# RESTRICTION VARIETIES AND GEOMETRIC BRANCHING RULES

IZZET COSKUN

Abstract. This paper develops a new method for studying the cohomology of orthogonal flag varieties. Restriction varieties are subvarieties of orthogonal flag varieties defined by rank conditions with respect to (not necessarily isotropic) flags. They interpolate between Schubert varieties in orthogonal flag varieties and the restrictions of general Schubert varieties in ordinary flag varieties. We give a positive, geometric rule for calculating their cohomology classes, obtaining a branching rule for Schubert calculus for the inclusion of the orthogonal flag varieties in Type-A flag varieties. Our rule, in addition to being an essential step in finding a Littlewood-Richardson rule, has applications to computing the moment polytopes of the inclusion of $SO(n)$ in $SU(n)$, the asymptotic of the restrictions of representations of $SL(n)$ to $SO(n)$ and the classes of the moduli spaces of rank two vector bundles with fixed odd determinant on hyperelliptic curves. Furthermore, for odd orthogonal flag varieties, we obtain an algorithm for expressing a Schubert cycle in terms of restrictions of Schubert cycles of Type-A flag varieties, thereby giving a geometric (though not positive) algorithm for multiplying any two Schubert cycles.

## Contents



## 1. Introduction

Let $Q$ denote a non-degenerate, symmetric bilinear form on a vector space $W$ of dimension $n$. Let $0 < k_1 < \cdots < k_h$ be non-negative integers such that $2k_h \leq n$. Let $OF(k_1, \ldots, k_h; n)$ denote the

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orthogonal partial flag variety parameterizing subspaces

$$W_1 \subset \cdots \subset W_h$$

of $W$ isotropic with respect to $Q$, where $W_i$ has dimension $k_i$. A restriction variety is the intersection of $OF(k_1, \ldots, k_h; n)$ with a Schubert variety in the ordinary flag variety $F(k_1, \ldots, k_h; n)$ defined by a flag satisfying certain tangency conditions with respect to $Q$. The main theorem of this paper is a positive, geometric rule for computing the cohomology class of a restriction variety in terms of Schubert cycles.

**Theorem 7.22.** *Algorithm 7.19 provides a positive, geometric rule for computing the cohomology class of a restriction variety. In particular, the algorithm computes the image of the map induced in cohomology by the natural inclusion*

$$i : OF(k_1, \ldots, k_h; n) \to F(k_1, \ldots, k_h; n).$$

An important special case, which we will treat first, describes the geometry of restriction varieties in orthogonal Grassmannians. Theorem 5.12 similarly gives a positive, geometric rule for computing the cohomology classes of restriction varieties in orthogonal Grassmannians.

Theorem 7.22 has many applications, most notably to calculating the moment polytopes for the inclusion of $SO(n)$ in $SU(n)$, the asymptotic of the restrictions of representations of $SL(n)$ to $SO(n)$ and the invariants of the moduli spaces of rank two vector bundles on hyperelliptic curves. Let $i : G' \to G$ be an inclusion of complex, reductive, connected Lie groups. Choose Borel subgroups $B' \subset G'$ and $B \subset G$ such that $i(B') \subset B$. Then the inclusion $i : G'/B' \to G/B$ induces a map in cohomology $i^* : H^*(G/B) \to H^*(G'/B')$. The structure coefficients of this map in terms of Schubert bases are called branching coefficients. Finding positive rules for calculating branching coefficients is a central problem (see [P] for references, a beautiful exposition of the subject and fundamental results). In the case of $SO(n)$ and $SL(n)$, the map $i$ is given by sending an isotropic flag $F_\bullet$ to the pair $(F_\bullet, F_\bullet^\perp)$. Our theorem calculates all the branching coefficients of $i^* : H^*(F(k_1, \ldots, k_h, n-k_h, \ldots, n-k_1; n)) \to H^*(OF(k_1, \ldots, k_h; n))$ for the classes that are pulled back from $F(k_1, \ldots, k_h; n)$ under the natural projection that sends $(F_\bullet, F_\bullet^\perp)$ to $F_\bullet$. These calculations have already found applications in the study of eigencones and the Belkale-Kumar product [RR].

Knowing the set of non-zero branching coefficients has important applications in symplectic geometry and representation theory. Let $K$ and $K'$ be the maximal compact subgroups of $G$ and $G'$, respectively. To each non-vanishing branching coefficient, in [BS], Berenstein and Sjamaar associate an inequality satisfied by the $K'$-moment polytope of a $K$-coadjoint orbit. Moreover, the totality of these inequalities gives a sufficient set of inequalities for the moment polytope. Similarly, non-vanishing branching coefficients determine which irreducible representations of $G'$ occur in the restriction of an irreducible representation of $G$ asymptotically. More precisely, let $V_\lambda$ be an irreducible representation of $G$ with highest weight $\lambda$ and let $V'_\mu$ be an irreducible representation of $G'$ with highest weight $\mu$. The answer to the question 'Does there exist a positive integer $N$, such that when the $G$-module $V_{N\lambda}$ is decomposed as a $G'$-module, the representation $V'_{N\mu}$ occurs as a component?' is characterized by non-vanishing branching coefficients (see [BS], [GS], [H], [P]).

Theorem 7.22 also has many geometric applications. For instance, using a Theorem of Desale and Ramanan [DR], we will compute the class of the moduli space of rank two vector bundles with fixed odd-degree determinant on a hyperelliptic curve of genus $g$ in $OG(g-1, 2g+2)$. In fact, we discover a recursion in $g$ for the class. However, the main purpose of this paper is to introduce a new point of view in calculating the cohomology classes of subvarieties of orthogonal Grassmannians and, more generally, orthogonal flag varieties. Theorem 5.12 is a first step for finding a positive, geometric rule for orthogonal flag varieties. We present it here separately in order to emphasize the simplicity of the geometric ideas without any combinatorial complications. In future work, we will give positive, geometric rules for calculating the class of the intersection of certain classes of Schubert varieties relying on the geometric principles discussed in this paper ([C4] and [C5]).

The study of the cohomology of isotropic flag varieties and geometric branching rules has a very long history. We mention a few representative results and refer the reader to [YT] and [Ta] for a more comprehensive set of references. Positive rules are known for multiplying arbitrary classes in maximal orthogonal Grassmannians and certain special classes in general (see, for instance, [FP], [BKT1], [YT], [CP] for references and results and [Dav] for a promising approach). Pragacz proved a positive combinatorial branching rule for the Lagrangian Grassmannian and maximal orthogonal Grassmannians of Type B [Pr1], [Pr2]. While this paper was in the refereeing process, Buch, Kresch and Tamvakis obtained Pieri rules for arbitrary isotropic Grassmannians [BKT2] and Giambelli rules for expressing classes in terms of their Pieri classes [BKT3]. The paper [Ta] discusses more general Giambelli rules and recent developments. It should also be mentioned that it is possible to obtain non-positive branching rules by first computing the pull-backs of the tautological bundles from the Type A flag manifold to the Type B or D flag manifolds. One can then use localization or the theory of Schubert polynomials to obtain branching rules. However, to the best of the author's knowledge, the rule presented in this paper remains the only known positive, geometric branching rule that applies to all partial flag varieties of Types B and D. The Type C case is simpler and will be exposed elsewhere.

The geometric point of view we present here has many advantages. It unifies different types. It gives a clear strategy for obtaining positive rules for calculations in the cohomology ring. It can be adapted to fields other than $\mathbb{C}$ and geometric situations more general than the intersection of two Schubert varieties. Most importantly, the calculation is at the level of cycles and not cycle classes. Hence, the information provided by the positive, geometric rules is much more refined than purely combinatorial rules.

The geometry of orthogonal homogeneous varieties is significantly more complicated than the geometry of Type-A homogeneous varieties. In this paper, we show that the computation of the branching coefficients can be reduced to four basic facts about quadric hypersurfaces. We now explain the strategy and recall these basic facts. For simplicity, we will discuss the case of orthogonal Grassmannians. The orthogonal Grassmannian $OG(k,n)$ parameterizes $k$-dimensional subspaces of $W$ that are isotropic with respect to $Q$. When $n=2k$, the isotropic linear spaces form two isomorphic connected components. It is customary to set $OG(k,2k)$ equal to one of these components. The cohomology of $OG(k,n)$ is generated by the classes of Schubert varieties.

The quadratic form $Q$ defines a smooth degree two hypersurface Q in $\mathbb{P}W$. We will interpret $OG(k,n)$ as the Fano variety of $(k-1)$-dimensional projective linear subspaces on Q. We will also need to study singular quadric hypersurfaces. Over the complex numbers, the projective equivalence class of a quadric hypersurface is determined by its dimension and corank. Let $Q_{d_i}^{r_i}$ denote a quadratic form of corank $r_i$ obtained by restricting $Q$ to a vector space of dimension $d_i$. Let $L_{n_j}$ denote an isotropic linear space of (vector space) dimension $n_j$. A restriction variety in $OG(k,n)$ is defined in terms of a sequence

$$L_{n_1} \subset \cdots \subset L_{n_s} \subset Q_{d_{k-s}}^{r_{k-s}} \subset \cdots \subset Q_{d_1}^{r_1}$$

of isotropic linear spaces and quadrics. (In Definitions 4.2 and 4.9, we will specify the conditions that these linear spaces and quadrics need to satisfy. For the purposes of the introduction we ignore these subtleties.) The restriction variety parameterizes the isotropic linear spaces that intersect $L_{n_j}$ in a subspace of dimension $j$ and $Q_{d_i}^{r_i}$ in a subspace dimension $k-i+1$ for every $1 \leq j \leq s$ and $1 \leq i \leq k-s$. Schubert varieties are examples of restriction varieties with the property that the quadrics in the sequence are as singular as possible (i.e., $d_i + r_i = n$). The strategy to calculate the class of a restriction variety is to specialize the quadrics in the sequence one at a time to become more singular until they are maximally singular. When we specialize the quadrics, the restriction variety breaks into a union of simpler restriction varieties. The process is governed by the following basic facts about quadrics.

• **The corank bound.** Let $Q_{d_2}^{r_2} \subset Q_{d_1}^{r_1}$ be two linear sections of $Q$ such that the singular locus of $Q_{d_1}^{r_1}$ is contained in the singular locus of $Q_{d_2}^{r_2}$. Then $r_2 - r_1 \leq d_1 - d_2$. In particular, the corank of a sub-quadric in $Q$ is bounded by its codimension.

• **The linear space bound.** The largest dimensional isotropic linear space with respect to a quadratic form $Q^r_d$ has dimension $\lfloor \frac{d+r}{2} \rfloor$. A linear space of dimension $j$ intersects the singular locus of $Q^r_d$ in a subspace of dimension at least $\max(0, j - \lfloor \frac{d-r}{2} \rfloor)$.

• **Irreducibility.** A sub-quadric $Q^{d-2}_d$ of $Q$ is reducible and equal to the union of two linear spaces of (vector space) dimension $d-1$ meeting along a linear space of dimension $d-2$. If $n = 2k$, then the linear spaces constituting $Q^{k-1}_{k+1}$ belong to two distinct connected components.

• **The variation of tangent spaces.** Let a quadric $Q^r_d$ be singular along a codimension $j$ linear subspace $M$ of a linear space $L$. Then the image of the Gauss map of $Q^r_d$ restricted to the smooth points of $L$ has dimension at most $j-1$. In other words, the tangent spaces to $Q^r_d$ along the smooth points of $L$ vary at most in a $(j-1)$-dimensional family.

The corank bound determines the order of the specialization. We increase the corank of the smallest dimensional quadric $Q^{r_i}_{d_i}$ that satisfies $d_i + r_i < d_{i-1} + r_{i-1}$ by one, i.e., we replace $Q^{r_i}_{d_i}$ in the sequence with $Q^{r_i+1}_{d_i}$. The algorithm is obtained by describing the flat limit of this specialization. Suppose that a general linear space parametrized by the restriction variety intersects the singular locus of $Q^{r_i}_{d_i}$ in a subspace of dimension $x_i$. The linear spaces parametrized by the flat limit intersect the singular locus of $Q^{r_i+1}_{d_i}$ in a subspace of dimension $x_i$ or $x_i+1$. The limit has more than one component when both cases are possible. 'The linear space bound' and 'the variance of tangent spaces' dictate which of the possibilities occur. In addition, if $r_i = d_i - 3$, then by the 'irreducibility' property, the new quadric $Q^{r_i+1}_{d_i}$ is reducible forcing the limit to possibly have more components. Surprisingly, each of these components occur with multiplicity one in the limit. The algorithm is obtained by inductively applying this specialization to each irreducible component. We refer the reader to §5 for the precise statement of the algorithm and detailed examples.

The organization of the paper is as follows. In §2, we recall general facts about the geometry of quadrics and orthogonal flag varieties. In §3, we state the Grassmannian rule purely combinatorially. In §4, we develop some of the basic properties of restriction varieties in orthogonal Grassmannians. In §5, we give the algorithm for computing the classes of restriction varieties in orthogonal Grassmannians. In §6, we give simple applications of the algorithm. In §7, we extend the algorithm to orthogonal flag varieties.

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## 2. Preliminaries

In this section, we recall the preliminaries about the geometry of quadric hypersurfaces and orthogonal Grassmannians. For a more detailed treatment we refer the reader to Chapter 6 of [GH].

2.1. **Preliminaries on quadrics.** Let $Q$ be a smooth quadric hypersurface in $\mathbb{P}^{n-1}$. Set $m = \lfloor \frac{n}{2} \rfloor$. The largest dimensional linear spaces contained in $Q$ have projective dimension $m-1$. If $n$ is odd, then the maximal dimensional linear spaces on $Q$ form an irreducible family of dimension $\frac{m(m+1)}{2}$. If $n$ is even, then the maximal dimensional linear spaces contained in $Q$ form two isomorphic families of dimension $\frac{m(m-1)}{2}$. Two linear spaces belong to the same irreducible component if and only if their dimension of intersection is equal to $m-1$ modulo 2 (see [GH] p. 735).

More generally, we will be interested in linear spaces on quadric hypersurfaces with singularities. A quadric hypersurface in $\mathbb{P}^{n-1}$ of corank $r$ (or, equivalently, with a singular locus of dimension $r-1$) is the cone over a smooth quadric hypersurface in $\mathbb{P}^{n-1-r}$ with vertex an $(r-1)$-dimensional projective linear space. If $Q$ is a quadric hypersurface of corank $r$ in $\mathbb{P}^{n-1}$, then the largest dimensional linear space on $Q$ has dimension $\lfloor \frac{n-r-2}{2} \rfloor + r$. The space of linear spaces of maximal dimension on $Q$ is irreducible if

$n-r$ is odd and has two irreducible components if $n-r$ is even. Setting $l=\frac{n-r-3}{2}$ in the former case and $l=\frac{n-r-2}{2}$ in the latter case, the dimension of each irreducible component of the space of maximal dimensional linear spaces is $\frac{(l+1)(l+2)}{2}$ and $\frac{l(l+1)}{2}$, respectively. In the latter case, two linear spaces belong to the same irreducible component if and only if their dimension of intersection is equal to $l+r$ modulo 2. These claims follow from the previous paragraph since $Q$ is a cone over a smooth quadric hypersurface in $\mathbb{P}^{n-1-r}$.

*Notation* 2.1. Denote the Fano variety of $s$-dimensional projective linear spaces contained in a quadric hypersurface $Q \in \mathbb{P}^{n-1}$ of corank $r$ by $F^r_{s,n}(Q)$.

Let $Q \subset \mathbb{P}^{n-1}$ be a quadric hypersurface of corank $r$. Let $s$ be a positive integer less than or equal to $\lfloor \frac{n-r-2}{2} \rfloor + r$. Consider the incidence correspondence of pairs of a point $p$ of $Q$ and an $s$-dimensional linear space containing $p$:

$$I = \{\ (x, [\Lambda]) \mid x \in \Lambda \subset Q\ \} \ \subset\ Q \times F^r_{s,n}(Q).$$

The automorphism group of $Q$ acts transitively on the smooth points of $Q$. The $s$-planes that contain a smooth point $p$ lie in the tangent linear space $H$ at $p$. $Q \cap H$ is a quadric hypersurface of corank $r+1$. The intersection with a hyperplane complementary to $p$ is a quadric hypersurface of corank $r$ and intersects all the $s$-planes containing $p$ in an $(s-1)$-dimensional linear space. We conclude that the space of $s$-dimensional linear spaces containing $p$ has the same dimension as the space of $(s-1)$-dimensional linear spaces lying on a quadric hypersurface in $\mathbb{P}^{n-3}$ of corank $r$. Therefore, by induction, we can calculate the general fiber dimension of the projection of $I$ to $Q$ and determine the dimension of $I$. The second projection maps $I$ onto $F^r_{s,n}(Q)$ with fiber dimension $s$. We thus obtain a recursion relation for the dimension of $F^r_{s,n}(Q)$.

A priori we need to check that the $s$-dimensional linear spaces that intersect the vertex in dimension greater than $s-1-\lfloor \frac{n-r-2}{2} \rfloor$ do not form another irreducible component (potentially of different dimension) of $F^r_{s,n}(Q)$. It is easy to see that linear spaces that intersect the vertex in larger than the expected dimension are limits of linear spaces that intersect the vertex in the expected dimension. Observe that every linear space on a quadric is contained in a maximal dimensional linear space. Take a linear space $\Lambda$ that intersects the vertex in the linear space $\Omega$. Assume that the dimension of $\Omega$ is larger than expected. Take a linear space $\Delta$ in $\Lambda$ complementary to $\Omega$. Take a linear space $\Gamma$ of dimension $\lfloor \frac{n-r-2}{2} \rfloor$ which contains $\Delta$, but does not intersect the vertex of $Q$. Since the Grassmannian of $s$-planes in the span of $\Gamma$ and $\Omega$ is irreducible, the claim follows.

In case $s < \frac{n-r-2}{2}$, the space of $s$-dimensional linear spaces on $Q$ is irreducible. If $s \geq \frac{n-r-2}{2}$ the recursion stops when we obtain a quadric of rank $r$ in $\mathbb{P}^{r+1}$ or $\mathbb{P}^r$ with multiplicity 2. The former case occurs if $n-r$ is even and the latter case occurs if $n-r$ is odd. This allows us to calculate the dimensions of the spaces of $s$-dimensional linear spaces on $Q$ recursively. It also proves that when $s \geq \frac{n-r-2}{2}$, the spaces of $s$-dimensional linear spaces on $Q$ is irreducible if $n-r$ is odd and has two components if $n-r$ is even. We have thus proved the following:

**Lemma 2.2.** *Let $Q$ be a quadric hypersurface in $\mathbb{P}^{n-1}$ of corank $r$. If $s < \frac{n-r-2}{2}$, then $F^r_{s,n}(Q)$ is irreducible of dimension*

$$(s+1)\ \frac{2n-3s-4}{2}.$$

*If $s \geq \frac{n-r-2}{2}$ and $n-r$ is even, then $F^r_{s,n}(Q)$ has two irreducible components each of dimension*

$$(s+1)\ \frac{n-2s+r-2}{2}\ +\ \frac{(n-r-2)\ (n-r)}{8}.$$

*If $s \geq \frac{n-r-2}{2}$ and $n-r$ is odd, then $F^r_{s,n}(Q)$ is irreducible of dimension*

$$(s+1)\ \frac{n-2s+r-3}{2}\ +\ \frac{(n-r-1)\ (n-r+1)}{8}.$$

2.2. **Preliminaries on orthogonal Grassmannians.** Let $W$ be an $n$-dimensional vector space endowed with a non-degenerate, symmetric, bilinear form $Q$. Set $m = \lfloor \frac{n}{2} \rfloor$. Let $0 < k \leq m$ denote a positive integer. Let $OG(k,n)$ denote the $k$-dimensional subspaces of $W$ isotropic with respect to the form $Q$, unless $n = 2k$. In the latter case, the parameter space of $k$-dimensional isotropic subspaces of $W$ has two isomorphic irreducible components. $OG(k,n)$ denotes one of these irreducible components.

The orthogonal Grassmannian $OG(k,n)$ is isomorphic to one irreducible component of the Fano variety $F^0_{k-1,n}(Q)$ of $(k-1)$-dimensional projective linear spaces on a smooth quadric hypersurface. The non-degenerate quadratic form $Q$ defines the smooth quadric hypersurface in $\mathbb{P}^{n-1}$. A linear space is isotropic with respect to $Q$ if and only if its projectivization is contained in the quadric hypersurface defined by $Q$. In particular, by the discussion in §2.1, the dimension of $OG(k,n)$ is

$$\frac{k(2n-3k-1)}{2}$$

By Ehresmann's Theorem [E], the cohomology of $OG(k,n)$ is generated by the classes of Schubert varieties. There are minor differences in the cohomology of $OG(k,n)$ depending on the parity of $n$ due to the fact that when $n$ is even, the half-dimensional isotropic subspaces form two connected components. For even $n$, the notation has to distinguish between these two connected components. For simplicity, we will first discuss the case of odd $n$, then describe the necessary modifications for even $n$.

