions in different orders one can solve more problems. For example, to find the number n of cubic scrolls in $\mathbb{P}^3$ that contain a fixed twisted cubic and five general points, we can count cubic scrolls in $\mathbb{P}^4$ that contain a twisted cubic, a point and meet 4 lines. If we specialize the point to the hyperplane of the twisted cubic, some of the limits become degenerate. Comparing the number to the one obtained from our algorithm, we conclude that $n=21$.

Remark 2. If we remove the non-degeneracy assumption in Theorem 6.8, there are components of D_H whose general point corresponds to a map with image a scroll S_0 of degree d_0 in H with many scrolls S_i outside H attached to it. The scrolls outside H can have contact of order m_i with H along their common lines with S_0. Moreover, the components do not have to remain balanced. New multiplicities appear: the divisors where the components of the scrolls have higher tangency with H appear with higher multiplicity. The limit of the directrices usually have tangency conditions with the limits of the hyperplane sections.

Even if we enlarge the class of problems to include these, at the next stage worse degenerations appear. Once the surface breaks again, we need to record the new hyperplane section which in turn can have various tangencies with both the directrix and the old hyperplane section. The analogue of Lemma 6.4 is not true for more than two curves. When the number of curves exceeds two, I do not know a complete list of the limits.

Remark 3. We can ask for the characteristic numbers of $S_{k,l}$ when $l-k>1$. Theorems 6.8 and 6.9 do not require the scrolls to be balanced. They determine the set-theoretic limits of unbalanced scrolls. In fact, Step 1 in the algorithm of Theorem 8.1 can be carried out for unbalanced scrolls the same way. However, the crucial observation that there are no matching conditions on the directrices of balanced scrolls no longer holds for unbalanced scrolls. After Step 1 of the algorithm we cannot reduce ourselves to the problem of counting smaller degree scrolls. In addition the limit of the hyperplane section has to meet the directrix along the special fiber.

One can reprove Theorems 6.8 and 6.9 by including this condition. The proof is identical, only the statement and the interpretation change. New divisors appear where the hyperplane section contains the special fiber or the directrix thus voiding the incidence condition. However, it becomes harder to trace this condition during a long degeneration.

Finally, the multiplicity statements become harder for unbalanced scrolls. Cones, especially, exhibit unexpected multiplicities. However, in small degree one can compute the characteristic numbers of unbalanced scrolls (see Example B2). We note that it is easy to see that each of the degenerations in Example B2 occur with multiplicity one by writing explicit first-order deformations, hence we omit a detailed argument.

9. The Gromov-Witten Invariants of $\mathbb{G}(1, N)$

In this section we explain the relation between the characteristic numbers of balanced scrolls and the Gromov-Witten invariants of $\mathbb{G}(1, N)$.

Gromov-Witten Invariants. Recall that $\overline{M}_{0,n}(X, \beta)$, the Kontsevich spaces of stable maps, come equipped with n evaluation morphisms $\rho_1, \cdots, \rho_n$ to X, where the i-th evaluation morphism takes the point $[C, p_1, \cdots, p_n, \mu]$ to the point $\mu(p_i)$ of X. Given classes $\gamma_1, \cdots, \gamma_n$ in the Chow ring A^*X of X, the *Gromov-Witten invariant* associated to these classes is defined by

$$I_\beta(\gamma_1, \cdots, \gamma_n) = \int_{\overline{M}_{0,n}^{\mathrm{virt}}(X,\beta)} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n).$$

If X is a homogeneous space $X = G/P$ and γ_i are fundamental classes of pure dimensional subvarieties Γ_i of X, then there is a close connection between the enumerative geometry of X and the Gromov-Witten invariants given by Lemma 14 in [FP]. We reproduce this lemma for the reader's convenience. Assume

$$\sum_{i=1}^{n} \operatorname{codim}(\Gamma_i) = \dim(X) + \int_\beta c_1(T_X) + n - 3.$$

Let $g\Gamma_i$ denote the g translate of Γ_i for some $g \in G$.

Lemma 9.1. *Let $g_1 \cdots, g_n \in G$ be general elements, then the scheme theoretic intersection*

$$\rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n) \tag{2}$$

is a finite number of reduced points supported in $M_{0,n}(X, \beta)$ and the Gromov-Witten invariant equals the cardinality of this set

$$I_\beta(\gamma_1, \cdots, \gamma_n) = \#\rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n).$$

In the case of $\overline{M}_{0,n}(\mathbb{G}(1, N), k+l)$ using Kleiman's theorem we can, in fact, conclude that the intersection in (2) is supported in the locus of maps to non-degenerate curves of directrix degree $\lfloor (k + l)/2 \rfloor$.

Assume the Γ_i are Schubert cycles of the form Σ_{a_i}, the cycle of lines meeting an a_i dimensional linear space. By Theorem 3.2 the cardinality of the intersection in (2) is equal to the number of balanced scrolls meeting general linear spaces of dimension a_i, $1 \leq i \leq n$. We thus obtain the following corollary to Theorem 8.1:

Corollary 9.2. *Let $\Gamma_i = \Sigma_{a_i}$. Assume a_i satisfy the conditions of Theorem 8.1. Then the algorithm described in Theorem 8.1 provides an algorithm for computing*

$$I_{k+l}(\gamma_1, \cdots, \gamma_n)$$

Remark. One has to exercise caution when translating the number of quadric surfaces to degree 2 Gromov-Witten invariants of Grassmannians. Our algorithm counts actual quadric surfaces. Since quadric surfaces can be seen as scrolls in two distinct ways depending on the choice of ruling, the Gromov-Witten invariant is twice the number of quadric surfaces.

A closer analysis of the algorithm in Theorem 8.1 shows that it computes the number of balanced scrolls of degree $k+l$ containing a section class of degree $k+l$ or $k+l+1$ subject to the non-degeneracy assumptions. By an argument similar to the one just given, our algorithm computes certain Gromov-Witten invariants of $\mathbb{F}(0,1;N)$. For a sample of different approaches to the Gromov-Witten invariants of Grassmannians and Flag manifolds see [Ci], [BKT] or [Tam].

We conclude with a table of characteristic numbers of surfaces. We will denote the number of $S_{k,l}$ in $\mathbb{P}^N$ that meet a_0 points, a_1 lines, ..., a_k k-planes by $n(N;k,l;a_0,a_1,\cdots,a_k)$.

$n(4;1,1;4,5)=1$	$n(4;1,2;9,0)=2$
$n(4;1,1;3,7)=9$	$n(4;1,2;8,2)=17$
$n(4;1,1;2,9)=64$	$n(4;1,2;7,4)=138$
$n(4;1,1;1,11)=430$	$n(4;1,2;6,6)=1140$
$n(4;0,2;4,4)=4$	$n(4;1,2;5,8)=9770$
$n(4;0,2;3,6)=30$	$n(5;1,2;4,5,1)=58$
$n(4;0,2;2,8)=190$	$n(5;1,2;4,4,3)=423$
$n(5;1,1;3,0,8)=48$	$n(5;2,2;9,1)=6$

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Mathematics Department, Harvard University, Cambridge, MA 02138

E-mail address: coskun@math.harvard.edu