We begin by describing the Schubert varieties in $OG(k, 2m+1)$. Let $\lambda$ denote a sequence

$$m \geq \lambda_1 > \lambda_2 > \cdots > \lambda_s > 0$$

of strictly decreasing integers, where $s \leq k$. Given $\lambda$, there is an associated sequence

$$m-1 \geq \tilde{\lambda}_{s+1} > \cdots > \tilde{\lambda}_m \geq 0$$

of strictly decreasing integers defined by requiring that there does not exist any parts $\lambda_i$ for which $\tilde{\lambda}_j + \lambda_i = m$. In other words, the associated partition is obtained by removing the integers $m-\lambda_1, \dots, m-\lambda_s$ from the sequence $m-1, m-2, \dots, 0$. For example, if $m=6$, then the partition associated to $(6,4)$ is $(5,4,3,1)$. The Schubert varieties in $OG(k, 2m+1)$ are parameterized by pairs $(\lambda, \mu)$, where $\lambda$ is a strictly decreasing partition of length $s$ and $\mu$

$$m-1 \geq \mu_{s+1} > \mu_{s+2} > \cdots > \mu_k \geq 0$$

is a subpartition of $\tilde{\lambda}$ (i.e., the parts of $\mu$ are a subset of the parts of $\tilde{\lambda}$) of length $k-s$. We will call such pairs of partitions *allowed pairs*. Observe that for maximal isotropic Grassmannians $OG(m, 2m+1)$, the partition $\mu = \tilde{\lambda}$ is uniquely determined by the partition $\lambda$. Consequently, in the literature it is standard to omit the sequence $\mu$ and parametrize Schubert varieties by strict partitions $\lambda$. We will find it useful to record the dimensions of all the flag elements where a jump in dimension occurs, so we add $\mu$ to the notation. For non-maximal Grassmannians there are several notations in use. The advantage of our notation is that it minimizes the amount of calculation needed to determine the dimensions of the flag elements where a jump in dimension occurs. Since $\mu$ is a subpartition of $\tilde{\lambda}$ we can assume that it occurs as $\tilde{\lambda}_{i_{s+1}}, \cdots, \tilde{\lambda}_{i_k}$. Given a pair $(\lambda, \mu)$, the discrepancy $\mathrm{dis}(\lambda, \mu)$ of the pair is defined by

$$\mathrm{dis}(\lambda, \mu) = (m-k)s + \sum_{j=s+1}^{k} (m-k+j-i_j).$$

Fix an isotropic flag $F_\bullet$

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_m \subset F_{m-1}^\perp \subset \cdots \subset F_1^\perp \subset W.$$

Here $F_i^\perp$ denotes the orthogonal complement of $F_i$ with respect to the bilinear form. In terms of the geometry of the quadric hypersurface $Q \subset \mathbb{P}^{n-1}$ we can describe $F_j^\perp$ as follows. A one-dimensional isotropic subspace corresponds to a point $p \in Q \subset \mathbb{P}^{n-1}$. The annihilator of that subspace corresponds to the tangent space to $Q$ at the point $p$. We can take $Q$ to be given by the equation $\sum_{i=1}^n X_i^2 = 0$. We can assume the isotropic subspace is generated by $v = (1, i, 0, \dots, 0)$. The annihilator of $v$ is given by vectors $(v_1, v_2, \dots, v_n)$ such that $v_1 + iv_2 = 0$. On the other hand, the tangent space to the quadric hypersurface at $p$ corresponding to $v$ is given by $X_1 + iX_2 = 0$. So the annihilator of a vector consists precisely of those vectors lying in the tangent hyperplane to the quadric at the point corresponding to the vector. To find $F_j^{\perp}$ we take the intersection of all the tangent hyperplanes at the points of $F_j$. The intersection is the projective linear space $\mathbb{P}^{n-1-j}$ everywhere tangent to $Q$ along the projectivization of $F_j$.

The Schubert variety $\Omega_\lambda^\mu(F_\bullet)$ is defined as the closure of the locus

$$\{[\Lambda] \in OG(k, 2m+1) |\dim(\Lambda \cap F_{m+1-\lambda_i}) = i \text{ for } 1 \leq i \leq s, \dim(\Lambda \cap F_{\mu_j}^{\perp}) = j \text{ for } s < j \leq k\}.$$

The codimension of a Schubert variety is given by $\sum_{i=1}^s \lambda_i + \mathrm{dis}(\lambda, \mu)$. We will denote the cohomology class of $\Omega_\lambda^\mu$ by $\sigma_\lambda^\mu$.

The description of the Schubert varieties in $OG(k, 2m)$ requires minor modifications to account for the fact that the space of $m$-dimensional isotropic subspaces have two irreducible components. Let $\lambda$ denote a sequence

$$m - 1 \geq \lambda_1 > \lambda_2 > \cdots > \lambda_s \geq 0$$

of strictly decreasing integers where $s \leq k$. When $k = m$ and $m$ is even (respectively, odd), we will assume that $s$ is even (respectively, odd). Given $\lambda$, we can define an associated sequence $\tilde{\lambda}$ of strictly decreasing integers

$$m - 1 \geq \tilde{\lambda}_{s+1} > \cdots > \tilde{\lambda}_m \geq 0$$

satisfying the condition that there does not exists $\lambda_i$ such that $\lambda_i + \tilde{\lambda}_j = m-1$. In other words, to obtain $\tilde{\lambda}$ remove from the sequence $m-1, \dots, 0$ the integers $m-1-\lambda_1, \dots, m-1-\lambda_s$. The Schubert varieties in $OG(k, 2m)$ are parameterized by pairs $(\lambda, \mu)$, where $\lambda$ is a strictly decreasing partition of length $s$ and $\mu$

$$m - 1 \geq \mu_{s+1} > \mu_{s+2} > \cdots > \mu_k \geq 0$$

is a subpartition of $\tilde{\lambda}$ of length $k - s$. We will call such pairs of partitions *allowed pairs*. As above, for maximal isotropic Grassmannians $OG(m, 2m)$, the partition $\mu = \tilde{\lambda}$ is uniquely determined by the partition $\lambda$, so it is often omitted from the notation. The pair $(\lambda, \mu)$ is a subpartition of a pair $(\lambda', \tilde{\lambda}')$ of total length $m$ defined as follows. If $m$ and $s$ have the same parity, then $\lambda = \lambda'$. If $m$ and $s$ have different parities, $\lambda'$ has length $s + 1$ and differs from $\lambda$ in that it includes the smallest number between 0 and $m - 1$ not already occurring in $\lambda$ and not adding to $m - 1$ with any of the parts in $\mu$. The discrepancy $\mathrm{dis}(\lambda, \mu)$ of the pair $(\lambda, \mu)$ is defined as follows: Since $(\lambda, \mu)$ is a subpartition of $(\lambda', \tilde{\lambda}')$, we can assume that the parts occur as $\lambda'_{i_1}, \dots, \lambda'_{i_s}, \tilde{\lambda}'_{i_{s+1}}, \cdots, \tilde{\lambda}'_{i_k}$. The discrepancy is defined as

$$\mathrm{dis}(\lambda, \mu) = \sum_{j=1}^k (m - k + j - i_j).$$

We will make the convention that $F_m$ denotes an $m$-dimensional isotropic subspace in one of the irreducible components. By abuse of notation, we will denote by $F_{m-1}^{\perp}$ an $m$-dimensional isotropic subspace in the other irreducible component. Note that strictly speaking the intersection of the quadric hypersurface with $F_{m-1}^{\perp}$ consists of the union of two $m$-dimensional isotropic subspaces one in each irreducible component. Our slight abuse of notation will make notation more compact. We will use this convention without further mention in the rest of the paper. The Schubert variety $\Omega_\lambda^\mu(F_\bullet)$ is defined as the closure of the locus

$$\{[\Lambda] \in OG(k, 2m) |\dim(\Lambda \cap F_{m-\lambda_i}) = i \text{ for } 1 \leq i \leq s,\ \dim(\Lambda \cap F_{\mu_j}^{\perp}) = j \text{ for } s < j \leq k\}.$$

The codimension of a Schubert variety is given by $\sum \lambda'_i + \mathrm{dis}(\lambda, \mu)$. We will denote the cohomology class of $\Omega_\lambda^\mu$ by $\sigma_\lambda^\mu$.

The cohomology classes $\sigma_\lambda^\mu$, as $(\lambda,\mu)$ varies over all allowed pairs, form an additive basis of the cohomology ring of $OG(k,n)$. Given an allowed pair $(\lambda,\mu)$ for $OG(k,2m+1)$, there is a dual allowed pair $(\lambda^c,\mu^c)$ defined by

$$\lambda_1^c = m-\mu_k,\ \dots,\ \lambda_{k-s}^c = m-\mu_{s+1},\ \mu_{k-s+1}^c = m-\lambda_s,\ \dots,\ \mu_k^c = m-\lambda_1.$$

Similarly, if $(\lambda,\mu)$ is an allowed pair for $OG(k,2m)$, define the dual pair $(\lambda^c,\mu^c)$ by setting

$$\lambda_1^c = m-1-\mu_k,\ \dots,\ \lambda_{k-s}^c = m-1-\mu_{s+1},\ \mu_{k-s+1}^c = m-1-\lambda_s,\ \dots,\ \mu_k^c = m-1-\lambda_1.$$

If $(\lambda,\mu)$ and $(\lambda^c,\mu^c)$ are dual allowed pairs, then $\sigma_\lambda^\mu \cdot \sigma_{\lambda^c}^{\mu^c}$ is equal to the Poincaré dual of the point class.

2.3. **Orthogonal flag varieties.** We now extend the discussion in the previous section to orthogonal flag varieties. We preserve the notation from the previous section. Let $0 < k_1 < \cdots < k_h \leq m$ be an increasing sequence of positive integers. The orthogonal flag variety $OF(k_1,\dots,k_h;n)$ parameterizes $h$-tuples

$$W_1 \subset \cdots \subset W_{k_h}$$

of isotropic subspaces of $W$, where $W_i$ has dimension $k_i$. When $2k_h = n$, this space has two isomorphic components and it is customary to let the orthogonal flag variety to be one of the components. $OF(k_1,\dots,k_h;n)$ admits a projection morphism to $OG(k_h,n)$ by forgetting the first $h-1$ linear spaces. The fibers of this projection are ordinary partial flag varieties $F(k_1,\dots,k_{h-1};k_h)$. The geometry of orthogonal partial flag varieties can be studied using this projection. For example, the dimension of $OF(k_1,\dots,k_h;n)$ is easily seen to be

$$\dim(OF(k_1,\dots,k_h;n)) = \dim(OG(k_h,n)) + \sum_{i=1}^{h-1} k_i(k_{i+1}-k_i).$$

The cohomology of $OF(k_1,\dots,k_h;n)$ is also generated by Schubert cycles. In order to parameterize Schubert cycles we need to enrich the partition notation from the previous section with the data of a color. Let $1 < \cdots < h$ be $h$ ordered colors. A colored partition $(\lambda,\mu,c_\bullet)$ for $OF(k_1,\dots,k_h;n)$ is an allowable pair $(\lambda,\mu)$ for $OG(k_h,n)$ where the parts $\lambda_1^{c_1} > \cdots > \lambda_s^{c_s}$ and $\mu_{s+1}^{c_{s+1}} > \cdots > \mu_{k_h}^{c_{k_h}}$ have been enriched by the data of a color such that $k_1$ of the parts are assigned the color 1, and $k_i - k_{i-1}$ of the parts are assigned the color $i$ for $1 < i \leq h$. The isotropic flag induces a complete flag $F_1 \subset \cdots \subset F_{k_h} = W_{k_h}$ on the $k_h$-dimensional isotropic linear space. On a Zariski open subset of a Schubert variety, for each flag element $F_i$, there exists a least index $c_i$ such that

$$\dim(F_i \cap W_{c_i}) = \dim(F_{i-1}\cap W_{c_i}) + 1.$$

We assign the index $c_i$ to the $i$-th part in the partition.

When $n$ is even (respectively, odd), the Schubert variety $\Omega_\lambda^\mu(F_\bullet, c_\bullet)$ is defined as the closure of the locus

$$\begin{aligned}\{\ [(W_1,\dots,W_{k_h})] \in OF(k_1,\dots,k_h;n)\ \mid\ &\dim(W_u \cap F_{m-\lambda_i^{c_i}(resp.,+1)}) = \#\{v \leq i \mid c_v \leq u\},\\ &\dim(W_u \cap F^{\perp}_{\mu_j^{c_j}}) = \#\{v\leq j \mid c_v \leq u\}\ \}.\end{aligned}$$

More generally, we will call a sequence $c_1,\dots,c_{k_h}$ of integers between 1 and $h$ such that $k_1$ of the terms are 1 and $k_i - k_{i-1}$ of the terms are $i$ a coloring scheme for the sequence $k_1,\dots,k_h$. For such a coloring scheme and a color $1 \leq u \leq h-1$ define $\mathrm{codim}(u)$, the codimension for the color $u$, to be the sum

$$\mathrm{codim}(u) = \sum_{1\leq i\leq k_h,\ c_i\leq u} \#\{j > i \mid c_j = u+1\}.$$

Define the color discrepancy $\mathrm{cdis}(c_\bullet)$ of a coloring scheme to be the sum

$$\mathrm{cdis}(c_\bullet) = \sum_{u=1}^{h-1}\mathrm{codim}(u)$$

Then the codimension of the Schubert variety corresponding to the colored partition $(\lambda, \mu, c_\bullet)$ is given by

$$\operatorname{codim}(\lambda, \mu) + \operatorname{cdis}(c),$$

where $\operatorname{codim}(\lambda, \mu)$ denotes the codimension of the Schubert variety $\Omega_\lambda^\mu$ in $OG(k_h, n)$. This is easily seen by considering the projection from $OF(k_1, \dots, k_h; n)$ to $OG(k_h, n)$.

## 3. Combinatorics

In this section, we present the rule for $OG(k, n)$ combinatorially without any explanations or motivation. The purpose of this subsection is to teach the reader the mechanics of the rule. The geometrically minded reader might prefer to read this section after reading §4 and §5.

*Notation* 3.1. A *sequence of n natural numbers with gaps* is a sequence of $n$ natural numbers written from left to right with a gap to the right of each number in the sequence. We will refer to the gap after the $i$-th number as the $i$-th position. For example, 1 1 2 3 0 0 0 0 is a sequence of 8 numbers with gaps.

**Definition 3.2.** *Let $0 \leq s \leq k < n$ be integers. A* sequence of brackets and braces *of type $(k, n)$ is a sequence of n natural numbers with gaps, s right brackets* ] *and $k - s$ right braces* } *such that:*

- *Every bracket or brace occupies a position and each position is occupied by at most one bracket or brace.*
- *Every number i in the sequence satisfies $0 \leq i \leq k - s$. The positive integers in the sequence are non-decreasing from left to right and are to the left of every zero in the sequence.*
- *Every bracket is to the left of every brace.*
- *If $2k = n$, a bracket in the k-th position may either be a bracket* ] *or a bracket decorated with a prime* ]′.

For example, 1]1]122]33]0000}00}00}000 is a sequence of brackets and braces of type $(7, 18)$ with $s = 4$. When we write our sequences, we often omit the gaps that are not occupied by a bracket or brace. To be concrete, the first rule forbids 0]]0, 0}}0 (two brackets or two braces in the same position), 00]}00 (a bracket and a brace in the same position), ]100 (a bracket that is not in a position). The second rule forbids the sequences of numbers that look like 1132 (3 is not allowed to be to the left of 2) or 11200300 (3 should be to the left of any zero). The third rule forbids 000}00]0 (a brace cannot be to the left of a bracket).

*Notation* 3.3. We order the brackets in the sequence from left to right and the braces in the sequence from right to left. In our example, $1]^1 1]^2 122]^3 33]^4 0000\}^3 00\}^2 00\}^1 000$ the small numbers above the brackets and braces indicate their order. Let $\rho(i, j)$ denote the number of integers to the right of the $i$-th brace and to the left of the $j$-th brace. Let $\rho(i, 0)$ denote the number of integers to the right of the $i$-th brace. In our example, $\rho(3, 2) = 2, \rho(2, 1) = 2, \rho(1, 0) = 3$. Let $p(]^i)$ and $p(\}^i)$ denote the number of integers to the left of the $i$-th bracket and $i$-th brace, respectively. These record the positions of the brackets and braces. In our running example, $p(]^1) = 1, p(]^2) = 2, p(]^3) = 5, p(]^4) = 7$ and $p(\}^3) = 11, p(\}^2) = 13, p(\}^1) = 15$. Let $l(i)$ denote the number of integers in the sequence that are equal to $i$. Let $l(\leq i)$ denote the number of *positive* integers in the sequence that are less than or equal to $i$. In our running example, $l(1) = 3, l(2) = 2, l(3) = 2, l(\leq 2) = 5, l(\leq 3) = 7$. When we are discussing more than one sequence, we will write $\rho_D, p_D$ and $l_D$ for the invariants of the sequence $D$.

We are now ready to define quadric diagrams, which are the main combinatorial objects of this paper. The first three conditions in the definition do not play a role in the algorithm. They are included for precision and the reader may ignore them in a first reading. The last three conditions are crucial and the reader should remember them.

**Definition 3.4.** *A* quadric diagram *for $OG(k, n)$ is a sequence of brackets and braces of type $(k, n)$ with s brackets such that the following conditions hold.*

(D1) $l(i) \leq \rho(i, i - 1)$ *for* $1 \leq i \leq k - s$.
(D2) $2p(]^s) \leq p(\}^{k-s}) + l(\leq k - s)$.

(D3) *Suppose that the integer $0 < i < k-s$ occurs in the sequence. If $i+1$ does not occur in the sequence, either $i=1$ and every position after a 1 is occupied by a bracket, or $l(j) = \rho(j, j-1)$ for every $j > i+1$ and $\rho(i+1, i) = 1$.*

(D4) *There are at least three zeros to the left of $\}^{k-s}$.*

(D5) *Let $x_i$ be the number of brackets such that $p(]^j) \leq l(\leq i)$. Then*

$$x_i \geq k - i + 1 - \frac{p(\}^i) - l(\leq i)}{2}.$$

(D6) *The two integers immediately to the left of a bracket are equal. If there is only one integer to the left of a bracket and $s < k$, then the integer is 1.*

*Remark* 3.5. Quadric diagrams index restriction varieties, which will be introduced in the next section and are the main geometric objects of study in this paper.

*Example* 3.6. Let us give a few examples to clarify the meaning of these conditions. The first condition says that the number of times $i$ appears in the sequence is less than or equal to the number of integers between the $i$-th and $(i-1)$-st braces. In particular, the following are forbidden 2220000}00}0 ($l(2) = 3$, but $\rho(2,1) = 2$), 11000}0 ($l(1) = 2$, but $\rho(1,0) = 1$). Let the right most bracket be at position $p(]^s)$ and the left most brace be at position $p(\}^{k-s})$. The second condition says that twice $p(]^s)$ is less than or equal to the sum of $p(\}^{k-s})$ and the number of positive integers in the sequence. For example, 00]00}0, 100]00}0 are allowed, but 000]00}0 is not ($2p(]^1) = 6 > p(\}^1) = 5$). The third condition is a consequence of the order in the algorithm. The reader does not have to pay attention to it except in a few places in the proof of the algorithm, where it simplifies the dimension counts. The rule says that if a positive integer occurs in the sequence, then all the larger integers (less than or equal to $k-s$) also occur in the sequence except in two very special cases. For example, 1]1]330000}00}0}00 (all the 1s are followed by brackets) and 1]1330000}00}0}00 (2 is missing, but $l(3) = \rho(3,2) = 2$ and $l(2) = \rho(2,1) - 1 = 0$) are allowed, but 1]130000}00}0}00 is not (2 is missing, but $l(3) = 1 \neq \rho(3,2) = 2$). These conditions are preserved during the algorithm. The reader may ignore them in a first reading.

The last three conditions are the important conditions that the reader has to remember. The fourth condition is self-evident. It allows 11]00]00}00 or 33000}00}00}0, but does not allow 1100}00. The sixth condition is also self-evident. It allows for 1]22]33]0000}00}0}0 or 22]22]2000}00000}0, but disallows 2]22]000}000}0 (there is only one integer to the left of $]^1$, but it is not 1) or 1234]0000}0}0}0}0 (the two numbers preceding the bracket are not equal). The fifth condition is the one that is hardest to visually verify without resorting to some counting. In words, it says that the number of integers that are to the right of the right-most $i$ and to the left of the $i$-th brace has to be at least twice the total number of brackets and braces that are at positions greater than $l(\leq i)$ and less than or equal to $p(\}^i)$. For example, it disallows 10]00}0 (There are three zeros to the right of the 1 that are to the left of }. There is one bracket and one brace in positions greater than 1 and less than or equal to 4. However, $3 \not\geq 4$).

We are now ready to state the algorithm. We begin by defining a new set of sequences of brackets and braces associated to $D$. The new sequences $D^a$ and $D^b$ defined below may fail to be quadric diagrams, but we address such instances below.

**Definition 3.7.** *If there exists an index $i$ in $D$ such that $l(i) < \rho(i, i-1)$, let $\kappa = \max(i \mid l(i) < \rho(i, i-1))$. Let $D^a$ be the sequence of brackets and braces obtained by changing the $(l(\leq \kappa)+1)$-st integer in the sequence $D$ to $\kappa$.*

*If $p_{D^a}(]^s) > l_{D^a}(\leq \kappa)$, let $\eta = \min(i \mid p_{D^a}(]^i) > l_{D^a}(\leq \kappa))$. Let $D^b$ be the sequence of brackets and braces obtained from $D^a$ by moving the bracket $]^\eta$ to the position $l_{D^a}(\leq \kappa)$.*

To clarify, let us give some examples. Let $D = 233]0000\}00\}0\}0$. Then $\kappa = 1$. We change the integer in the position $l(\leq 1)+1$ (in this case the left most 2) to 1 to obtain $D^a = 133]0000\}00\}0\}0$. We slide the first bracket in $D^a$ to the right of the 1 we added to the immediate right of it to obtain $D^b = 1]330000\}00\}0\}0$. Note that in this case both $D^a$ and $D^b$ are quadric diagrams.

Next let $D = 00]0]0000\}0$. Here $\kappa = 1$, so we turn the left most 0 into 1 to obtain $D^a = 10]0]0000\}0$. We slide the first bracket to the right of the 1 to its immediate right to obtain $D^b = 1]00]0000\}0$. Here note that $D^b$ is a quadric diagram, but $D^a$ fails condition (D6). We have to turn $D^a$ into a quadric diagram. Here is the algorithm that turns $D^a$ into a quadric diagram.

*Algorithm* 3.8. • If $D^a$ fails condition (D5), discard it. $D^a$ does not lead to any quadric diagrams.

• If $D^a$ satisfies condition (D5) but not condition (D6), change the $(l(\leq \kappa)+1)$-st integer in the sequence to $\kappa$ and move $\}^\kappa$ one position to the left. Repeat until you reach a sequence of brackets and braces that satisfies condition (D6). Label the resulting sequence $D^c$. If $D^c$ is a quadric diagram, we refer to it as a quadric diagram derived from $D^a$. Otherwise, proceed to the next step.

• If $D^a$ or $D^c$ satisfy conditions (D5) and (D6), but fail condition (D4), replace $D^a$ or $D^c$ with two identical diagrams $D^{a_1}$ and $D^{a_2}$ obtained by replacing $\}^{k-s}$ (in $D^a$ or $D^c$) with $]^{s+1}$ in position $p(\}^{k-s})-1$ and turning the digits equal to $k-s$ to 0. If $2p(]^{s+1}) = n$, then we use $(]')^{s+1}$ instead of $]^{s+1}$ in $D^{a_2}$. We refer to $D^{a_1}$ and $D^{a_2}$ as quadric diagrams derived from $D^a$. Furthermore, if $2k = n$ and $2p(]^{s+1}) = n$, then discard the diagram with $]^{s+1}$ (respectively, $(]')^{s+1}$) if $s+1 \neq k \mod (2)$ (respectively, if $s+1 = k \mod (2)$).

In our example, we first turn $D^a = 10]0]0000\}0$ to $11]0]000\}00$. This diagram still fails condition (6), so we repeat to obtain $11]1]00\}000$. Now condition (D6) is satisfied, but condition (D4) fails. Since $n = 8 = 2\cdot 4$, we obtain the two diagrams $00]0]0]0000$ and $00]0]0]'0000$. These are the two diagrams derived from $D^a$.

Let $D = 000\}000\}000\}$, then $\kappa = 3$. We turn the left most 0 into 3 to obtain $D^a = 300\}000\}000\}$. In this case, there are no brackets to the left of the 3, so there is no $D^b$. The sequence $D^a$ fails condition (D4). Since $n$ is odd, we replace $D^a$ with two identical quadric diagrams $D^{a_1} = 00]0000\}000\}$ and $D^{a_2} = 00]0000\}000\}$.

Let $D = 00]0000\}00\}0$. Then $D^a = 20]0000\}00\}0$ and $D^b = 2]00000\}00\}0$. Neither of these diagrams satisfy condition (D6). We already know that we should replace $D^a$ with $22]000\}000\}0$. Here is how to modify $D^b$.

*Algorithm* 3.9. • If $D^b$ does not satisfy condition (D6), let $]^j$ be the bracket for which it fails. Let $i$ be the integer immediately to the left of $]^j$. Replace $i$ with $i-1$ and move $\}^{i-1}$ one position to the left. As long as the resulting sequence does not satisfy condition (D6), repeat this process either until the resulting sequence is a quadric diagram (in which case this is the quadric diagram derived from $D^b$) or two braces occupy the same position. In the latter case, no quadric diagrams are derived from $D^b$.

In our example, we replace $D^b = 2]00000\}00\}0$ with $1]00000\}0\}00$, which is a quadric diagram. If our example had been $D = 00]0000\}0\}0$, then $D^b = 2]00000\}0\}0$. Replacing 2 with 1 and moving $\}^1$ to the left would produce $1]00000\}\}00$. Hence, in this case no quadric diagrams are derived from $D^b$.

We need one final definition. Given a sequence of brackets and braces such that $p(]^s) > l(\kappa)$ , let $y_{x_\kappa+1} = \max(i \mid l(\leq i) \leq p(]^{x_\kappa+1}))$ or set $y_{x_\kappa+1} = k-s+1$ if $l(\leq i) < p(]^{x_\kappa+1})$ for all $i$. $y_{x_\kappa+1}$ is the largest integer that occurs to the right of $]^{x_\kappa+1}$, which is the first bracket occurring in a position greater than $l(\leq \kappa)$. The condition $p(]^{x_\kappa+1}) - l(\leq\kappa) - 1 = y_{x_\kappa+1} - \kappa$ will play an important role. In words, this condition says that the number of integers larger than $\kappa$ to the left of $]^{x_\kappa+1}$ is one more than the cardinality of the set of integers greater than $\kappa$ occurring to the left of $]^{x_\kappa+1}$. In view of condition (D3), a sequence satisfying this equality looks like $\cdots\kappa+1\ \kappa+2\ \cdots\kappa+l-1\ \kappa+l\ \kappa+l]\cdots$ or $\cdots\kappa+1\ \kappa+2\ \cdots\kappa+l-1\ 00]\cdots$, where we have drawn the part of the sequence starting with the left most $\kappa+1$ and ending with $]^{x_\kappa+1}$. We are now ready to state the algorithm.

*Algorithm* 3.10. Let $D$ be a quadric diagram. If $l(i) = \rho(i, i-1)$ for every $1 \leq i \leq k-s$, then return $D$ and stop. Otherwise, let $D^a$ and $D^b$ be as above.

(1) If $p(]^{x_\kappa+1}) - l(\leq\kappa) - 1 > y_{x_\kappa+1} - \kappa$ or $p(]^s) \leq l(\kappa)$ in $D$, then return the quadric diagrams that are derived from $D^a$.
(2) If $D^a$ violates condition (D5), then return the quadric diagrams that are derived from $D^b$.
(3) Otherwise, return the quadric diagrams that are derived from both $D^a$ and $D^b$.

*Remark* 3.11. In the proof of Theorem 5.12, we will check in detail that Algorithm 3.10 always returns at least one quadric diagram. Briefly, $D^a$ does not lead to a quadric diagram only if it violates condition

(D5). In that case, by the definition of $\kappa$, there has to be equality in condition (D5) for all indices $\kappa \leq i \leq k-s$ in the diagram $D$. Then, condition (D4) implies that there has to be a bracket to the right of $\kappa$ in $D^a$; and condition (D6) implies that $p(]^{x_\kappa+1}) - l(\leq \kappa) - 1 = y_{x_\kappa+1} - \kappa$ in $D$. Finally, while running Algorithm 3.9, if two braces occupy the same position, then condition (D5) is violated for the index $\kappa - 1$ in the diagram $D$. These considerations imply that there is a quadric diagram derived from $D^b$ by Algorithm 3.10 (see paragraph 6 of the proof of Theorem 5.12 for more details).

The reader can turn to the beginning of §5 for examples. In the next two sections we will explain the geometric meaning behind this algorithm.

## 4. Restriction varieties in the orthogonal Grassmannians

In this section, we introduce restriction varieties in orthogonal Grassmannians and discuss their basic properties. Restriction varieties are subvarieties of $OG(k,n)$ that parameterize isotropic $k$-planes that intersect elements of a given flag in specified dimensions. We do not require the flag to be isotropic; however, we need to impose some basic numerical restrictions in order to obtain geometrically meaningful subvarieties.

*Notation* 4.1. Let $W$ be a vector space of dimension $n$. Let $Q$ be a non-degenerate, symmetric bilinear form on $W$. We denote an isotropic linear space of (vector space) dimension $n_j$ by $L_{n_j}$. In case $2n_j = n$, $L_{n_j}$ and $L'_{n_j}$ denote isotropic linear spaces in different connected components. Let $Q_{d_i}^{r_i}$ denote a sub-quadric of corank $r_i$ cut out by a $d_i$-dimensional linear section of $Q$. We denote the singular locus of $Q_{d_i}^{r_i}$ by $Q_{d_i}^{r_i,sing}$. For convenience, we let $r_0 = 0$ and $d_0 = n$.

**Definition 4.2.** *A sequence of linear spaces and quadrics $(L_\bullet, Q_\bullet)$ associated to $OG(k,n)$ is a totally ordered set*

$$L_{n_1} \subsetneq L_{n_2} \subsetneq \cdots \subsetneq L_{n_s} \subsetneq Q_{d_{k-s}}^{r_{k-s}} \subsetneq \cdots \subsetneq Q_{d_1}^{r_1}$$

*of isotropic linear spaces $L_{n_j}$ (or possibly $L'_{n_s}$ in case $2n_s = n$) and sub-quadrics $Q_{d_i}^{r_i}$ of $Q$ such that*

1. $2n_s \leq d_{k-s} + r_{k-s}$.
2. $2(k-i+1) \leq r_i + d_i$ *for every* $1 \leq i \leq k-s$.
3. $r_{i+1} + d_{i+1} \leq r_i + d_i \leq n$ *for every* $1 \leq i < k-s$.
4. $Q_{d_{i-1}}^{r_{i-1},sing} \subseteq Q_{d_i}^{r_i,sing}$ *for every* $1 < i \leq k-s$.
5. $\dim(L_{n_j} \cap Q_{d_i}^{r_i,sing}) = \max(n_j, r_i)$.
6. *Let $x_1$ denote the number of isotropic subspaces in the sequence contained in the singular locus of $Q_{d_1}^{r_1}$. For every $1 \leq i \leq k-s$, either $r_i = r_1 = x_1$, or $r_l - r_i \geq l - i - 1$ for every $l > i$. Furthermore, if $r_l = r_{l-1} > x_1$ for some $l$, then $d_i - d_{i+1} = r_{i+1} - r_i$ for all $i \geq l$ and $d_{l-1} - d_l = 1$.*

*Remark* 4.3. Conditions (1), (2) and (3) express basic facts about quadrics. Conditions (1) and (2) express the "Linear space bound" that the dimension of an isotropic linear space with respect to a quadratic form of corank $r$ in $d$ variables is at most half of $d+r$. Since $L_{n_s} \subset Q_{d_{k-s}}^{r_{k-s}}$, Condition (1) needs to be satisfied. Below, in defining restriction varieties, we will require the isotropic $k$-planes to intersect $Q_{d_i}^{r_i}$ in a subspace of dimension $k-i+1$. Hence, Condition (2) needs to be satisfied. Condition (3) expresses the "Corank bound" that a hyperplane section of a quadric of corank $r$ can have corank at most $r+1$. Conditions (4) and (5) express that the singular loci of the quadrics $Q_{d_i}^{r_i}$ are in the most special position. The singular locus of the quadric $Q_{d_i}^{r_i}$ contains the singular locus of all the larger dimensional quadrics in the sequence. Furthermore, isotropic linear spaces in the sequence of dimension greater (resp., less) than $r_i$ contain (resp., are contained in) the singular locus of $Q_{d_i}^{r_i}$. Finally, Condition (6) is a technical condition: If a quadric $Q_{d_i}^{r_i}$ is more singular than the linear spaces in the sequence force it to be, then each quadric contained in $Q_{d_i}^{r_i}$ is more singular than the one larger quadric containing it except in a very special case detailed in Condition (6). These conditions will automatically hold for all the varieties in our algorithm, hence the reader does not need to remember these conditions to implement the algorithm.

We will use sequences of brackets and braces introduced in the previous section for representing the geometric sequences.

**Definition 4.4.** *Let $(L_\bullet, Q_\bullet)$ be a sequence for $OG(k,n)$. The sequence of brackets and braces associated to $(L_\bullet, Q_\bullet)$ is a sequence of non-negative integers of length $n$, $s$ right brackets and $k-s$ right braces such that*

(1) *The sequence consists of $r_i - r_{i-1}$ integers equal to $i$ for $1 \leq i \leq k-s$ placed in increasing order followed by a sequence of $n - r_{k-s}$ zeros.*

(2) *The right square brackets are placed after the $n_j$-th integer in the sequence for $1 \leq j \leq s$ and the right braces are placed after the $d_i$-th integer in the sequence for $1 \leq i \leq k-s$.*

*In case $2n_s = n$, we distinguish between $L_{n_s}$ and $L'_{n_s}$ by writing $]$ and $]'$, respectively, for the right bracket after the $n_s$-th digit.*

*Example* 4.5. The sequence of brackets and braces 1]22]000}00}0 represents the sequence $L_1 \subset L_3 \subset Q_6^3 \subset Q_8^1$. To determine the (vector space) dimension $d_i$ of the span of the quadric $Q_{d_i}^{r_i}$, we count the number of digits to the left of the $i$-th brace. For example, there are 8 digits to the left of the right most brace, so $d_1 = 8$. There are six digits to the right of the second brace, so $d_2 = 6$. To determine $r_i$, we count the number of positive digits less than or equal to $i$. In this example, there are 3 positive digits less than or equal to 2, so $r_2 = 3$. There is a unique one, so $r_1 = 1$. Finally, to determine $n_j$, we count the number of digits to the left of the $j$-th square bracket. In this example, $n_1 = 1$, $n_2 = 3$. The reader will notice that the Zariski closure of the subvariety of $OG(4,9)$ parameterizing isotropic subspaces $\Lambda$ that satisfy

$$\dim(\Lambda \cap L_1) = 1, \dim(\Lambda \cap L_3) = 2, \dim(\Lambda \cap Q_6^3) = 3, \dim(\Lambda \cap Q_8^1) = 4$$

is the Schubert variety $\Omega_{4,2}^{3,1}$. Note that the sequence $\mu$ in our notation for Schubert varieties denotes the codimensions (equivalently, coranks) of the quadrics defining the variety, so it is very easy to read from the diagram.

The sequence of brackets and braces associated to $(L_\bullet, Q_\bullet)$ is a sequence of brackets and braces in the sense of the previous section. Since $n_1 < \cdots < n_s < d_{k-s} < \cdots < d_1$, the brackets and braces occupy different positions. Since the quadrics contain the linear spaces, the brackets are to the left of all the braces. The positive integers are increasing and less than or equal to the number of braces and they are all to the left of the zeros by construction. The position of a bracket $p(]^j)$ is equal the dimension $n_j$ of the linear space $L_{n_j}$. The position of a brace $p(\}^i)$ is equal to the dimension of the span $d_i$ of the quadric $Q_{d_i}^{r_i}$. The dimension $r_i$ of the singular locus of $Q_{d_i}^{r_i}$ is the number of positive integers $l(\leq i)$ less than or equal to $i$. Finally, $l(i)$ is $r_i - r_{i-1}$ and $\rho(i, i-1) = d_{i-1} - d_i$. Hence, these sequences satisfy conditions (D1) (which is equivalent to Condition (3)), (D2) (which is equivalent to condition (2)) and (D3) (which is equivalent to Condition (6)).

**Definition 4.6.** *Given a sequence $(L_\bullet, Q_\bullet)$, let $x_i$ denote the number of isotropic linear spaces $L_{n_j}$ of the sequence contained in $Q_{d_i}^{r_i, sing}$. Similarly, let $y_j$ be the integer such that $r_{y_j - 1} < n_j \leq r_{y_j}$. If $r_i < n_j$ for every $1 \leq i \leq k-s$, set $y_j = k-s+1$.*

*Remark* 4.7. We will require the $(k-i+1)$-dimensional subspace contained in $Q_{d_i}^{r_i}$ to intersect $Q_{d_i}^{r_i, sing}$ in a subspace of dimension $x_i$. The index $y_j$ is the smallest index $i$ such that $L_{n_j}$ is contained in the singular locus of $Q_{d_i}^{r_i}$. By conditions (4) and (5), every quadric of index at least $y_j$ will be everywhere singular along $L_{n_j}$.

We need some further assumptions on the sequence $(L_\bullet, Q_\bullet)$ before it reflects the properties of the corresponding variety.

*Example* 4.8. Consider the sequence $L_3 \subset Q^1_5 \subset Q^1_6$ depicted by

$$100]00\}0\}0.$$

By 'the linear space bound', any isotropic 3-plane in $OG(3,7)$ which is contained in $Q^1_6$ necessarily must contain the singular point of $Q^1_6$. (Geometrically, any plane in a five dimensional quadric cone contains the vertex.) Hence the sequence $L_1 \subset Q^1_5 \subset Q^1_6$

$$1]0000\}0\}0$$

better reflects the geometric properties of isotropic 3-planes contained in $Q^1_6$. Similarly, consider the sequence $Q^2_4 \subset Q^0_6$ depicted by

$$2200\}00\}0.$$

The quadric $Q^2_4$ is reducible. (Geometrically, a quadric surface which is singular along a line is the union of two planes.) Hence, two sequences of the form $L_3 \subset Q^0_6$

$$000]000\}0$$

better reflect the geometry of the corresponding variety.

These examples motivate the following definition.

**Definition 4.9.** *A sequence $(L_\bullet, Q_\bullet)$ associated to $OG(k,n)$ is* admissible *if the linear spaces and quadrics satisfy the following additional conditions:*

(7) $r_{k-s} \leq d_{k-s} - 3$.

(8) *For every $1 \leq i \leq k-s$,*

$$x_i \geq k - i + 1 - \frac{d_i - r_i}{2}.$$

(9) *For any $1 \leq j \leq s$, there does not exist $1 \leq i \leq k-s$ such that $n_j - r_i = 1$.*

*Remark* 4.10. If Condition (7) is violated, then $Q^{r_{k-s}}_{d_{k-s}}$ would either be reducible or non-reduced. Condition (8) expresses the fact that in a quadric $Q^{r_i}_{d_i}$, a linear space of dimension $k-i+1$ has to intersect the singular locus in dimension at least $k-i+1-\frac{d_i-r_i}{2}$ (see Remark 4.7). Condition (9) expresses the fact that if $n_j - r_i = 1$ for some pair, then the tangent spaces to $Q^{r_i}_{d_i}$ would be constant along $L_{n_j}$. Hence the $(k-i+1)$-dimensional subspace contained in $Q^{r_i}_{d_i}$ would actually be contained in $Q^{r_i+1}_{d_i-1}$ with singular locus $L_{n_j}$. The reader should remember these three conditions in order to implement the algorithm.

**Lemma 4.11.** *The sequence of brackets and braces associated to an admissible sequence is a quadric diagram. Conversely, every quadric diagram corresponds to an admissible sequence $(L_\bullet, Q_\bullet)$.*

*Proof.* We already saw that the sequence associated to $(L_\bullet, Q_\bullet)$ is a sequence of brackets and braces that satisfies the conditions (D1), (D2) and (D3). Conditions (7), (8) and (9) translate to the conditions (D4), (D5) and (D6). If $r_{k-s} \leq d_{k-s} - 3$, then there are at least three zeros to the left of $\}^{k-s}$ since the total number of positive integers in the sequence $(r_{k-s})$ is three less than the position of $\}^{k-s}$. Using the facts that $d_i = p(\}^i)$ and $r_i = l(\leq i)$, Conditions (8) and (D5) are direct translations of each other. Finally, if the two digits preceding a bracket $]^j$ are $a < b$, then $n_j - r_a = 1$ contradicting Condition (9). If a bracket is at the first position, then $n_1 = 1$. If $r_1 = 0$, then $n_1 - r_1 = 1$ contradicting Condition (9). Hence, the digit preceding $]^1$ must be 1. We conclude that conditions (D6) and (9) are equivalent. Finally, observe that Condition (8) implies Condition (2). We have included Condition (2) to simplify certain statements in the proof of the algorithm. We conclude that the data defining quadric diagrams and admissible sequences are equivalent. $\square$

**Definition 4.12.** *Let $(L_\bullet, Q_\bullet)$ be an admissible sequence for $OG(k,n)$. A* restriction variety $V(L_\bullet, Q_\bullet)$ *is the subvariety of $OG(k,n)$ defined as the Zariski closure of the following quasi-projective variety*

$$V(L_\bullet, Q_\bullet)^0 := \{\ [W] \in OG(k,n) \mid \dim(W \cap L_{n_j}) = j, \dim(W \cap Q_{d_i}^{r_i}) = k-i+1, \dim(W \cap Q_{d_i}^{r_i,sing}) = x_i\ \}.$$

*Example* 4.13. Schubert varieties in $OG(k,n)$ are restriction varieties defined with respect to sequences satisfying $d_i + r_i = n$ for all $1 \leq i \leq k-s$ (see Lemma 4.18). The intersection of a general Schubert variety in $G(k,n)$ with $OG(k,n)$ (when non-empty) is a restriction variety associated to a sequence where $s = 0$ and $r_i = 0$ for $1 \leq i \leq k$ (see Proposition 6.2 for the precise statement). Hence, restriction varieties are a class of varieties that interpolate between the restriction of Schubert varieties in $G(k,n)$ and Schubert varieties in $OG(k,n)$.

*Remark* 4.14. A restriction variety does not have to be irreducible. For example,

000}0}0

in $OG(2,5)$ consists of two irreducible components. (Geometrically, the corresponding restriction variety parametrizes lines on a smooth quadric surface in $\mathbb{P}^3$.) When the inequality in Condition (8) is an equality for an index $i$, then the $(d_i+r_i)/2$-dimensional linear spaces in $Q_{d_i}^{r_i}$ form two irreducible components. The $(k-i+1)$-dimensional subspaces contained in $Q_{d_i}^{r_i}$ may be distinguished by their parity of the dimension of their intersection with linear spaces in each of these components.

**Definition 4.15.** *Let $(L_\bullet, Q_\bullet)$ be an admissible sequence. An index $1 \leq i \leq k-s$ such that*

$$x_i = k - i + 1 - \frac{d_i - r_i}{2}$$

*is called a special index. For each special index, a marking $m_\bullet$ of $(L_\bullet, Q_\bullet)$ designates one of the irreducible components of $\frac{d_i+r_i}{2}$-dimensional linear spaces of $Q_{d_i}^{r_i}$ as even and the other one as odd, such that*

- *If $d_{i_1} + r_{i_1} = d_{i_2} + r_{i_2}$, for two special indices $i_1 < i_2$, and the component containing a linear space $V$ is designated even for $i_2$, then the component containing $V$ is designated even for $i_1$ as well; and*
- *If $2n_s = d_i + r_i$ for a special index $i$, then the component to which $L_{n_s}$ belongs is assigned the parity of $s$; and*
- *If $n = 2k$, $m_\bullet$ assigns the component containing $L_k$ the parity that characterizes the component $OG(k,2k)$.*

*A marked restriction variety $V(L_\bullet, Q_\bullet, m_\bullet)$ is the Zariski closure of the subvariety of $V(L_\bullet, Q_\bullet)^0$ parameterizing $k$-dimensional isotropic subspaces $W$, where, for each special index $i$, $W$ intersects subspaces of dimension $\frac{d_i+r_i}{2}$ of $Q_{d_i}^{r_i}$ designated even (respectively, odd) by $m_\bullet$ in a subspace of even (respectively, odd) dimension.*

**Proposition 4.16.** *The marked restriction variety $V(L_\bullet, Q_\bullet, m_\bullet)$ associated to a marked admissible sequence is an irreducible variety of dimension*

$$\dim(V(L_\bullet, Q_\bullet, m_\bullet)) = \sum_{j=1}^{s}(n_j - j) + \sum_{i=1}^{k-s}(d_i + x_i - 2s - 2i) \tag{1}$$

*Proof.* We prove this proposition by induction on $k$. Suppose $k = 1$. If $s = 1$, then clearly the variety is isomorphic to projective space of dimension $n_1 - 1$ and the proposition holds. If $s = 0$, then the variety is isomorphic to a quadric hypersurface in $\mathbb{P}^{d_1-1}$ singular along a linear space of codimension at least three (by Condition (7) in Definition 4.9). Since such a quadric is irreducible of dimension $d_1 - 2$, the base case of the induction follows.

Now suppose that the proposition holds up to $k-1$. If $k - s = 0$, then the proposition is immediate. In that case, the isotropic subspaces are contained in the Grassmannian $G(k, n_k)$ and the restriction variety is an ordinary Schubert variety $(\Sigma_{n_k-n_1-k+1,\dots,n_k-n_{k-1}-1})$ in $G(k,n_k)$. The irreducibility and the dimension follow from these considerations. We may assume that $k-s>0$. Let $(L_\bullet, Q'_\bullet)$ be the sequence for $OG(k-1,n)$ obtained from $(L_\bullet, Q_\bullet)$ by omitting $Q^{r_1}_{d_1}$ from the sequence (and subtracting one from the indices of the quadrics). Observe that $(L_\bullet, Q'_\bullet)$ is also an admissible sequence: Conditions (1)-(9) remain valid when we omit the largest quadric. Let $m'_\bullet$ be the restriction of the marking $m_\bullet$ to this new sequence, where $m'$ designates the same components of linear spaces as even if $r_i+d_i<r_1+d_1$ and swaps the designation for linear spaces with $r_i+d_i=r_1+d_1$. Let $V(L_\bullet,Q_\bullet,m_\bullet)^0$ denote the intersection of $V(L_\bullet,Q_\bullet,m_\bullet)$ with $V(L_\bullet,Q_\bullet)^0$, the Zariski open set used to define $V(L_\bullet,Q_\bullet)$. We then have a morphism $f: V(L_\bullet,Q_\bullet,m_\bullet)^0 \to V(L_\bullet,Q'_\bullet,m'_\bullet)^0$ by taking the intersection of the linear spaces of dimension $k$ in $V(L_\bullet,Q_\bullet,m_\bullet)^0$ with $Q^{r_2}_{d_2}$. By induction, we can assume that the the image is an irreducible variety of dimension predicted by the proposition. We now study the fibers of this morphism. Fix a point $[W]$ in the image. By assumption, the dimension of intersection of $W$ with the singular locus of $Q^{r_1}_{d_1}$ is $x_1$. Then any $k$-dimensional linear space containing $W$ has to be contained in the quadric $Q'$ cut out on $Q_1$ by the linear space everywhere tangent to $W$. This is a quadric of corank $r_1+k-1-x_1$ in a linear space of dimension $d_1-(k-1-x_1)$. We have to choose a $k$-plane containing $W$. We can choose a linear section $Q''$ of $Q'$ complementary to $W$. Choosing a $k$-plane is equivalent is to choosing a point on $Q''$. Hence, the dimension of the fiber is $d_1-k+1+x_1-2-k+1$. Furthermore, by Condition (8)

$$x_1 \geq k - \frac{d_1-r_1}{2}.$$

If the inequality is strict, it follows that

$$r_1+k-1-x_1 < (d_1-k+1+x_1)-2,$$

hence $Q''$ and consequently the fiber is irreducible. If equality holds, then $Q''$ is a union of two linear spaces. The marking $m_\bullet$ selects one of these components by specifying the parity of the dimension of intersection with the $k$-dimensional linear space. Hence, the fiber is irreducible. This concludes the proof. ∎

*Remark* 4.17. Since Equation 1 does not depend on the marking $m_\bullet$, every irreducible component of the restriction variety $V(L_\bullet,Q_\bullet)$ has dimension

$$\sum_{j=1}^{s}(n_j-j)+\sum_{i=1}^{k-s}(d_i+x_i-2s-2i).$$

Observe that $V(L_\bullet,Q_\bullet)$ has an irreducible component for every marking $m_\bullet$. The markings $m_\bullet$ parameterize the irreducible components of $V(L_\bullet,Q_\bullet)$. Correspondingly, given a sequence $D$ of brackets and braces, we define $\dim(D)$ by the expression

$$\sum_{j=1}^{s}(p(]^j)-j)+\sum_{i=1}^{k-s}(p(\}^i)+x_i-2s-2i).$$

**Lemma 4.18.** *Schubert varieties in $OG(k,n)$ are the restriction varieties where the admissible sequence defining the restriction variety satisfies $r_i+d_i=n$ for every $1\leq i\leq k-s$. When $n=2k$, we also require that the $k$-dimensional linear spaces to intersect the $k$-dimensional linear space $L_k$ in the sequence in a subspace of the correct parity.*

*Proof.* Set $\alpha=\lfloor (n-1)/2\rfloor$. Let the sequence $\lambda$ be defined by setting $\lambda_j=\alpha+1-n_j$. Let the sequence $\mu$ be given by setting $\mu_{k-i+1}=r_i$. We claim that the restriction variety $V(L_\bullet,Q_\bullet)$ is the Schubert variety $\Omega^{\mu}_{\lambda}$. Since the sequence satisfies Conditions (4) and (5), it suffices to show that there does not exist $n_j$ and $r_i$ such that $n_j-r_i=1$ for any $i$ and $j$. This is guaranteed by Condition (9) defining admissible sequences. When $2k=n$, we require that the length of $\lambda$ have the same parity as $k$ (alternatively, we could interpret a restriction variety with the wrong parity as a Schubert variety for the other connected component of $OG(k,2k)$). Note that under the assumptions of the lemma, the restriction variety is automatically irreducible. Suppose equality holds in Condition (8) for some $i_0$. Then $n$ is even. Condition (9) and the assumption on the sequence imply that equality holds for every $i \geq i_0$. In particular, equality must hold for the index $k-s$. Combining Conditions (9) and equality in Condition (8), we have

$$n_s \geq r_{k-s} + 1 + s - \left( s + 1 - \frac{d_{k-s} - r_{k-s}}{2} \right) = \frac{d_{k-s} + r_{k-s}}{2} = \frac{n}{2}.$$

Using Condition (1), we deduce that $n_s = n/2$. Hence, the marking is uniquely determined by the sequence. □

## 5. The Algorithm for computing the classes of restriction varieties in Grassmannians

5.1. **The strategy and examples.** The strategy to calculate the class of a restriction variety $V(L_\bullet, Q_\bullet)$ is to specialize it into a union of Schubert varieties by successively making the quadrics in the sequence more singular. By the corank bound (Condition (3)), if $r_i + d_i = r_{i-1} + d_{i-1}$, then $Q_{d_i}^{r_i}$ is as singular as it can be given that it is contained in $Q_{d_{i-1}}^{r_{i-1}}$, so its corank cannot be increased. If $r_i + d_i = n$ for all $1 \leq i \leq k-s$, then $V(L_\bullet, Q_\bullet)$ is a Schubert variety and there is nothing further to do. Otherwise, there is a smallest dimensional quadric whose corank can be increased. We increase the corank of this quadric (fixing all the other linear spaces and quadrics) by one by specializing the quadric in a pencil. As a result of this specialization, the restriction variety breaks into a union of restriction varieties each with multiplicity one. In the rest of this section, we will describe the components and show that they occur with multiplicity one. We first discuss several fundamental examples that illustrate the possibilities.

*Example* 5.1. We first compute the class of $V(Q_4^0)$ depicted by

$$0000\}0$$

in $OG(1,5)$. Projectively, $V(Q_4^0)$ parametrizes points on a smooth quadric hypersurface $Q$ in $\mathbb{P}^4$ that are contained in a smooth hyperplane section $Q_4^0$. We specialize the hyperplane section until it becomes tangent to $Q$. This specialization replaces $Q_4^0$ with $Q_4^1$ (singular at the point of tangency). In the process, the restriction variety is transformed to

$$1000\}0.$$

This is the quadric diagram $D^a$ described in §3. Observe that if the linear spaces had to intersect the singular locus, then they would just be the singular point of $Q_4^1$. The singular point has smaller dimension than the quadric $Q_4^1$. That's why in these cases the quadric diagrams derived from $D^b$ do not occur. The cohomology class of a smooth hyperplane section is the same as that of a singular hyperplane section, hence $V(Q_4^0)$ and $V(Q_4^1)$ have the same cohomology class. Since $V(Q_4^1)$ is a Schubert variety with class $\sigma^1$ in $OG(1,5)$, this concludes the calculation.

During this process, a quadric may become reducible. As a slight variation, we compute the class of $V(Q_3^0)$ depicted by

$$000\}0$$

in $OG(1,4)$. Projectively, $V(Q_3^0)$ parametrizes points contained in a smooth conic in a smooth quadric surface $Q$ in $\mathbb{P}^3$. We specialize the plane of the conic until it becomes tangent to $Q$, replacing $Q_3^0$ with $Q_3^1$. Note that $Q_3^1$ violates Condition (7) (its corank is two less than its ambient dimension). Geometrically, a singular conic is a union of two lines which belong to two different rulings on the quadric surface $Q$. The sequence of brackets and braces $100\}0$ fails condition (D4). We replace it by the two quadric diagrams $00]00$ and $00]'00$ according to §3. Hence, the restriction variety corresponding to the diagram $000\}0$ is replaced by the two restriction varieties corresdponding to

$$00]00 \quad \text{and} \quad 00]'00.$$

Geometrically, the class of a conic is the sum of the classes of two lines on the quadric one in each ruling. This concludes the calculation since the latter two varieties are Schubert varieties with classes $\sigma_0$ and $\sigma^2$, respectively. Hence, the class of $V(Q_3^0)$ in $OG(1,4)$ is $\sigma_0 + \sigma^2$. This example shows that in the algorithm, we have to replace a quadric by two linear spaces if the specialization forces the quadric to become reducible (or, equivalently, violate Condition (7)).

*Example* 5.2. Next we calculate the class of the restriction variety $V(L_2 \subset Q_4^0)$ in $OG(2,5)$ depicted by

$$00]00\}0.$$

Geometrically, this example corresponds to calculating the class of the inclusion of $OG(2,4)$ in $OG(2,5)$. More concretely, we calculate the class of the space of lines contained in the smooth quadric surface $Q_4^0$ and intersect the line $L_2$ (in a point). Note that $Q_4^0$ is a smooth hyperplane section of the ambient quadric $Q$ in $\mathbb{P}^4$. We specialize it until it becomes tangent to $Q$ at a point on $L_2$. This replaces $Q_4^0$ with $Q_4^1$, the quadric cone singular at the point of tangency. This is depicted by $D^a = 10]00\}0$, which violates condition (D5). By the linear space bound (Condition (8)), lines contained in a quadric cone in $\mathbb{P}^3$ all pass through the vertex of the cone. Hence, the degeneration replaces $V(L_2 \subset Q_4^0)$ with the restriction variety $V(L_1 \subset Q_4^1)$

$$1]000\}0.$$

This is the quadric diagram $D^b$ defined in §3. This restriction variety is the Schubert variety with class $\sigma_2^1$. Note that in this case, this is the diagram derived from $D^b$ and $D^a$ does not lead to any diagrams since it violates condition (D5). Geometrically, this corresponds to the fact that the lines are required to pass through the singular point.

*Example* 5.3. Finally, consider the variety $V(L_2 \subset Q_6^0)$ in $OG(2,7)$.

$$\begin{array}{l} 00]0000\}0 \to 1]00000\}0 \\ \downarrow \\ 11]000\}00 \end{array}$$

Geometrically, this variety parametrizes lines on a smooth quadric $Q$ in $\mathbb{P}^6$ that intersect a line $L_2$ and are contained in a smooth hyperplane section $Q_6^0$ of $Q$. As before, we specialize the linear space defining $Q_6^0$ to be tangent to $Q$ along a point of the line $L_2$, replacing $Q_6^0$ with $Q_6^1$. In the limit, there are two possibilities. In the first case, the lines may all pass through the singular point of $Q_6^1$. This case ($V(L_1 \subset Q_6^1)$) is depicted by the quadric diagram $D^b = 1]00000\}0$. In the second, case the lines intersect $L_2$ in a point other than the vertex. This is denoted by the sequence of brackets and braces $D^a = 10]0000\}0$. Note that this sequence fails condition (D6). By "the variation of tangent spaces", the tangent spaces to $Q_6^1$ are constant along the line $L_2$. Therefore, the lines in $Q_6^1$ that intersect $L_2$ in a point other than the singular point have to be contained in the quadric $Q_5^2$ given by the intersection of $Q$ with the linear space everywhere tangent to $Q$ along $L_2$. This possibility ($V(L_2 \subset Q_5^2)$) is depicted by $11]000\}00$, which is the quadric diagram derived from $D^a$ as in §3. Both of these varieties are Schubert varieties and occur with multiplicity one in the limit. Hence, the class of $V(L_2 \subset Q_6^0)$ is $\sigma_3^1 + \sigma_2^2$.

Example 5.3 shows the basic branching. When we increase the corank of the quadric, the linear spaces intersect the new singular locus either in a larger dimensional vector space (unless this possibility leads to a smaller dimensional variety as in Example 5.1) or in the same dimensional vector space (unless this possibility is excluded by the linear space bound (Condition (8)) as in Example 5.2). Additional branching occurs when one of the quadrics become reducible (as in Example 5.1). The general rule is obtained by repeating these three fundamental examples. In fact, these examples capture all the geometric complexity of restriction varieties in orthogonal Grassmannians. Next we give a complicated example that illustrates the inductive structure of the Algorithm.

*Example* 5.4. Consider the restriction variety $V = V(Q_4^0 \subset Q_6^0 \subset Q_8^0)$ in $OG(3,9)$. Concretely, $V$ is the intersection of $OG(3,9)$ with a general Schubert variety $\Sigma_{3,2,1}(F_\bullet)$ in $G(3,9)$. We calculate the class of $V$ in terms of Schubert classes in $OG(3,9)$ as follows.

0000}00}00}0 $\to$ 3000}00}00}0 $\xrightarrow{\times 2}$ 000]000}00}0 $\to$ 200]000}00}0 $\xrightarrow{\times 2}$ 000]0]0000}0 $\to$ 100]0]0000}0

$\downarrow$

22]0000}00}0 $\to$ 1]20000}00}0 $\to$ 1]22000}00}0

$\downarrow$

11]0000}0}00 $\to$ 11]2000}0}00

We explain the salient features of this example. In the first two steps, we increase the corank of the smallest dimensional quadric $Q_4^0$ by one. After the second step, we obtain $Q_4^2$, which is a reducible quadric equal to the union of two linear spaces of dimension three. (in terms of the combinatorics of quadric diagrams 3300}00}00}0 violates condition (D4), so has to be replaced by two copies of 000]000}00}0.) Correspondingly, the restriction varieties breaks into two irreducible components both isomorphic to the restriction variety $V(L_3 \subset Q_6^0 \subset Q_8^0)$. The symbol $\times 2$ above the right arrow indicates that there are two components of the limit with the same class (though they are distinct varieties, each occurring with multiplicity one). In the next two steps, we increase the corank of the quadric $Q_6^0$ by one. After the second step, either the linear spaces intersect the singular locus of $Q_6^2$ and we get the restriction variety indicated by the down arrow or the linear spaces do not intersect the singular locus of $Q_6^2$. In the latter case, the tangent spaces to $Q_6^2$ are constant along $L_3$. Hence, these linear spaces must intersect the quadric $Q_5^3$ everywhere tangent along the three dimensional linear space in a two-dimensional subspace. Note that the latter quadric $Q_5^3$ is reducible, the union of two linear spaces. Hence, in this case there are two components which are indicated after the right arrow. (In terms of the combinatorics of quadric diagrams we have $D^a =$ 220]000}00}0 and $D^b =$ 22]0000}00}0. $D^b$ is a quadric diagram, but $D^a$ fails condition (D6), so we replace it with $D^c =$ 222]00}000}0, which fails condition (D4). We have to replace $D^c$ by two copies of 000]0]0000}0). The rest of the calculation is similar to the previous examples. We conclude that the class of the variety is equal to

$$4\sigma_{2,1}^{1} + 2\sigma_{4}^{3,1} + 2\sigma_{3}^{3,2}.$$

5.2. **The algorithm.** We now give the algorithm for computing the class of the variety $V(L_\bullet, Q_\bullet)$ in terms of Schubert classes in $OG(k,n)$. First, we begin with a slogan that can help guide the reader through the combinatorics.

**The Rule in Slogan Form:** Increase the dimension of the singular locus of the smallest dimensional quadric allowed by the corank bound (Condition (3)) by one. The linear spaces intersect the new singular locus either in a subspace of the same dimension as before or in one larger dimension, unless one of these possibilities leads to a smaller dimensional variety or is precluded by the linear space bound (Condition (8)).

This section and §3 make this slogan precise.

**Definition 5.5.** *Let $(L_\bullet, Q_\bullet)$ be an admissible sequence. We say that the quadric $Q_{d_i}^{r_i}$ is saturated if $r_i = n - d_i$. $V(L_\bullet, Q_\bullet)$ is saturated if every quadric $Q_{d_i}^{r_i}$, $1 \leq i \leq k-s$, in its definition is saturated. If the admissible sequence contains a quadric which is not saturated, define the active index $\kappa$ to be the largest index $i$ for which $r_i - r_{i-1} < d_{i-1} - d_i$ (where, by convention, we set $r_0 = 0$ and $d_0 = n$).*

*Remark* 5.6. By Lemma 4.18, a saturated restriction variety is a Schubert variety. If a quadric $Q_{d_i}^{r_i}$ in the definition of a restriction variety is not saturated, then $Q_{d_j}^{r_j}$ is not saturated for any $j \geq i$. In particular, the smallest dimensional quadric $Q_{d_{k-s}}^{r_{k-s}}$ is not saturated. The quadric $Q_{d_\kappa}^{r_\kappa}$ is the smallest dimensional quadric in the sequence $(L_\bullet, Q_\bullet)$ which is not maximally singular given the larger quadrics containing it.

We will compute the class of $V(L_\bullet, Q_\bullet)$ by successively increasing $r_\kappa$ by one, where $\kappa$ is the active index. This corresponds to a specialization of the flag defining the restriction variety. In the process, $V(L_\bullet, Q_\bullet)$ will specialize into a union of restriction varieties. Applying the degeneration to each of the resulting varieties, we will be able to decompose any restriction variety into a union of Schubert varieties.

*Degeneration* 5.7. Let $Sing(Q)$ denote the singular locus of a quadric $Q$. To avoid multiple indices set $L = L_{n_{x_\kappa+1}}$. Let $p \in L \cap Sing(Q^{r_{\kappa+1}}_{d_{\kappa+1}})$. Suppose that $p \notin Sing(Q^{r_\kappa}_{d_\kappa})$. Recall that $L$ is the smallest dimensional isotropic linear space in $(L_\bullet, Q_\bullet)$ that is not entirely contained in $Sing(Q^{r_\kappa}_{d_\kappa})$. It is understood that if $\kappa = k-s$, the condition that $p \in Sing(Q^{r_{\kappa+1}}_{d_{\kappa+1}})$ is vacuous. Similarly, if $x_\kappa = s$, then $p \in Q^{r_\kappa}_{d_\kappa} \cap Sing(Q^{r_{\kappa+1}}_{d_{\kappa+1}})$, but $p \notin Sing(Q^{r_\kappa}_{d_\kappa})$.

Let $S = Span(Q^{r_\kappa}_{d_\kappa})$ and let $U = Sing(Q^{r_\kappa}_{d_\kappa})$. Let $T = T_p Q^{r_{\kappa-1}}_{d_{\kappa-1}}$ be the tangent space to $Q^{r_{\kappa-1}}_{d_{\kappa-1}}$ at $p$. By Condition (5), $Q^{r_{\kappa-1}}_{d_{\kappa-1}}$ is smooth at $p$ so the tangent hyperplane exists. Moreover, since $p$ is not a singular point of $Q^{r_\kappa}_{d_\kappa}$, $T$ cannot contain $Q^{r_\kappa}_{d_\kappa}$. We conclude that $T \cap S := M$ is a codimension one linear space. On the other hand, since $Q^{r_{\kappa+1}}_{d_{\kappa+1}}$ is singular at $p$, $M$ automatically contains $Q^{r_{\kappa+1}}_{d_{\kappa+1}}$. Let $N = Span(M, U)$. Note that $N$ has dimension $d_\kappa$. Consider the pencil of linear spaces determined by $N$ and $S$. Since $N$ and $S$ have $M$ in common, they span a linear space of dimension $d_\kappa + 1$. In this linear space and in appropriate coordinates, this pencil can be expressed as $tx + (1-t)y$, where $y = 0$ defines $N$ and $x = 0$ defines $S$. This pencil of linear spaces cut out a pencil $Q^{r_\kappa(t)}_{d_\kappa}(t)$ of sub-quadrics on $Q$. When $t = 1$, this is the original quadric $Q^{r_\kappa}_{d_\kappa}$. When $t = 0$, it is a quadric of corank $r_\kappa + 1$. Note that all of these quadrics contain $Q^{r_{\kappa+1}}_{d_{\kappa+1}}$ and are contained in $Q^{r_{\kappa-1}}_{d_{\kappa-1}}$. Consequently, there exists a one-parameter family of sequences $(L_\bullet(t), Q_\bullet(t))$, where only the quadric $Q^{r_\kappa(t)}_{d_\kappa}(t)$ varies in the pencil just constructed. At a general $t$, the sequence is projectively equivalent to the original sequence. At the special point $t = 0$, the sequence $(L_\bullet(0), Q_\bullet(0))$ is equivalent to a sequence where $r_\kappa$ has been replaced by $r_\kappa + 1$. Correspondingly, there is a one-parameter family of restriction varieties $V(t)$ defined with respect to the flags $(L_\bullet(t), Q_\bullet(t))$. As long as $t \neq 0$, these varieties are isomorphic. Hence, they form a flat family. By the properness of the Hilbert scheme, there exists a flat limit $V(0)$. Our algorithm is obtained by describing $V(0)$.

*Notation* 5.8. For the rest of the paper, we will always use Degeneration 5.7. Given an admissible sequence $(L_\bullet, Q_\bullet)$, $(L_\bullet(t), Q_\bullet(t))$ will denote the position of the flag at time $t$ under this degeneration. To simplify notation, we will use $(L^a_\bullet, Q^a_\bullet)$ to denote the special position of the flag at $t = 0$. The dimension of the linear spaces and the dimension and corank of the quadrics in $(L^a_\bullet, Q^a_\bullet)$ will be denoted by $n'_j, d'_i$ and $r'_i$, respectively. Note that except for $r'_\kappa$, these invariants equal to those of $(L_\bullet, Q_\bullet)$ and $r'_\kappa = r_\kappa + 1$.

Observe that the sequence of brackets and braces associated to $(L^a_\bullet, Q^a_\bullet)$ is $D^a$ defined in §3. The degeneration increases $r_\kappa$ by one. This is represented by changing the integer in the $(r_\kappa + 1)$-st place in the quadric diagram corresponding to $(L_\bullet, Q_\bullet)$ to $\kappa$.

The reader should note that the sequence $(L^a_\bullet, Q^a_\bullet)$ does not have to be admissible. The algorithm will consist of decomposing $(L^a_\bullet, Q^a_\bullet)$ into a collection of admissible sequences $(L^j_\bullet, Q^j_\bullet)$. The flat limit will be supported along the union of the restriction varieties corresponding to these sequences. We replace $V(L_\bullet, Q_\bullet)$ by a collection of restriction varieties $V(L^j_\bullet, Q^j_\bullet)$ each occurring with multiplicity one. Hence, the cohomology class of $V(L_\bullet, Q_\bullet)$ is the sum of the cohomology classes of $V(L^j_\bullet, Q^j_\bullet)$. The varieties $V(L^j_\bullet, Q^j_\bullet)$ will be "closer" to Schubert varieties. By "closer" we mean that the admissible sequence $(L^j_\bullet, Q^j_\bullet)$ will have either $s^j = s + 1$ (one more linear space and one fewer quadric); or $r^j_i \geq r_i$ with strict inequality for at least one $i$ (one of the quadrics will have a higher dimensional singularity). If we keep applying the algorithm to each of the varieties that are output, the varieties will eventually become saturated. Hence, we will express the class of $V(L_\bullet, Q_\bullet)$ as a sum of Schubert cycles.

*A reminder about our notation:* Recall that $\kappa$ denotes the active index of $(L_\bullet, Q_\bullet)$. $x_i$ denotes the number of isotropic subspaces of the sequence contained in the singular locus of $Q^{r_i}_{d_i}$. In particular, if $x_i < s$, then $L_{n_{x_i+1}}$ denotes the smallest dimensional isotropic space in the sequence strictly containing $Q^{r_i, sing}_{d_i}$ (in the quadric diagram notation, $L_{n_{x_i+1}}$ is depicted by the left most bracket such that one of the digits to its left is zero or greater than $i$). $y_j$ denotes the index of the largest dimensional quadric containing $L_{n_j}$

in its singular locus or $y_j = k - s + 1$ if there are none (in terms of quadric diagrams, $y_j$ is the positive digit to the immediate left of the $j$-th bracket or $y_j = k - s + 1$ if this digit is zero.) The condition $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$ means that the codimension of $Q_{d_\kappa}^{r_\kappa, sing}$ in $L_{n_{x_\kappa+1}}$ is one more than the number of quadrics in the sequence that contain $Q_{d_\kappa}^{r_\kappa}$ but do not contain $L_{n_{x_\kappa+1}}$ in their singular locus.

*Algorithm* 5.9. We now give the algorithm that describes the maximal dimensional components of the flat limit of Degeneration 5.7.

**Step 1.** If $V(L_\bullet, Q_\bullet)$ is saturated (i.e., a Schubert variety), output $V(L_\bullet, Q_\bullet)$ and stop. The algorithm terminates. Otherwise,

- Let $(L_\bullet^a, Q_\bullet^a)$ be the sequence obtained by replacing $Q_{d_\kappa}^{r_\kappa}$ in $(L_\bullet, Q_\bullet)$ with $Q_{d_\kappa}^{r_\kappa+1} = Q_{d'_\kappa}^{r'_\kappa}$.
- If $x_\kappa < s$, then let $(L_\bullet^b, Q_\bullet^b)$ be the sequence obtained by replacing $L_{n_{x'_\kappa+1}}$ in $(L_\bullet^a, Q_\bullet^a)$ with $L_{r'_\kappa}$ (the singular locus of $Q_{d'_\kappa}^{r'_\kappa}$).

and proceed to Step 2.
**Step 2.** Depending on the case, replace $V(L_\bullet, Q_\bullet)$ by the following union of restriction varieties and stop.
• If $x_\kappa = s$ or if $n_{x_\kappa+1} - r_\kappa - 1 > y_{x_\kappa+1} - \kappa$ in the sequence $(L_\bullet, Q_\bullet)$, replace $V(L_\bullet, Q_\bullet)$ with the restriction varieties obtained by running Algorithm 5.10 on $(L_\bullet^a, Q_\bullet^a)$.

• If $(L_\bullet^a, Q_\bullet^a)$ violates Condition (8) (i.e., $x'_\kappa < k - \kappa + 1 - \frac{d'_\kappa - r'_\kappa}{2}$), then replace $V(L_\bullet, Q_\bullet)$ with the restriction varieties obtained by running Algorithm 5.10 on $(L_\bullet^b, Q_\bullet^b)$.

• If $x_\kappa < s$, $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$ in the sequence $(L_\bullet, Q_\bullet)$ and Condition (8) is satisfied for $(L_\bullet^a, Q_\bullet^a)$ (i.e., $x'_\kappa \geq k - \kappa + 1 - \frac{d'_\kappa - r'_\kappa}{2}$), then replace $V(L_\bullet, Q_\bullet)$ with the restriction varieties obtained by running Algorithm 5.10 on both sequences $(L_\bullet^a, Q_\bullet^a)$ and $(L_\bullet^b, Q_\bullet^b)$.

*Algorithm* 5.10 (Normalizing the sequence). Given a sequence $(L_\bullet^\alpha, Q_\bullet^\alpha)$ equal to $(L_\bullet^a, Q_\bullet^a)$ or $(L_\bullet^b, Q_\bullet^b)$ defined in Algorithm 5.9, run the following loop on the sequence. We will call the process of replacing the sequence $(L_\bullet, Q_\bullet)$ by the sequences produced by this algorithm *normalizing the sequence*.

i. If the sequence $(L_\bullet^\alpha, Q_\bullet^\alpha)$ is admissible, output the sequence $(L_\bullet^\alpha, Q_\bullet^\alpha)$ and stop. Otherwise, proceed to [ii].
ii. If $r_{k-s} + 2 \geq d_{k-s}$ (i.e., Condition (7) is violated) in $(L_\bullet^\alpha, Q_\bullet^\alpha)$, replace $(L_\bullet^\alpha, Q_\bullet^\alpha)$ by two sequences $(L_\bullet^i, Q_\bullet^i)$ for $i = 1, 2$, where $(L_\bullet^i, Q_\bullet^i)$ is the sequence obtained from $(L_\bullet^\alpha, Q_\bullet^\alpha)$ by replacing $Q_{d_{k-s}}^{r_{k-s}}$ with $L_{d_{k-s}-1}$ unless $2(d_{k-s} - 1) = n$. If $2(d_{k-s} - 1) = n$, then in one of the sequences replace $Q_{d_{k-s}}^{r_{k-s}}$ with $L_{d_{k-s}-1}$ and in the other with $L'_{d_{k-s}-1}$. If in addition $2k = n$, discard the sequence that parameterizes linear spaces that has the wrong parity for the dimension of intersection with $L_k$. For each of the sequences $(L_\bullet^i, Q_\bullet^i)$, return to Step [i] and run the loop again setting $(L_\bullet^\alpha, Q_\bullet^\alpha) = (L_\bullet^i, Q_\bullet^i)$. If $r_{k-s} + 2 < d_{k-s}$ (i.e., Condition (7) holds), proceed to [iii].
iii. If Condition (9) is violated for $(L_\bullet^\alpha, Q_\bullet^\alpha)$, while Condition (9) is violated, let $\mu$ be the largest index for which it is violated. Form a new sequence $(L_\bullet^\beta, Q_\bullet^\beta)$ by replacing $Q_{d_\mu}^{r_\mu}$ in $(L_\bullet^\alpha, Q_\bullet^\alpha)$ with $Q_{d_\mu-1}^{r_\mu+1}$. Discard the sequence $(L_\bullet^\beta, Q_\bullet^\beta)$ if $d_{\mu+1} = d_\mu - 1$ in $(L_\bullet^\alpha, Q_\bullet^\alpha)$. If there are no remaining sequences, the algorithm terminates and does not our put any sequences. If $(L_\bullet^\beta, Q_\bullet^\beta)$ satisfies Condition (9), proceed to Step [i] and run the loop again setting $(L_\bullet^\alpha, Q_\bullet^\alpha) = (L_\bullet^\beta, Q_\bullet^\beta)$.

We already observed that the sequence $(L_\bullet^a, Q_\bullet^a)$ is represented by the sequence of brackets and braces $D^a$ defined in §3. Next observe that $(L_\bullet^b, Q_\bullet^b)$ is represented by $D^b$ defined in §3. $(L_\bullet^b, Q_\bullet^b)$ is obtained from $(L_\bullet^a, Q_\bullet^a)$ by replacing the smallest dimensional linear space containing the singular locus of $Q_{d_\kappa}^{r'_\kappa}$ with the singular locus of $Q_{d_\kappa}^{r'_\kappa}$. This corresponds to shifting the left most bracket whose position is greater than $l_{D^a}(\leq \kappa)$ to the position $l_{D^a}(\leq \kappa)$.

The problem, as observed in §3, is that $D^a$ and $D^b$ need not be quadric diagrams. Equivalently, $(L^a_\bullet, Q^a_\bullet)$ and $(L^b_\bullet, Q^b_\bullet)$ may fail to be admissible. Algorithm 5.10 replaces them by admissible sequences. The sequence $(L^a_\bullet, Q^a_\bullet)$ may fail to satisfy Conditions (7), (8), or (9). If it fails to satisfy Condition (8), this sequence does not lead to a variety supported on the flat limit. If it fails to satisfy Condition (7), then Algorithm 5.10 in Step (ii) replaces the sequence by two sequences. The geometric meaning of this step is that the quadric $Q^{r'_\kappa}_{d_\kappa}$ is reducible consisting of a union of two linear spaces. When $n$ is even and the linear spaces have dimension $n/2$, they belong to two different connected components. These are distinguished in the algorithm.

When $(L^a_\bullet, Q^a_\bullet)$ fails to satisfy Condition (9) such as in the sequence represented by 10]0]0]00000}0, the loop in Step iii of Algorithm 5.10 increases the dimension of the singular locus of the quadric failing Condition (9) by one and decreases its dimension by one until Condition (9) is satisfied. In this case, the loop would produce the sequences represented by 11]0]0]0000}00, 11]1]0]000}000 and 11]1]1]00}0000, which satisfies Condition (9). Note however that Condition (7) may now fail to be satisfied, hence needs to be checked again. In Algorithm 5.10, it would have made more sense to swap Steps ii and iii. We write it this way for consistency with the case of flag varieties.

The sequence $(L^b_\bullet, Q^b_\bullet)$ may also fail to satisfy Condition (9). For example, the sequence represented by 3]0000}00}0} fails Condition (9). The loop in Step iii of Algorithm 5.10 increases the dimension of the singular loci and decreases the dimension of the quadrics containing the quadric failing Condition (9) successively. In this case, the loop would produce the sequences represented by 2]0000}0}00} and 1]0000}0}0}0, successively.

The geometric meaning of Step iii in Algorithm 5.10 is as follows. When $r_i = n_j - 1$, by "the variation of tangent spaces", the tangent space to $Q^{r_i}_{d_i}$ is constant along $L_{n_j}$. Hence, if a linear space intersects $L_{n_j}$, then it must be contained in this fixed tangent space. Therefore, the subspaces that are contained in $Q^{r_i}_{d_i}$ are already contained in the codimension one quadric cut out on $Q^{r_i}_{d_i}$ by the linear space everywhere tangent to $Q^{r_i}_{d_i}$ along $L_{n_j}$. The dimension of this quadric is one smaller and its singular locus contains $L_{n_j}$. Step iii of the Algorithm 5.10 replaces $Q^{r_i}_{d_i}$ with this quadric.

The geometric meaning of Algorithm 5.9 is apparent. Step 1 checks whether a given restriction variety is a Schubert variety. If so, the algorithm stops. Otherwise, we increase the corank of $Q^{r_\kappa}_{d_\kappa}$ by one using Degeneration 5.7. There are two possibilities. Either the linear spaces intersect the new singular locus of $Q^{r_\kappa+1}_{d_\kappa}$ in a vector space of dimension $x_\kappa$ (this possibility corresponds to the sequence $(L^a_\bullet, Q^a_\bullet)$ and is depicted by $D^a$) or they intersect the singular locus in a subspace of dimension $x_\kappa + 1$ (this possibility is depicted by the sequence $(L^b_\bullet, Q^b_\bullet)$ and is depicted by $D^b$). Under the first condition in Step 2, the variety corresponding to $(L^b_\bullet, Q^b_\bullet)$ has smaller dimension than the original variety $V(L_\bullet, Q_\bullet)$. Therefore, the sequence $(L^b_\bullet, Q^b_\bullet)$ does not lead to a component of the flat limit of the Degeneration 5.7. We replace the original sequence by sequences obtained from $(L^a_\bullet, Q^a_\bullet)$. In the second case, $(L^a_\bullet, Q^a_\bullet)$ violates Condition (8), hence the dimension of intersection of the linear spaces with the singular locus $Q^{r'_\kappa}_{d_\kappa}$ has to increase. Therefore, the only possibilities are derived from the sequence $(L^b_\bullet, Q^b_\bullet)$. In the final case, sequences derived from both sequences $(L^a_\bullet, Q^a_\bullet)$ and $(L^b_\bullet, Q^b_\bullet)$ give components of the flat limit of the Degeneration 5.7. This is the geometric branching.

From our description of the two algorithms, it is clear that Algorithm 5.9 and Algorithm 3.10 are the same algorithm, one phrased in terms of admissible sequences and the other in terms of the quadric diagrams representing them. In the rest of the section, we will work with the geometric algorithm.

We will check shortly that Algorithm 5.9 replaces a restriction variety with restriction varieties. Hence, we can apply the algorithm to each of the resulting varieties until the end result is a collection of Schubert varieties. Before proceeding, we urge the reader to work through the examples in the beginning of this section.

**Definition 5.11.** *A degeneration path for $V_1$ is a sequence of restriction varieties $V_1 \to V_2 \to \cdots \to V_m$ starting with $V_1$ and ending with a Schubert variety $V_m$ such that $V_{i+1}$ is one of the varieties assigned to $V_i$ by Algorithm 5.9.*

**Theorem 5.12.** *The class of a restriction variety $V$ is equal*

$$[V] = \sum [V_i]$$

*where $V_i$ are the restriction varieties produced by Algorithm 5.9. In particular, the coefficient $c_\lambda^\mu$ in*

$$[V] = \sum c_\lambda^\mu \sigma_\lambda^\mu$$

*is the number of degeneration paths starting with $V$ and ending in a variety with cohomology class $\sigma_\lambda^\mu$. Furthermore, the algorithm respects the marking of restriction varieties.*

*Proof.* We prove the theorem in three steps. We first check that Algorithm 5.9 transforms restriction varieties into a collection of restriction varieties of the same dimension. Then we interpret replacing $Q_{d_\kappa}^{r_\kappa}$ by $Q_{d_\kappa}^{r_\kappa+1}$ in Step 1 of Algorithm 5.9 as applying Degeneration 5.7. Using a dimension count, we show that the flat limit is supported along the varieties produced by the algorithm. Finally, we check that the flat limit is reduced at the generic point of each of these varieties. Theorem 5.12 follows. We begin by analyzing each case in the algorithm separately.

• If the sequence $(L_\bullet, Q_\bullet)$ is saturated, then Lemma 4.18 implies that $V(L_\bullet, Q_\bullet)$ is a Schubert variety. In this case, there is nothing further to do. Accordingly, Algorithm 5.9 terminates. From now on we may assume that $(L_\bullet, Q_\bullet)$ is not saturated.

The new sequences $(L_\bullet^a, Q_\bullet^a), (L_\bullet^b, Q_\bullet^b)$ formed in Step 1 may fail to be admissible. However, Conditions (1)-(6) are satisfied for them and for any of the sequences output by Algorithm 5.9. We begin by verifying this for $(L_\bullet^a, Q_\bullet^a)$. Conditions (4) and (5) hold by construction. Since Conditions (1) and (2) hold for $(L_\bullet, Q_\bullet)$ and replacing $r_\kappa$ by $r_\kappa + 1$ can only increase the left-hand-side of the inequalities, Conditions (1) and (2) also hold for $(L_\bullet^a, Q_\bullet^a)$. The active index $\kappa$ is chosen so that $Q_{d_\kappa}^{r_\kappa}$ satisfies the strict inequality $d_\kappa + r_\kappa < d_{\kappa-1} + r_{\kappa-1} \leq n$ in Condition (3). Increasing $r_\kappa$ by one can at worst turn these inequalities into equalities and improves the corresponding inequality for the index $\kappa$. Therefore, Condition (3) holds for $(L_\bullet^a, Q_\bullet^a)$. The sequence $(L_\bullet^a, Q_\bullet^a)$ satisfies Condition (6) by the choice of $\kappa$. The ranks of the quadrics $r_i$ remain unchanged for indices $i \neq \kappa$. The choice of $\kappa$ implies that $j - i \leq d_i - d_j = r_j - r_i$ for every $j > i \geq \kappa$ in $(L_\bullet, Q_\bullet)$. Hence, replacing $r_\kappa$ with $r_\kappa + 1$ ensures the inequality $r_i' - r_\kappa' \geq i - \kappa - 1$ for $i > \kappa$. The inequalities for $r_\kappa' - r_i'$ improve by one for $\kappa > i$. Finally, the second half of Condition (6) is also immediate from the choice of $\kappa$. We conclude that Conditions (1)-(6) hold for $(L_\bullet^a, Q_\bullet^a)$.

Next we note that the sequence $(L_\bullet^b, Q_\bullet^b)$ is obtained from $(L_\bullet^a, Q_\bullet^a)$ by replacing the linear space $L_{n_{x'_\kappa+1}}$ with the smaller dimensional linear space $L_{r'_\kappa}$. Replacing a linear space by a smaller dimensional one clearly preserves Conditions (1)-(4) and (6). Since all the quadrics with corank $r_i \leq r'_\kappa$ are singular along $L_{r'_\kappa}$, Condition (5) also holds. Hence, the sequence $(L_\bullet^b, Q_\bullet^b)$ satisfies Conditions (1)-(6).

Finally, we analyze how Algorithm 5.10 affects Conditions (1)-(6). We make the observation that if Condition (9) fails for $(L_\bullet^a, Q_\bullet^a)$, then it fails only for the index $\kappa$. If Condition (9) fails for $(L_\bullet^b, Q_\bullet^b)$, then it can only fail for indices $i < \kappa$.

- In Step ii of Algorithm 5.10, the quadric $Q_{d_{k-s}}^{r_{k-s}}$ is replaced by a linear space $L_{d_{k-s}-1}$ of dimension $d_{k-s} - 1$. Conditions (2)-(6) are unaffected by this change. By assumption, we have $d_{k-s} \leq r_{k-s} + 2$. Hence $2n_{s+1} = 2(d_{k-s} - 1) \leq d_{k-s} + r_{k-s} \leq d_{k-s-1} + r_{k-s-1}$. This verifies Condition (1).
- In Step iii of Algorithm 5.10, a quadric $Q_{d_i}^{r_i}$ is replaced by a quadric of corank $r_i + 1$ and ambient dimension $d_i - 1$. Note that all the inequalities in (1)-(3) are invariant under this transformation. Conditions (4) and (5) hold by construction. Condition (9) may fail to be satisfied for the index $\kappa$ in $(L_\bullet^a, Q_\bullet^a)$ or for some indices $i < \kappa$ in $(L_\bullet^b, Q_\bullet^b)$. In the former case the loop increases the rank $r_\kappa$ to that of at most $r_{\kappa+1}$ and it is clear that the resulting sequence satisfies Condition (6). If $(L_\bullet^b, Q_\bullet^b)$ violates Condition (9), it either violates it for all $1 \leq i \leq \kappa$ or only for $\kappa - 1$. In the first case, in $(L_\bullet, Q_\bullet)$ we must have $r_1 = r_\kappa = x_\kappa$ in $(L_\bullet, Q_\bullet)$ and the loop produces a sequence that satisfies the same equalities. Else $r_{\kappa-1} = r_\kappa$ in $(L_\bullet, Q_\bullet)$ and by Condition (6) for $(L_\bullet, Q_\bullet)$,

$d_{\kappa-1} - 1 = d_\kappa$ and the loop in Step iii, does not produce any sequences. Hence, the sequences produced in Step iii satisfy Conditions (1)-(6).

We conclude that all the sequences occurring in the algorithm satisfy Conditions (1)-(6).

If Conditions (7)-(9) hold in a sequence for all the indices $i \geq \alpha$, then Algorithm 5.10 does not modify the quadrics with these indices. Hence, every intermediate sequence formed during Algorithm 5.10 also satisfies Conditions (7)-(9) for indices $i \geq \alpha$.

Condition (8) may fail to hold for $(L^a_\bullet, Q^a_\bullet)$. However, since $(L_\bullet, Q_\bullet)$ is admissible, Condition (8) can only fail in $(L^a_\bullet, Q^a_\bullet)$ for the index $\kappa$ and the right hand side of the inequality can exceed the left hand side by at most 1/2. Moreover, in $(L_\bullet, Q_\bullet)$, we must have the equality

$$x_\kappa = k - \kappa + 1 - \frac{d_\kappa - r_\kappa}{2}.$$

The choice of $\kappa$ implies that equality holds in Condition (8) for all the indices $i > \kappa$ in the sequence $(L_\bullet, Q_\bullet)$. Since $r_i + d_i = r_\kappa + d_\kappa$ for all $i > \kappa$, we can rewrite the inequality in Condition (8) for the index $i$ as

$$x_i \geq x_\kappa + r_i - r_\kappa + \kappa - i.$$

By Condition (9), $r_{\kappa+1} - r_\kappa - 1 \geq x_{\kappa+1} - x_\kappa$. Hence, we see that equality holds for the index $\kappa+1$. By induction, it follows that equality holds for all the indices $\kappa \leq i \leq k-s$. Furthermore, $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$ in $(L_\bullet, Q_\bullet)$. Finally, note that if $x_\kappa = s$, then equality for the index $k-s$ implies that $d_{k-s} = r_{k-s} + 2$ contradicting Condition (7) for $(L_\bullet, Q_\bullet)$. We conclude that if Condition (8) fails for $(L^a_\bullet, Q^a_\bullet)$, then $x_\kappa < s$ and $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$. Therefore, the cases in Algorithm 5.9 are exhaustive and mutually exclusive. We may assume from now on that the sequence $(L^a_\bullet, Q^a_\bullet)$ satisfies Condition (8). We also conclude that the sequence $(L^b_\bullet, Q^b_\bullet)$ satisfies both Conditions (7) and (8). Since $x_\kappa < s$, Condition (7) has to hold for $(L^a_\bullet, Q^a_\bullet)$. Replacing a linear space with a smaller linear space does not affect Condition (7). Replacing $L_{x'_\kappa+1}$ with $L_{r'_\kappa}$ increases the left hand side of the inequality in Condition (8) by one without affecting the right hand side.

Therefore, $(L^b_\bullet, Q^b_\bullet)$ is either admissible or fails Condition (9). As we observed while verifying Step iii of Algorithm 5.10 preserves Conditions (1)-(6), no new sequences are formed unless Condition (9) fails for all the indices $1 \leq i \leq \kappa - 1$. In this case, any sequence formed in Step iii of Algorithm 5.10 clearly satisfies Condition (9), hence is admissible. Hence, every sequence formed in Step 4 of Algorithm 5.9 is admissible.

Condition (7) may fail to hold for $(L^a_\bullet, Q^a_\bullet)$ or while running Step iii of Algorithm 5.10 on $(L^a_\bullet, Q^a_\bullet)$. This may happen in only one of two ways. The sequence $(L_\bullet, Q_\bullet)$ either has $d_{k-s} = r_{k-s} + 3$ and $\kappa = k - s$; or $d_{k-s} = r_{k-s} + 3 + 2\alpha$, $\kappa = k-s$ and $(L_\bullet, Q_\bullet)$ has $\alpha$ linear spaces of dimensions $r_{k-s}+2, r_{k-s}+3, \dots, r_{k-s}+\alpha+1 = n_s$. By the observation three paragraphs above, $\kappa = k - s$. Hence, either Condition (7) is directly violated for $(L^a_\bullet, Q^a_\bullet)$ or Condition (7) is violated after applying Step iii of Algorithm 5.10 for the index $\kappa$ $\alpha$ times. The equality $d_{k-s} = r_{k-s} + 3 + 2\alpha$ follows by combining Condition (8)

$$s - \alpha > s + 1 - \frac{d_{k-s} - r_{k-s}}{2}$$

for $(L_\bullet, Q_\bullet)$ with the inequality $d_{k-s} - r_{k-s} - 2\alpha \leq 3$ that expresses that Condition (7) is violated after $\alpha$-steps. In either of the two cases, Step ii of Algorithm 5.10 outputs admissible sequences.

Finally, if Condition (9) fails for the sequence $(L^a_\bullet, Q^a_\bullet)$, then, as observed above, it fails only for the index $\kappa$. Applying Step iii of Algorithm 5.10 either produces a sequence which is admissible or which violates Condition (7). In the latter case, running Step ii of Algorithm 5.10 outputs an admissible sequence. We conclude that all the sequences output by Algorithm 5.9 are admissible. We now analyze the dimensions of the corresponding varieties.

- The expression in Equation (1) for the dimension of a restriction variety remains unchanged when we replace $Q^{r_\kappa}_{d_\kappa}$ with $Q^{r_\kappa+1}_{d_\kappa}$.

- In Cases 2 and of the Algorithm, we have the equality $n_{x'_\kappa+1} - r'_\kappa = y_{x'_\kappa+1} - \kappa$. Hence, when we replace $L_{n_{x'_\kappa+1}}$ with $L_{r'_\kappa}$ to form $(L^b_\bullet, Q^b_\bullet)$, $x_i$ increases by one for the indices $\kappa \leq i < y_{x_\kappa+1}$. The dimension of the linear space with index $x'_\kappa+1$ decreases by $n'_{x_\kappa+1} - r'_\kappa$. All other terms in the expression in Equation (1) remain unchanged.
- Step iii of Algorithm 5.10 increases $x_\mu$ by one and decreases $d_\mu$ by one, hence preserves the expression in Equation (1).
- Finally, replacing $Q^{d_{k-s}-2}_{d_{k-s}}$ with $L_{d_{k-s}-1}$ in Step ii of Algorithm 5.10 increases the first sum in Equation (1) by $d_{k-s} - s - 2$. It changes the second sum by $-x_s - d_{k-s} + 2s + 2$. Since we must have $x_s = s$, we conclude that this step also preserves the expression in Equation (1).

Combining these observations, we conclude that every sequence produced by Algorithm 5.9 is admissible and gives rise to a restriction variety of the same dimension as $V(L_\bullet, Q_\bullet)$. The algorithm can be recursively applied to each of the resulting restriction varieties. It is clear that the algorithm must terminate in a collection of Schubert varieties. At each stage of the algorithm, either the corank of a quadric in the sequence increases by at least one or the number of quadrics in the sequence decreases. Since there are finitely many quadrics in the sequence and the corank of the quadrics are bounded above, eventually the sequence must become saturated. Then the resulting varieties are Schubert varieties.

We now analyze Degeneration 5.7 to conclude that the support of the flat limit is the union of restriction varieties replacing $V(L_\bullet, Q_\bullet)$ in Algorithm 5.9. In order to restrict the possible irreducible components of the support of the flat limit, we write down conditions that the linear spaces in the limit have to satisfy. We then observe that these conditions already cut out varieties of dimension equal to the dimension of $V(L_\bullet, Q_\bullet)$. The following observation puts strong restrictions on the support of the flat limit.

*Observation* 5.13. The linear spaces parameterized by the restriction varieties $V(L_\bullet(t), Q_\bullet(t))$ intersect the linear spaces $L_{n_j}(t)$ in a subspace of dimension at least $j$ and the quadrics $Q^{r_i}_{d_i}(t)$ in a linear space of dimension at least $k-i+1$. Similarly, they intersect $Q^{r_i,sing}_{d_i}(t)$ in a linear space of dimension at least $x_i$. Since intersecting a proper variety in at least a given dimension is a closed condition, the linear spaces parameterized by the flat limit $V(0)$ have to intersect the linear spaces $L_{n_j}(0)$ in a subspace of dimension at least $j$ and the quadrics $Q^{r_i}_{d_i}(0)$ in a subspace of dimension at least $k-i+1$. Furthermore, they intersect $Q^{r_i,sing}_{d_i}(0)$ in a subspace of dimension at least $x_i$.

A quick inspection of the algorithm will reveal that in each of the limits either the linear spaces intersect the vertex of $Q^{r_\kappa+1}_{d_\kappa}(0)$ in a subspace of dimension $x_\kappa+1$ and otherwise remain as unconstrained as possible given Observation 5.13; or the linear spaces continue to intersect $Q^{r_\kappa+1}_{d_\kappa}(0)$ in a subspace of dimension $x_\kappa$ and only satisfy the constraints imposed by Observation 5.13. A priori in the limit the linear spaces could become more special. However, we claim that these loci have strictly smaller dimension and do not form an irreducible component of the support of the flat limit. We now verify this claim.

Let $Y$ be an irreducible component of the support of the flat limit of Degeneration 5.7. Then we can build a sequence of consisting of $k$ linear spaces and quadrics such that the closure of the locus of linear spaces intersecting the $i$-th element in the sequence (counting in increasing dimension) in dimension $i$ contains $Y$. We complete the linear spaces and quadrics in the sequence $(L^a_\bullet, Q^a_\bullet)$ to a set of linear spaces and quadrics whose dimensions increase by one at each stage making sure that Conditions (4) and (5) of Definition 4.2 are satisfied. We then select those linear spaces and quadrics that have a jump in the dimension of intersection with a general linear space parameterized by $Y$. We thus obtain a set of $k$ linear spaces and quadrics. By construction the closure of the locus $X$ of linear spaces that intersect the $i$-th one in dimension $i$ contains $Y$. Observation 5.13 implies that the $i$-th linear space or quadric in the sequence thus obtained has dimension less than or equal to the $i$-th linear space or quadric (counting in increasing dimension) in the sequence $(L^a_\bullet, Q^a_\bullet)$. By Proposition 4.16, Equation (1) gives an upper bound on the dimension of $X$ (note that we used the fact that the sequence is admissible in the proof only to deduce the equality).

We now estimate the dimension of $X$. We obtain the sequence defining $X$ by replacing linear spaces and quadrics in $(L^a_\bullet Q^a_\bullet)$ by smaller dimensional ones in increasing order. We will do this in greater generality in preparation for the discussion of orthogonal flag varieties.

- If we replace a linear space of dimension $n'_{i+j}$ in the $(i+j)$-th position with a linear space of dimension $n^*_i$ in the $i$-th position not contained in $(L^a_\bullet, Q^a_\bullet)$, then according to Equation (1) the dimension changes as follows. Let $y'_{i+j}$ and $y^*_i$ be the smallest index quadrics containing the corresponding linear spaces in their singular locus. The dimension decreases by $n'_{i+j} - n^*_i + j + y'_{i+j} - y^*_i$. Since Conditions (6) and (9) hold for $(L_\bullet, Q_\bullet)$, we have that $n'_{i+j} - n^*_i + y'_{i+j} - y^*_i \geq 0$. Consequently the decrease in the dimension is at least $j$ with equality when $n'_{i+j} - n^*_i + y'_{i+j} - y^*_i = 0$.
- If we replace the $i$-th largest quadric in a vector space of dimension $d'_i$ by the $(i+j)$-th largest quadric in a vector space of dimension $d^*_{i+j}$ , then according to Equation (1) the dimension decreases by $d'_i - d^*_{i+j} + x'_i - x^*_{i+j}$. This decrease is at least $j$ with strict inequality unless Condition (9) fails for $r'_i$.
- Finally, if we replace the quadric $Q^{r'_i}_{d'_i}$ with the linear space $L_{n^*_j}$, then the first sum in Equation (1) changes by $n'_j - s - 1$. The second sum changes by $-d'_i + (k - s - y^*_j - x'_i) + (2s+2)$. Hence, the total change is
$$-d'_i + n^*_j + k + 1 - x'_i - y^*_j,$$
where $y^*_{n_j}$ denotes the index of the largest dimensional quadric containing $L^*_{n_j}$ in its singular locus. We rewrite this expression as follows:
$$(k - i + 1 - \lfloor \frac{d'_i - r'_i}{2} \rfloor - x'_i) + (n^*_j - \lceil \frac{d'_i + r'_i}{2} \rceil + k - s - y^*_j) + (-k + s + i).$$
The sum in the first parentheses is strictly negative unless Condition (8) is violated or there is equality in Condition (8); otherwise it is zero. The sum in the second parentheses is strictly negative unless $j = s+1$ and $d'_i + r'_i = d_{k-s} + r_{k-s}$; otherwise it is zero. Finally, the third sum is strictly negative unless $i = k - s$; otherwise it is zero.

Since our degeneration is flat, $Y$ has to have the same dimension as $V(L_\bullet, Q_\bullet)$. Since $X$ contains $Y$, our dimension calculation puts strong restrictions on $X$.

First, suppose $x_\kappa = s$ in $(L_\bullet, Q_\bullet)$. Then by Conditions (6) and (9) for $(L_\bullet, Q_\bullet)$, $n'_l - r'_i + y'_l - i > 0$ for every $l$ with $n'_l > r'_i$ in $(L^a_\bullet, Q^a_\bullet)$. Furthermore, Condition (9) holds for $(L^a_\bullet, Q^a_\bullet)$. If $d'_{k-s} - r'_{k-s} > 2$, then replacing any linear space or quadric with a smaller dimensional one strictly decreases the dimension. Note also that $(L^a_\bullet, Q^a_\bullet)$ is admissible. In this case, we conclude that $X$ has to be $V(L^a_\bullet, Q^a_\bullet)$. Since $V(L^a_\bullet, Q^a_\bullet)$ and $V(L_\bullet, Q_\bullet)$ have the same dimension, we conclude that $Y$ has to be a component of $V(L^a_\bullet, Q^a_\bullet)$. If $d'_{k-s} - r'_{k-s} = 2$, then $Q^{r'_{k-s}}_{d'_{k-s}}$ is necessarily reducible consisting of two linear spaces of dimension $d'_{k-s} - 1$. If $2(d'_{k-s} - 1) = n$, then these linear spaces belong to two different connected components. We can therefore replace $Q^{r'_{k-s}}_{d'_{k-s}}$ with either of these linear spaces to obtain two sequences. Note that replacing any other linear space or quadric with a smaller dimensional one strictly decreases the dimension. Hence $X$ has to be the variety corresponding to one of these sequences. Since $X$ has the same dimension as $V(L_\bullet, Q_\bullet)$, we conclude that $Y$ has to be an irreducible component of $X$. Observe that Algorithm 5.9 selects the sequences corresponding to $X$.

Next, suppose that $x_\kappa < s$ and $n_{x_\kappa+1} - r_\kappa - 1 > y_{x_\kappa+1} - \kappa$ in $(L_\bullet, Q_\bullet)$. Then the sequence $(L^a_\bullet, Q^a_\bullet)$ is admissible. Furthermore, by our dimension calculation, replacing any linear space or quadric in $(L^a_\bullet, Q^a_\bullet)$ leads to a strictly smaller dimensional locus. We conclude that $X = V(L^a_\bullet, Q^a_\bullet)$ and $Y$ has to be an irreducible component of $V(L^a_\bullet, Q^a_\bullet)$.

Next, suppose that $x_\kappa < s$ and Condition (8) is violated for $(L^a_\bullet, Q^a_\bullet)$ for $\kappa$. Note that in that case, there must be an equality in Condition (8) in $(L_\bullet, Q_\bullet)$ for the index $\kappa$. Hence, by Conditions (6) and (8), $r_{\kappa-1} < r_\kappa$ in $(L_\bullet, Q_\bullet)$. By the "linear space bound", every linear space of dimension $k - \kappa + 1$ contained in

$Q^{r'_\kappa}_{d'_\kappa}$ must intersect the singular locus of this quadric in dimension at least $x_\kappa + 1$. Hence, we can replace the sequence $(L^a_\bullet, Q^a_\bullet)$ with the sequence $(L^b_\bullet, Q^b_\bullet)$. By our dimension calculation, replacing any linear space or quadric by a smaller dimensional one results in a variety of strictly smaller dimension. Hence, we conclude that $Y$ has to be an irreducible component of $V(L^b_\bullet, Q^b_\bullet)$. As we observed above, $(L^b_\bullet, Q^b_\bullet)$ may fail Condition (9) for $i < \kappa$. In that case, by the "variation of tangent spaces", any linear space of dimension $k - i + 1$ intersecting $L_{r'_\kappa}$ in dimension $x_i$ is necessarily contained in the quadric everywhere tangent to $Q$ along $L_{r'_\kappa}$. Hence, we can replace the sequence $(L^b_\bullet, Q^b_\bullet)$ as in Step iii of Algorithm 5.10 to obtain an equivalent definition of the same variety (Note that since $r_{\kappa-1} < r_\kappa$ in $(L_\bullet, Q_\bullet)$, the definition of $\kappa$ ensures that $d_\mu - 1 < d_{\mu-1}$ while running Step iii of Algorithm 5.10).

Finally, suppose that $x_\kappa < s$ and $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$ in $(L_\bullet, Q_\bullet)$ and $(L^a_\bullet, Q^a_\bullet)$ satisfies Condition (8). Let $i \leq \kappa$ be the smallest index such that $n_{x'_i+1} - r'_i = y_{x'_i+1} - i$ in $(L^a_\bullet, Q^a_\bullet)$. Note that by Conditions (6) and (9) for $(L_\bullet, Q_\bullet)$, there may be such indices precisely when $r_{\kappa-1} = r_\kappa$, $r_i \geq x_\kappa$ and $r_i = r_\kappa - \kappa + i + 1$ in $(L_\bullet, Q_\bullet)$. By our dimension counts, replacing $L_{x'_\kappa+1}$ with $L_{r'_j}$ for an index $i \leq j \leq \kappa$ can result in a sequence that has the same dimension as $V(L_\bullet, Q_\bullet)$. Replacing any other linear space or quadric with a smaller dimensional one, gives a smaller dimensional variety. The rest of the analysis of this case is more subtle. We need to argue that unless $j = \kappa$, these loci do not occur in the limit. For a general linear space $W_t \in V(L_\bullet(t), Q_\bullet(t))$, let $W_{t,j} = Q^{r_j}_{d_j}(t) \cap W_t$ for $i \leq j < \kappa$. The tangent space to $Q^{r_j}_{d_j}$ along $W_{t,j}$ intersects $L_{n_{x_\kappa+1}}$ in a subspace of dimension $r_j + 1$. By semi-continuity, this must be true for every linear space contained in $V(L_\bullet(t), Q_\bullet(t))$ and also in the limit $V(L_\bullet(0), Q_\bullet(0))$. However, the tangent space to $Q^{r_j}_{d_j}$ at a general linear space parameterized by the variety associated to the sequence obtained from $(L^a_\bullet, Q^a_\bullet)$ by replacing $L_{x'_\kappa+1}$ with $L_{r'_j}$ intersects $L_{n_{x_\kappa+1}}$ in dimension $r_j = r'_j$. We conclude that the support of $Y$ cannot equal such a locus. Hence $X$ is the locus associated to one of the sequences $(L^a_\bullet, Q^a_\bullet)$ or $(L^b_\bullet, Q^b_\bullet)$. These sequences may fail to satisfy Condition (9). In that case, Step iii of Algorithm 5.10 replaces them by equivalent varieties unless for $(L^b_\bullet, Q^b_\bullet)$ we have $d_{\kappa-1} - 1 = d_\kappa$. In the latter case, by "the variation of tangent spaces", the $(k - \kappa + 2)$-dimensional subspaces of the linear spaces $W$ parameterized by $X$ have to be contained in $Q^{r'_\kappa}_{d'_\kappa}$. In other words, we have to replace $Q^{r'_{\kappa-1}}_{d'_{\kappa-1}}$ by a smaller quadric. By our dimension counts, such a locus has strictly smaller dimension, hence cannot support $Y$.

In order to conclude the proof, we need to verify that the limits all occur and are reduced at the generic point of each of these loci. This is a straightforward local calculation. Let $U$ be the Zariski open set of our family of restriction varieties parameterizing linear spaces $W(t)$ such that $\dim(W(t) \cap Q^{r_\kappa(t)}_{d_\kappa}(t)) = k - \kappa + 1$. Let $Z$ be the family of restriction varieties obtained by applying Degeneration 5.7 to the admissible sequence obtained from $(L_\bullet, Q_\bullet)$ by omitting the quadrics $Q^{r_1}_{d_1}, \dots, Q^{r_\kappa - 1}_{d_{\kappa-1}}$. Then there exists a natural morphism $f : U \to Z$ sending $W(t)$ to $W(t) \cap Q^{r_\kappa(t)}_{d_\kappa}(t)$, which is smooth at the generic point of each of the irreducible components of the fiber of $Z$ at $t = 0$. We may, therefore, assume that $\kappa = 1$. Furthermore, without loss of generality, we may assume that $n = d_\kappa + r_\kappa + 1$ and $x_\kappa = 0$. We will check that the multiplicity is one by exhibiting cycles that intersect $V(L_\bullet, Q_\bullet)$ in one point and exactly one of the potential limits in one point. This will allow us to conclude that each of the limits occur with multiplicity one. There is a Schubert cycle in the class of the variety $V(L^a_\bullet, Q^a_\bullet)$ (respectively, $V(L^b_\bullet, Q^b_\bullet)$) that occurs with coefficient one and does not occur in the class of $V(L^b_\bullet, Q^b_\bullet)$ (respectively, $V(L^a_\bullet, Q^a_\bullet)$). We use the dual of these Schubert cycles for our computation. Note that by our assumptions on $\kappa$ and $n$, $d_i + r_i = d_\kappa + r_\kappa = n - 1$ for every $i \geq \kappa$. Hence, $n - d_{k-s} + 1 = r_{k-s} + 2$ and $2(r_{k-s} + 2) \leq r_{k-s} + d_{k-s} + 1 = n$.

First, suppose $x_\kappa = s(= 0)$ and $d_\kappa = r_\kappa + 3$ in $(L_\bullet, Q_\bullet)$. In this case, this is the standard family of a quadric breaking into a union of two linear spaces. Both occur in the limit with multiplicity one. In this case there is nothing to check. Next suppose $x_\kappa = s(= 0)$ and $d_\kappa > r_\kappa + 3$ in $(L_\bullet, Q_\bullet)$. Let $\beta_i = n - d_i + 1$. Let $S$ be the Schubert variety defined with respect to a general isotropic flag

$$L_{\beta_1} \subset \cdots \subset L_{\beta_{k-s}}.$$

In case $2\beta_{k-s} = n$, we will always define a second Schubert variety $S'$ by replacing $L_{\beta_{k-s}}$ with $L'_{\beta_{k-s}}$. Note that under our assumptions $V(Q_\bullet)$ is irreducible. Both $V(Q_\bullet)$ and $V(Q^\alpha_\bullet)$ intersect $S$ (and $S'$ when appropriate) in a reduced point. The spans $Span(Q^{r_i}_{d_i})$ and $Span(Q^{r'_i}_{d_i})$ intersect the linear space $L_{\beta_i}$ in a one dimensional subspace for every $1 \leq i \leq k$. Any $k$-dimensional linear space contained in $V(Q_\bullet) \cap S$ or $V(Q^\alpha_\bullet) \cap S$ must contain these one-dimensional subspaces. Hence, the $k$-dimensional linear space is uniquely determined as the $Span((Q^{r_i}_{d_i} \cap L_i), 1 \leq i \leq k)$ or $Span((Q^{r'_i}_{d_i} \cap L_i), 1 \leq i \leq k)$, respectively. By Kleiman's Transversality Theorem [K], we conclude that the intersection of the two varieties consists of a single reduced point. When $2\beta_{k-s} = n$, two general linear spaces in the class $L = L_{\beta_{k-s}}$ intersect in a unique point if $n = 0$ modulo 4 and are otherwise disjoint. A general linear space in the class $L$ and a general linear space in the class $L' = L'_{\beta_{k-s}}$ intersect in a unique point if $n = 2$ modulo 4 and are otherwise disjoint. When $V(Q'_\bullet)$ has two components, repeating the argument with $S'$, we conclude that both components occur with multiplicity one.

Next, suppose that $x_\kappa(=0) < s$ and $(L^a_\bullet, Q^a_\bullet)$ fails to satisfy Condition (8). Let $\alpha_{x_\kappa+1} = \alpha_1 = n - r_\kappa$. Let $\alpha_j = n - n_{j-1}$ for $j > x_{\kappa+1}$. Let $\beta_i = n - d_i + 1$. Let $S$ be the Schubert variety defined with respect to the linear spaces and quadrics

$$L_{\beta_1} \subset \cdots \subset L_{\beta_{k-s}} \subset Q^{n-\alpha_s}_{\alpha_s} \subset \cdots \subset Q^{n-\alpha_1}_{\alpha_1}.$$

Proposition 4.18 implies that $S$ is a Schubert variety. We claim that $S$ intersects both $V(L_\bullet, Q_\bullet)$ and $V(L^b_\bullet, Q^b_\bullet)$ in a unique, reduced point. The linear spaces $L_{\beta_i}$ intersect the quadrics $Q^{r_i}_{d_i}$ and $Q^{r^b_i}_{d^b_i}$ in unique points. Any $k$-dimensional linear space in the intersection of $S$ and $V(L_\bullet, Q_\bullet)$ or $S$ and $V(L^b_\bullet, Q^b_\bullet)$ must contain the $(k-s)$-dimensional linear space $\Lambda$ spanned by these points. In $S \cap V(L_\bullet, Q_\bullet)$, the quadrics everywhere tangent to $\Lambda$ determine unique points in $Q^{n-\alpha_j}_{\alpha_j} \cap L_{n_j}$ for $j > 0$. In $S \cap V(L^b_\bullet, Q^b_\bullet)$, the quadrics everywhere tangent to $\Lambda$ determine unique points in $Q^{n-\alpha_j}_{\alpha_j} \cap L_{n_j}$ for $j > 1$ and furthermore the $k$-plane has to contain the point $L_{r^b_\kappa} \cap Q^{n-\alpha_1}_{\alpha_1}$ (which is contained in the singular locus of all the quadrics). Hence, in both cases, the $k$-dimensional linear space in the intersection is uniquely determined. We proved above that $n_s = n/2$ in this case. Hence, both $V(L_\bullet, Q_\bullet)$ and $V(L^b_\bullet, Q^b_\bullet)$ are irreducible. Therefore, $V(L^b_\bullet, Q^b_\bullet)$ occurs in the limit with multiplicity one.

Next, suppose $x_\kappa(=0) < s$ and $n_{x_\kappa+1} - r_\kappa - 1 > y_{x_\kappa+1} - \kappa$. In this case, let $i_0$ denote the smallest index for which equality holds in Condition (8) in $(L_\bullet, Q_\bullet)$. If there is no such index, set $i_0 = 0$ and $r_{i_0} = 0$. For $n_j \leq r_{i_0}$, let $\alpha_j = n - n_j + 1$. For $n_j > r_{i_0}$, set $\alpha_j = n - n_{j-1}$. Next, for each index $i < i_0$, let $l_i$ be the largest positive integer such that $r_i + l_i + 1 = n_{x_i + l_i}$. If there does not exist such $l_i$, set $l_i = 0$. Let $\beta_i = n - d_i + l_i + 1$ for $i < i_0$ and let $\beta_i = n - d_i + 1$ for $i \geq i_0$. Let $S$ be the Schubert variety defined by the sequence

$$L_{\beta_1} \subset \cdots \subset L_{\beta_{k-s}} \subset Q^{n-\alpha_s}_{\alpha_s} \subset \cdots \subset Q^{n-\alpha_1}_{\alpha_1}.$$

When $2\beta_{k-s} = n$, define $S'$ by replacing $L_{\beta_{k-s}}$ with $L'_{\beta_{k-s}}$. Note that Proposition 4.18 implies $S$ is a Schubert variety. As in the previous cases, it is straightforward to see that $S$ intersects both $V(L_\bullet, Q_\bullet)$ and $V(L^a_\bullet, Q^a_\bullet)$ in a unique, reduced point. When appropriate, the same holds for $S'$. We conclude that $V(L^a_\bullet, Q^a_\bullet)$ occurs in the limit with multiplicity one.

Finally, suppose $x_\kappa(=0) < s$, $n_{x_\kappa+1} - r_\kappa - 1 = y_{x_\kappa+1} - \kappa$ and Condition (8) is satisfied for $(L^a_\bullet, Q^a_\bullet)$. Then duals for $V(L^a_\bullet, Q^a_\bullet)$ and $V(L^b_\bullet, Q^b_\bullet)$ are obtained as in the previous cases. Let $S$ be the Schubert variety defined exactly as in the previous paragraph. Let $T$ be the Schubert variety defined by replacing $\alpha_{x_\kappa+1} = n - n_{x_\kappa+1} + 1$ in the definition of $S$ with $\alpha_{x_\kappa+1} = n - r_\kappa$. Then it is straightforward to see that both $S$ and $T$ intersect $V(L_\bullet, Q_\bullet)$ in a unique reduced point. $S$ (respectively, $T$) intersects $V(L^a_\bullet, Q^a_\bullet)$ (respectively, $V(L^b_\bullet, Q^b_\bullet)$) in a unique, reduced point and has empty intersection with $V(L^b_\bullet, Q^b_\bullet)$ (respectively, $V(L^a_\bullet, Q^a_\bullet)$). It follows that both limits occur with multiplicity one. Finally, by replacing $S$ with $S'$ and $T$ with $T'$ when appropriate, it is easy to see that in case these varieties are reducible, both components occur with multiplicity one and that the algorithm preserves marking. This concludes the proof of the theorem. $\square$

*Remark* 5.14. From the analysis in the proof of Theorem 5.12, it follows that at each stage of the degeneration a restriction variety breaks into at most three irreducible components.

## 6. Applications of Algorithm 5.9

In this section we discuss a couple of immediate applications of Algorithm 5.9. The Introduction discusses other applications.

6.1. **The moduli space of vector bundles on hyperelliptic curves.** There is a beautiful, classical construction that associates to a general pencil of quadric hypersurfaces in $\mathbb{P}^{2g+1}$ a hyperelliptic curve $C$ of genus $g$. In fact, every smooth hyperelliptic curve of genus $g$ arises this way [GH, §6], [DR]. We recall the construction for the reader's convenience.

Let $Q_1$ and $Q_2$ be general quadric hypersurfaces in $\mathbb{P}^{2g+1}$. Let $tQ_1 + uQ_2$ be the pencil generated by $Q_1$ and $Q_2$. Consider the incidence correspondence $I$ parameterizing pairs $(Q, C)$, where $Q$ is a quadric hypersurface contained in the pencil and $C$ is a connected component of the space of $g$-dimensional projective linear spaces on $Q$. The incidence correspondence $I$ is irreducible and maps to $\mathbb{P}^1$ by the first projection $\pi_1$. When $Q$ is a smooth quadric, the space of $g$-dimensional projective linear spaces on $Q$ has two connected components. Hence, $I$ is a double cover of $\mathbb{P}^1$. When $Q$ has corank one, then the space of $g$-dimensional projective spaces has only one component. Hence, $\pi_1$ is ramified at the $2g+2$ points in the pencil that are quadrics of corank one. By the Riemann-Hurwitz formula, we conclude that $I$ is a hyperelliptic curve of genus $g$.

To see that there are $2g+2$ corank one quadrics in a general pencil, observe that the pencil can be identified with a $(2g+2)\times(2g+2)$ symmetric matrix whose entries are linear homogeneous polynomials in $t$ and $u$. The quadrics of corank one correspond to matrices with zero determinant. Since the determinant is a homogeneous polynomial of degree $2g+2$ in $t$ and $u$, it will have $2g+2$ roots in $\mathbb{P}^1$. If the pencil is general, these roots will be distinct and the corresponding symmetric matrix will have corank exactly one. Furthermore, it is clear from this description that one can construct a pencil with any $2g+2$ distinct roots. Hence, every smooth hyperelliptic curve of genus $g$ arises via this construction.

Let $C$ be a smooth hyperelliptic curve of genus $g \geq 2$. Let $MV_{2,o}(C_g)$ denote the moduli space of rank two vector bundles with a fixed determinant of odd-degree on $C$. Realize $C$ as a double cover of a pencil of quadric hypersurfaces in $\mathbb{P}^{2g+1}$. By a celebrated theorem of Desale and Ramanan [DR], $MV_{2,o}(C_g)$ is isomorphic to the space of $(g-2)$-dimensional projective linear spaces contained in this pencil of quadric hypersurfaces in $\mathbb{P}^{2g+1}$. Equivalently, if $Q_1$ and $Q_2$ are two smooth quadric hypersurfaces that generate the pencil, $MV_{2,o}(C_g)$ is isomorphic to the space of $(g-2)$-dimensional projective linear spaces contained in both $Q_1$ and $Q_2$.

We can view the space $X$ parameterizing $(g-2)$-dimensional projective linear spaces contained in $Q_1$ as the orthogonal Grassmannian $OG(g-1, 2g+2)$, which naturally includes in $G(g-1, 2g+2)$. We can also view the space of $(g-2)$-dimensional projective linear spaces contained in $Q_2$ as a subvariety $Y$ of $G(g-1, 2g+2)$. Of course, $Y$ is isomorphic to $X$; however, its embedding in $G(g-1, 2g+2)$ differs from that of $X$ by translation with an element of $\mathbb{P}GL(2g+2)$. By Kleiman's Transversality Theorem, $X$ and $Y$ intersect transversally. Therefore, the class of the intersection $Y \cap OG(g-1, 2g+2)$ in $H^*(OG(g-1, 2g+2), \mathbb{Z})$ is the pull-back of the class of $Y$ in $H^*(G(g-1, 2g+2), \mathbb{Z})$ under the map induced by inclusion.

The class of $Y$ in $G(g-1, 2g+2)$ is well-known to be $2^{g-1}\sigma_{g-1,g-2,\dots,2,1}$. There are several ways of calculating this class. First, it is the top Chern class of the vector bundle $Sym^2(S^*)$ on $G(g-1, 2g+2)$, where $S^*$ denotes the dual of the tautological bundle of $G(g-1, 2g+2)$. Calculating the top Chern class of $Sym^2(S^*)$ is a standard exercise in using the splitting principle. Alternatively, one can use degenerations for a more pleasant calculation. Very briefly, break the quadric into a union of two linear spaces using a general pencil $Q + tL_1L_2$. The flat limit of the space of $(g-2)$-dimensional projective linear spaces contained in $Q$ is the space of $(g-2)$-dimensional projective linear spaces contained in $L_1$ or $L_2$ that intersect $Q \cap L_1 \cap L_2$ in $(g-3)$-dimensional projective linear spaces (see [C1] or [C2]). Now inductively break $Q \cap L_1 \cap L_2$ into a union of linear spaces using a general pencil. Continuing this process for $(g-1)$ steps, we obtain $2^{g-1}$ flags of the form

$$\mathbb{P}^{2g} \supset \mathbb{P}^{2g-2} \supset \mathbb{P}^{2g-4} \supset \cdots \supset \mathbb{P}^4,$$

where $\mathbb{P}^{2g-2i}$ is one of the two linear spaces obtained by degenerating the $(2g-2i)$-dimensional quadric. Inductively, the flat limit of $Y$ is the space of $(g-2)$-dimensional projective linear spaces that intersect $\mathbb{P}^{2g-2i}$ in a projective space of dimension $g-2-i$. We conclude that the class of $Y$ is is $2^{g-1}\sigma_{g-1,g-2,\dots,2,1}$ in the cohomology of $G(g-1,2g+2)$.

In conclusion, the class of $MV_{2,o}(C_g)$ is $2^{g-1}$ times the class of the restriction variety associated to the Schubert class $\sigma_{g-1,g-2,\dots,1}$ in $G(g-1,2g+2)$. More explicitly, the class of $MV_{2,o}(C_g)$ in $OG(g-1,2g+2)$ is equal to $2^{g-1}$ times the class of the restriction variety associated to the admissible sequence

$$Q_5^0 \subset Q_7^0 \subset \cdots \subset Q_{2g-1}^0 \subset Q_{2g+1}^0.$$

Using Algorithm 5.9 the class can be easily computed. Here we give the class for the first few genera.

(1) $[MV_{2,o}(C_2)] = 2\sigma^1$
(2) $[MV_{2,o}(C_3)] = 4\sigma_0^1 + 4\sigma^{3,1}$
(3) $[MV_{2,o}(C_4)] = 16\sigma_2^{3,1} + 16\sigma_{1,0}^1 + 16\sigma_1^{4,1}$
(4) $[MV_{2,o}(C_5)] = 64\sigma_{3,1}^{3,1} + 64\sigma_{2,1,0}^1 + 64\sigma_{2,1}^{5,1} + 32\sigma_{3,0}^{4,1} + 32\sigma_3^{5,4,1} + 32\sigma_{5,0}^{3,1} + 32\sigma_5^{5,3,1} + 32\sigma_{4,0}^{3,2} + 32\sigma_4^{5,3,2}$

More generally, one obtains a recursion in the genus for the class. Suppose that the class of $MV_{2,o}(C_{g-1})$ in $OG(g-3,2g)$ is given by

$$[MV_{2,o}(C_{g-1})] = \sum c_{\lambda,\mu}[\Omega_\lambda^\mu],$$

where $\Omega_\lambda^\mu$ is defined with respect to a sequence $(L_\bullet^\lambda, Q_\bullet^\mu)$. Let $t$ be the largest index of a linear space in the sequence such that $n_t = t$. Define a new sequence $(\tilde{L}_\bullet^\lambda, \tilde{Q}_\bullet^\mu)$ by setting $\tilde{L}_{n_j}^\lambda = L_{n_j}^\lambda$ for all $1 \le j \le s$ and $\tilde{Q}_{d_{i+1}}^{\mu,r_{i+1}} = Q_{d_i}^{\mu,r_i}$ for $1 \le i \le g-3-s$. Set $\tilde{Q}_{d_1}^{\mu,r_1} = Q_{2g+1-t}^t$. Then we have that

$$[MV_{2,o}(C_g)] = 2\sum c_{\lambda,\mu}[V(\tilde{L}_\bullet^\lambda, \tilde{Q}_\bullet^\mu)].$$

*Remark* 6.1. When $g=2$, $MV_{2,o}(C_2)$ is a complete intersection of two quadric hypersurfaces in $\mathbb{P}^5$ [GH, §6]. Ciprian Manolescu (in private correspondence) posed the question whether $MV_{2,o}(C_g)$ can be a complete intersection for $g>2$. In fact, one can ask for a much weaker property. Can $MV_{2,o}(C_g)$ be a complete intersection of ample divisors in $OG(g-1,2g+2)$? The codimension of $MV_{2,o}(C_g)$ in $OG(g-1,2g+2)$ is $\frac{g(g-1)}{2}$. The codimension of the Schubert variety $\sigma_g^{g-2,g-3,\dots,2,1}$ is $g+1$. Hence, the sum of the codimensions of these two varieties is $\frac{g^2+g}{2}+1$. If $g>2$, this is less than the dimension of $OG(g-1,2g+2)$. Hence, if $MV_{2,o}(C_g)$ were a complete intersection of ample divisors, $\sigma_g^{g-2,g-3,\dots,2,1}\cdot[MV_{2,o}(C_g)] \neq 0$. However, the cup product of these classes is clearly zero since the one-dimensional vector space defining the Schubert variety can be chosen to not be contained in $Q_{2g+1}^0$ defining the restriction variety. Hence, we conclude that for $g>2$, $MV_{2,o}(C_g)$ cannot be a complete intersection of ample divisors even in $OG(g-1,2g+2)$, let alone in $G(g-1,2g+2)$ or $\mathbb{P}^{\binom{2g+2}{g-1}-1}$.

6.2. **A geometric algorithm for computing the product of arbitrary Schubert cycles.** The pull-back of a Schubert class under the inclusion $j : OG(k,n) \to G(k,n)$ can be expressed as a sum of classes of restriction varieties. Consider a Schubert cycle $\Sigma_{\lambda_1,\dots,\lambda_k}$ defined with respect to a general partial flag

$$F_{n-k+1-\lambda_1} \subset F_{n-k+2-\lambda_2} \subset \cdots \subset F_{n-\lambda_k}.$$

The intersection of this flag with the quadric hypersurface $Q$ leads to the sequence of quadrics

$$Q_{n-k+1-\lambda_1}^0 \subset Q_{n-k+2-\lambda_2}^0 \subset \cdots \subset Q_{n-\lambda_k}^0.$$

Note that since none of the quadrics are singular, the Conditions (3)-(6) of Definition 4.2 are automatically satisfied. Similarly, since there are no linear spaces in the sequence, Condition (1) is automatically satisfied. However, Condition (2) may be violated. In that case, the corresponding variety is empty and the pull-back is zero. From now on we assume that the sequence satisfies all the conditions in Definition 4.2. If the sequence is admissible, then the pull-back of the Schubert cycle is the class of the corresponding restriction variety. However, the sequence may fail to be admissible and thus the pull-back maybe the sum of classes of restriction varieties. We now describe how to express the pull-back as a sum of these. Since Condition (2) in Definition 4.2 is satisfied, $n-k+i-\lambda_i \geq 2i$ for all $i$. Suppose that equality holds for $i \leq \alpha$ and the inequality is strict for $i = \alpha+1$. Then the quadric $Q_{n-k+1-\lambda_1}$ consists of two points $p_1, p_2$. The linear spaces have to contain one of the $p_i$ and be contained in the tangent space to $Q$ along $p_i$. Then $Q^1_{n-k+2-\lambda_2}$ consists of two lines intersecting at $p_i$. The linear spaces containing $p_i$ have to contain one of these lines. Continuing we deduce the following proposition.

**Proposition 6.2.** *Let $\sigma_{\lambda_1,\dots,\lambda_k}$ be a Schubert cycle in $G(k,n)$. Let $j : OG(k,n) \to G(k,n)$ be the natural inclusion. Then*

(1) *$j^*\sigma_{\lambda_1,\dots,\lambda_k} = 0$ unless $n-k-i \geq \lambda_i$ for every $1 \leq i \leq k$.*
(2) *Suppose that $n-k-i = \lambda_i$ for $i = 1,\dots,\alpha$ and $n-k-i > \lambda_i$ for $i = \alpha+1$. Further suppose that if $2k = n$, then $\alpha \neq k$. Let $(L_\bullet, Q_\bullet)$ be the admissible sequence*

$$L_1 \subset L_2 \subset \cdots \subset L_{\alpha-1} \subset L_\alpha \subset Q^\alpha_{n-k+\alpha+1-\lambda_{\alpha+1}} \subset \cdots \subset Q^\alpha_{n-1-\lambda_{k-1}} \subset Q^\alpha_{n-\lambda_k}.$$

*Then $j^*\sigma_{\lambda_1,\dots,\lambda_k} = 2^\alpha[V(L_\bullet, Q_\bullet)]$, where $[V(L_\bullet, Q_\bullet)]$ denotes the cohomology class of the restriction variety $V(L_\bullet, Q_\bullet)$. If $2\alpha = 2k = n$, then the class is $2^{\alpha-1}$ times the Poincaré dual of a point.*

Theorem 5.12 gives a geometric algorithm for computing the product of any two Schubert cycles in the cohomology ring of the orthogonal Grassmannian $OG(k,n)$ when $n$ is odd. When $n$ is even, the quadric $Q$ has an involution exchanging the half-dimensional isotropic linear spaces. The same method gives an algorithm for computing the invariant part of the cohomology ring. For simplicity we assume that $n$ is odd. We can reverse the algorithm to express any Schubert variety in the orthogonal Grassmannian as a linear combination of the restriction of general Schubert varieties in the ordinary Grassmannian. This algorithm is of independent interest and may be interpreted as a Giambelli-like formula, which expresses an arbitrary Schubert cycle as a linear combination of restrictions of Schubert cycles from the ordinary Grassmannian. We can then multiply the Schubert varieties in the ordinary Grassmannian and use Proposition 6.2 to restrict back the product to the orthogonal Grassmannian.

*Example* 6.3. For example, we can express the Schubert variety $\sigma^{3,1}_{4,2}$ in $OG(4,9)$ as follows.

$$1]22]000\}00\}0 \leftarrow 1]00]0000\}0\}0 \leftarrow \frac{1}{2}\ 1]000\}000\}0\}0 \leftarrow \frac{1}{4}\ 00\}000\}000\}0\}$$

The Schubert variety $\sigma^{3,1}_{4,2}$ in $OG(4,9)$ is a quarter of the restriction of the Schubert cycle $\sigma_{4,2}$ in the ordinary Grassmannian $G(4,9)$. Similarly, we can express the Schubert variety $\sigma^{2,0}_{3,1}$ in $OG(4,9)$ as follows.

$$22]00]000\}00\} \leftarrow 00]00]0000\}0\} \leftarrow \frac{1}{2}33]000\}000\}0\} \leftarrow \frac{1}{2}00]0000\}00\}0\} - \frac{1}{2}1]00000\}0\}0\}0$$
$$\leftarrow \frac{1}{4}000\}000\}00\}0\} - \frac{1}{4}00\}00000\}0\}0\}$$

The Schubert variety $\sigma^{2,0}_{3,1}$ in $OG(4,9)$ is a quarter of the difference of the restriction of the Schubert cycles $\sigma_{3,1}$ and $\sigma_4$ in the ordinary Grassmannian $G(4,9)$. The reader might enjoy verifying that $\sigma^{3,1}_{4,2} \cdot \sigma^{2,0}_{3,1} = \sigma_{4,3,2,1}$ by multiplying the corresponding cycles in $G(4,9)$ and then restricting the product back to $OG(4,9)$.

*Algorithm* 6.4 (Reversing Algorithm 5.9). Let $V(L_\bullet, Q_\bullet)$ be a restriction variety in $OG(k,n)$ with $n$ odd.

(1) If the class of $V(L_\bullet, Q_\bullet)$ is a fraction of a restriction of a Schubert cycle in $G(k,n)$ (In Proposition 6.2, we determined that this happens precisely when $r_i = n_{x_1} = x_1 = s$ for all $i$), let $(Q'_\bullet)$ be the sequence consisting of the linear sections defining the corresponding Schubert variety in $G(k,n)$.

(2) If in the sequence $(L_\bullet, Q_\bullet)$, $r_i = n_{x_1} = x_1$ for every $i$, but $s \neq x_1$, then let $\alpha$ be the largest non-negative integer with $n_{s-\alpha} = n_s - \alpha$. Let $(L^\alpha_\bullet, Q^\alpha_\bullet)$ be the sequence obtained from $(L_\bullet, Q_\bullet)$ by replacing $L_{n_s}$ with $Q^{n_s-\alpha-2}_{n_s+\alpha+1}$.

(3) If in the sequence $(L_\bullet, Q_\bullet)$, $r_i \neq x_1$ for some $i$, let $i$ be the largest index for which $r_i > x_1$ and there does not exist a smaller index $l$ such $r_l = r_{l-1} > x_1$. If $r_i \neq n_j$ for any $j$, let $(L^\alpha_\bullet, Q^\alpha_\bullet)$ be the sequence obtained from $(L_\bullet, Q_\bullet)$ by replacing $r_i$ with $r_i - 1$. If $r_i = n_j$ for some $j$, let $\alpha$ be the largest non-negative integer for which $n_{j-\alpha} = n_j - \alpha$. Let $(L^\alpha_\bullet, Q^\alpha_\bullet)$ be the sequence obtained from $(L_\bullet, Q_\bullet)$ by replacing $Q^{r_i}_{d_i}$ with $Q^{r_i-\alpha-2}_{d_i+\alpha+1}$.

In Case (1), $[V(L_\bullet, Q_\bullet)]$ is already a fraction of the restriction of a Schubert cycle in $G(k,n)$. There is nothing further to do.

In Case (2), by the Algorithm 5.9 we can express

$$[V(L^\alpha_\bullet, Q^\alpha_\bullet)] = 2[V(L_\bullet, Q_\bullet)] + \text{other terms}.$$

In Case (3), by the Algorithm 5.9 we can express

$$[V(L^\alpha_\bullet, Q^\alpha_\bullet)] = [V(L_\bullet, Q_\bullet)] + \text{other terms}.$$

In both Cases (2) and (3), the other terms have the property that the sum of the dimension of the linear spaces is strictly smaller (as is the sum of the ranks of the quadrics) than those in $(L_\bullet, Q_\bullet)$. On the other hand, $(L^\alpha_\bullet, Q^\alpha_\bullet)$ has the property that either it has fewer linear spaces than $(L_\bullet, Q_\bullet)$ or the sum of the ranks of the quadrics is smaller than those of $(L_\bullet, Q_\bullet)$. We can solve for the class $[V(L_\bullet, Q_\bullet)]$, then apply the algorithm to each of the terms. It is clear that this eventually terminates expressing the class as a linear combination of the classes of restriction of Schubert varieties in the Grassmannian. We thus obtain a geometric (though non-positive) algorithm for multiplying arbitrary Schubert cycles in the cohomology ring (see the beginning of the subsection for an example).

## 7. Restriction varieties in orthogonal flag varieties

In this subsection, we extend the discussion of restriction varieties from orthogonal Grassmannians to orthogonal flag varieties. We first begin by describing the combinatorics, we then give the geometric explanation. We preserve the notation from §3 and §4.

**Definition 7.1.** *A* coloring $c_\bullet$ *for $OF(k_1, \dots, k_h; n)$ is a sequence of $k_h$ positive integers $1 \leq c_i \leq h$ such that $k_1$ of the integers are equal to 1 and $k_j - k_{j-1}$ of them are equal to $j$ for $2 \leq j \leq h$. A* colored sequence of brackets and braces $(D, c_\bullet)$ *for $OF(k_1, \dots, k_h; n)$ is a sequence of brackets and braces $D$ of type $(k_h, n)$ together with a coloring $c_\bullet$ such that the $i$-th bracket or brace in the sequence counting from left to right is assigned the color $c_i$.*

We denote the coloring in a colored sequence of brackets and braces by placing the color as a subscript on the brackets and braces. For example, $11]_1 22]_2 000\}_3 00\}_1 0$ is a colored sequence for $F(2,3,4;10)$. The coloring can be determined by reading the subscripts under the brackets and braces from left to right. In this case, the coloring is $(1,2,3,1)$. Given a colored sequence of brackets and braces, there is an associated sequence of brackets and braces obtained by forgetting the coloring.

**Definition 7.2.** *A colored sequence for $OF(k_1, \dots, k_h; n)$ is called a* colored quadric diagram *if the underlying sequence is a quadric diagram for $OG(k_h, n)$.*

Define the dimension of a coloring $c_\bullet$ by the equation:

$$\dim(c_\bullet) = \sum_{u=1}^{h-1} \sum_{i \mid c_i \leq u} \#\{j < i \mid c_j = u+1\}.$$

The dimension of a colored sequence of brackets and braces $(D, c_\bullet)$ is defined by

$$\dim(D, c_\bullet) = \dim(D) + \dim(c_\bullet).$$

The algorithm is very similar to the algorithm in the Grassmannian case. In order to keep the exposition brief, we will state the combinatorial and geometric versions simultaneously. We now start explaining the geometric meaning behind colored quadric diagrams.

**Definition 7.3.** *A sequence* $(L_\bullet, Q_\bullet, c_\bullet)$

$$L_{n_1}[c_1] \subset \cdots \subset L_{n_s}[c_s] \subset Q^{r_{k_h-s}}_{d_{k_h-s}}[c_{s+1}] \subset \cdots \subset Q^{r_1}_{d_1}[c_{k_h}]$$

*for the orthogonal flag variety* $OF(k_1, \dots, k_h, n)$ *consists of a sequence* $(L_\bullet, Q_\bullet)$ *for* $OG(k_h, n)$ *together with the assignment of a color between* $1$ *and* $h$ *to each of these linear spaces and quadrics such that* $k_1$ *of the colors are one, and* $k_i - k_{i-1}$ *of the colors are* $i$ *for* $2 \leq i \leq h$*. The sequence is called admissible if the underlying sequence for* $OG(k_h, n)$ *is admissible.*

Admissible sequences in $OF(k_1, \dots, k_h; n)$ allow us to define restriction varieties in orthogonal flag varieties.

**Definition 7.4** (Restriction varieties)**.** *Let* $(L_\bullet, Q_\bullet, c_\bullet)$ *be an admissible sequence for* $OF(k_1, \dots, k_h; n)$*. Then the restriction variety* $V(L_\bullet, Q_\bullet, c_\bullet)$ *is defined as the Zariski closure of the locus in* $OF(k_1, \dots, k_h; n)$ *parameterizing*

$$\{(W_1, \dots, W_h) \in OF(k_1, \dots, k_h; n) \mid \textit{for every } 1 \leq u \leq h,\ \dim(W_u \cap L_{n_j}) = \#\{l \leq j \mid c_l \leq u\},$$
$$\dim(W_u \cap Q^{r_i}_{d_i}) = \#\{l \leq k_h - i + 1 \mid c_l \leq u\},\ \dim(W_u \cap Q^{r_i, sing}_{d_i}) = \#\{l \leq x_i \mid c_l \leq u\}\ \}$$

We can depict restriction varieties in $OF(k_1, \dots, k_h; n)$ by colored quadric diagrams.

**Definition 7.5.** *The colored quadric diagram associated to the restriction variety* $V(L_\bullet, Q_\bullet, c_\bullet)$ *in the orthogonal flag variety* $OF(k_1, \dots, k_h; n)$ *is the quadric diagram associated to* $V(L_\bullet, Q_\bullet)$ *in* $OG(k_h, n)$*, where the* $i$*-th right bracket or right brace counting from left to right is decorated by the integer* $c_i$*.*

*Example* 7.6. For example, the colored quadric diagram associated to $L_1[1] \subset L_3[2] \subset Q^3_6[1] \subset Q^1_8[2]$ in $OF(2, 4; 9)$ is

$$1]_1 22]_2 000\}_1 00\}_2 0.$$

If we ignore the subscripts under the brackets and the braces, we recover the quadric diagram in Example 4.5. The subscripts read from left to right is the sequence $c_\bullet$ (in this case $1, 2, 1, 2$). Geometrically, this diagram records the flag elements for which the dimension of intersection with some $W_i$ jumps. The flag elements where the jump for $W_i$ occurs are depicted by the brackets and braces that have a subscript less than or equal to $i$. For instance, in this example, the brackets and braces that have a subscript of 1 correspond to $L_1$ and $Q^3_6$. These are the flag elements where a dimension jump occurs for $W_1$. The reader will have noticed that this restriction variety is the Schubert variety $\sigma^{3^1, 1^2}_{1^1, 3^2}$ in $OF(2, 4; 9)$.

In view of our discussion in §4, it is clear that colored quadric diagrams and colored admissible sequences record exactly the same data.

**Definition 7.7.** *A marking* $m_\bullet$ *of a colored sequence is a marking of the underlying sequence* $(L_\bullet, Q_\bullet)$*. The marked restriction variety* $V(L_\bullet, Q_\bullet, m_\bullet, c_\bullet)$ *is the component of the restriction variety* $V(L_\bullet, Q_\bullet, c_\bullet)$ *whose image under the natural projection*

$$\pi : OF(k_1, \dots, k_h; n) \to OG(k_h, n)$$

*is the marked restriction variety* $V(L_\bullet, Q_\bullet, m_\bullet)$ *in* $OG(k_h, n)$*.*

The geometric properties of restriction varieties in flag varieties follow from the properties of the restriction varieties in orthogonal Grassmannians by studying the natural projection morphism

$$\pi : OF(k_1, \dots, k_h; n) \to OG(k_h, n).$$

**Proposition 7.8.** *Let $(L_\bullet, Q_\bullet, m_\bullet, c_\bullet)$ be an admissible sequence for $OF(k_1, \dots, k_h; n)$ with marking $m_\bullet$. Then the marked restriction variety $V(L_\bullet, Q_\bullet, m_\bullet, c_\bullet)$ is irreducible of dimension*

$$\dim(V(L_\bullet, Q_\bullet, c_\bullet, m_\bullet)) = \sum_{j=1}^{s}(n_j - j) + \sum_{i=1}^{k-s}(d_i + x_i - 2s - 2i) + \dim(c_\bullet). \tag{2}$$

*In particular, every component of the restriction variety $V(L_\bullet, Q_\bullet, c_\bullet)$ has the same dimension.*

*Proof.* The restriction variety $V(L_\bullet, Q_\bullet, m_\bullet, c_\bullet)$ on the Zariski open set used in its definition admits a projection to $V(L_\bullet, Q_\bullet, m_\bullet)$ in $OG(k_h, n)$. The fibers of the projection are Zariski open subsets of Schubert varieties $\sigma_{c_\bullet}$ in $F(k_1, \dots, k_{h-1}; k_h)$. The irreducibility and the dimension follow from the irreducibility and dimension of restriction varieties in $OG(k_h, n)$ and standard facts about Schubert varieties in ordinary flag varieties. □

**Definition 7.9.** *Given a sequence $(L_\bullet, Q_\bullet, c_\bullet)$ (whether admissible or not), we will refer to the expression in Equation (2) as the dimension of the sequence.*

The expression in Equation (2) not surprisingly is the same as the dimension of the colored sequence of brackets and braces defined before.

*Remark* 7.10. As in the case of the orthogonal Grassmannians, the Schubert varieties are precisely the restriction varieties associated to sequences where all the quadrics are saturated (i.e., they satisfy $d_i + r_i = n$). The proof is identical to the proof of Lemma 4.18.

Next, we would like to extend the results of §4 to orthogonal flag varieties. The algorithm for computing the classes of restriction varieties in orthogonal flag varieties is very similar to the case of orthogonal Grassmannians. We will increase the corank of the quadrics in the sequence using Degeneration 5.7. The order will be the same as in the Grassmannian case. The limits will also have a very similar description. However, there are a few new phenomena that one needs to take into account. In particular, some geometric possibilities that we discarded because they led to smaller dimensional varieties now may have the same dimension. We give some typical examples. The reader who prefers to know the rule before seeing the examples should skip the next two examples and return to them after reading the rule.

*Example* 7.11. Consider the following three closely related examples.

$$00]00000\}0\}0 \to 22]0000\}00\}0$$

The first example is the restriction variety $V(L_2 \subset Q_7^0 \subset Q_8^0)$ in the orthogonal Grassmannian $OG(3, 9)$. According to Algorithm 5.9, we increase the corank of $Q_7^0$ by one. If the three dimensional linear spaces do not intersect the singular locus of $Q_7^1$, then according to the variation of tangent spaces, we get the limit depicted. If the three dimensional linear spaces do intersect the singular locus, then we would get $1]00000\}0\}00$. However, note that the dimension of this variety is less than the dimension of the original variety, hence cannot be a component of the support of the flat limit (see Step iii of Algorithm 5.10). In contrast, consider the restriction variety $V(L_2[2] \subset Q_7^0[1] \subset Q_8^0[2])$ in $OF(1, 3; 9)$.

$$\begin{array}{l} 00]_2 00000\}_1 0\}_2 0 \to 22]_2 0000\}_1 00\}_2 0 \\ \downarrow \\ 1]_2 20000\}_2 0\}_1 00 \end{array}$$

In this case, when we increase the corank of $Q_7^0$, the three dimensional linear spaces can intersect the singular locus of $Q_7^1$. Although the dimension of the image of the projection to $OG(3, 9)$ decreases by one, the fiber dimension increases by one as well. Geometrically, this flag variety parametrizes pointed planes. Although the plane becomes special in this limit, the point has more room to vary. Hence, now this limit has the same dimension as the original variety. In contrast, if we repeat the calculation for the restriction variety $V(L_2[2] \subset Q^0_7[2] \subset Q^0_8[1])$, the fiber dimension does not increase and we again get only one limit.

$$00]_2 00000\}_2 0\}_1 0 \to 22]_2 0000\}_2 00\}_1 0$$

These examples illustrate the principle that in the algorithm when certain linear spaces or quadrics coincide, the dimension of the image of the projection $\pi$ to $OG(k_h, n)$ decreases. However, depending on the ordering of the colors, the fiber dimension may increase. It is not hard to see that the increase in the fiber dimension is at most the decrease in the dimension of the image of $\pi$. The limits of the degeneration will consist of the limits of the image of $\pi$ described in §5 together with the limits where the decrease in the dimension of the image of $\pi$ is exactly compensated by the increase in the fiber dimension. The limits all occur with multiplicity one.

The next example demonstrates a few subtleties that occur when a quadric becomes reducible.

*Example* 7.12. Consider the restriction variety $V(Q^0_4[1], Q^0_5[2], Q^0_6[3])$ in $OF(1,2,3;6)$. (Recall that by convention $OF(1,2,3;6)$ is only one of the two irreducible components of the space of flags.) Before explaining a few salient features, let us write out the entire calculation.

$$\begin{array}{llll}
0000\}_1 0\}_2 0\}_3 \to & 3000\}_1 0\}_2 0\}_3 \to & 1]_3 000\}_1 0\}_2 0 \to & 1]_3 00]'_1 00\}_2 0 \\
 & \downarrow & & \searrow \\
 & 2000\}_1 0\}_2 0\}_3 \to & 200]_1 00\}_2 0\}_3 & 0]_3 0]_2 0]_1 000 \\
 & \downarrow \qquad \searrow & & \\
 & 0]_3 0]_2 0]_1 000 & 00]_2 0]'_1 000\}_3 &
\end{array}$$

The class of this restriction variety is given by

$$2\sigma_{2^3,1^2,0^3} + \sigma^{2^1,1^2}_{2^3} + \sigma^{1^2,0^3}_{0^1} + \sigma^{2^1,0^3}_{1^2}.$$

In this example two points are worth noting. When we increase the corank of $Q^0_5$ in the restriction variety $V(Q^1_4[1] \subset Q^0_5[2] \subset Q^0_6[3])$ depicted by $3000\}_1 0\}_2 0\}_3$, one possible limit is the restriction variety $V(L_1[3] \subset Q^1_4[1] \subset Q^1_5[2])$ depicted by $1]_3 000\}_1 0\}_2 0$. In the Grassmannian case, requiring a linear space to intersect the singular locus of a quadric instead of a quadric always led to smaller dimensional varieties. This is not necessarily the case for partial flag varieties. Geometrically, flags consisting of points, lines and planes on a quadric $Q$ in $\mathbb{P}^5$ can specialize to be contained in the singular hyperplane section $Q^1_5$. The family of planes contained in $Q$ is three dimensional. As long as a plane intersects the quadrics $Q^1_4[1]$ and $Q^1_5[2]$ properly, the smaller dimensional flag elements (i.e., the point and the line) are determined. The family of planes contained in $Q^1_5$ is one-dimensional. However, if the plane is contained in $Q^1_5[2]$, then the smaller flag elements are no longer determined and are free to vary in a two-dimensional family. We thus get a new type of limit that we did not see in the Grassmannian case. Similarly, for orthogonal flag varieties, the limits that occur when a quadric becomes reducible are much more subtle. For instance, when we increase the corank of $Q^1_4$ by one in $2000\}_1 0\}_2 0\}_3$, the quadric $Q^2_4$ becomes reducible. However, the planes parameterized by the limiting variety may intersect these linear spaces in a point, in a line or in a plane. All three cases occur in this example. In the Grassmannian case, those that intersected the linear space in a line or a plane would lead to smaller dimensional varieties. Also note that the Schubert variety $\sigma^{2^1,1^2}_{2^3}$ depicted by $1]_3 00]'_1 00\}_2 0$ is not a limit of this degeneration, although it has the same dimension as the original variety. The space of lines in $Q^1_5$ has two irreducible components. The flag elements $W_2$ (i.e., the lines) parametrized by $1]_3 00]'_1 00\}_2 0$ and $2000\}_1 0\}_2 0\}_3$ belong to two different irreducible components on $Q^1_5$. In fact, the coefficient of the Schubert variety $\sigma^{2^1,1^2}_{2^3}$ in the class of $2000\}_1 0\}_2 0\}_3$ is zero. This example demonstrates that we will have to keep careful track of the irreducible components that contain different flag elements.

We now give the algorithm for orthogonal flag varieties. We preserve the terminology from the previous sections. We will say that a sequence $(L^\alpha_\bullet, Q^\alpha_\bullet)$ is derived from a colored sequence $(L_\bullet, Q_\bullet, c_\bullet)$ if, for every

$1 \leq i \leq k_h$, the $i$-th linear space or quadric (in increasing order by dimension) in the sequence $(L^\alpha_\bullet, Q^\alpha_\bullet)$ has dimension less than or equal to the $i$-th linear space or quadric in $(L_\bullet, Q_\bullet)$. Let $j_1 < \cdots < j_{k_h}$ be the positions of the brackets and braces in $D$. Let $j^\alpha_1 < \cdots < j^\alpha_{k_h}$ be the positions of brackets and braces in $D^\alpha$. Equivalently, a sequence of brackets and braces $D^\alpha$ is derived from $(D, c_\bullet)$ if the positions of the brackets and braces satisfy $j^\alpha_i \leq j_i$ for all $1 \leq i \leq k_h$.

We begin with an algorithm for assigning a coloring to a sequence $(L^\alpha_\bullet, Q^\alpha_\bullet)$ derived from a colored sequence $(L_\bullet, Q_\bullet, c_\bullet)$. It is convenient to introduce auxiliary notation. We will say that a quadric or a linear space is smaller (respectively, larger) than another quadric or linear space if its dimension is smaller (respectively, larger). We will denote by $X_{\alpha,i}$ the $\alpha$-th largest linear space or quadric in the sequence $(L_\bullet, Q_\bullet)$ to which $c_\bullet$ assigns the color $i$. For example, $X_{1,i}$ is the largest linear space or quadric with color $i$. $X_{1,1} \supset X_{2,1} \supset \cdots$ are the linear spaces and quadrics of color 1 in decreasing order, etc.

*Algorithm* 7.13. [Algorithm for assigning a coloring] Let $(L^\alpha_\bullet, Q^\alpha_\bullet)$ be a sequence derived from $(L_\bullet, Q_\bullet, c_\bullet)$. We assign the coloring $c^\alpha_\bullet$ to this sequence as follows. Let $c^\alpha_\bullet$ assign the color 1 to the largest linear space or quadric in $(L^\alpha_\bullet, Q^\alpha_\bullet)$ whose dimension does not exceed the dimension of $X_{1,1}$ in $(L_\bullet, Q_\bullet, c_\bullet)$. Proceed to the index pair $(2,1)$ or $(1,2)$ depending on whether $k_1 > 1$ or $k_1 = 1$, respectively. Suppose the algorithm has proceeded to assign a color up to the index pair $(\alpha, i)$. Let $c^\alpha_\bullet$ assign the color $i$ to the largest linear space or quadric in $(L^\alpha_\bullet, Q^\alpha_\bullet)$ whose dimension does not exceed that of $X_{\alpha,i}$ in $(L_\bullet, Q_\bullet, c_\bullet)$ and to which $c^\alpha_\bullet$ does not yet assign a color. Proceed to the index pair $(\alpha+1, i)$ or $(1, i+1)$ depending on whether $k_i - k_{i-1} > \alpha$ or $k_i - k_{i-1} = \alpha$, respectively. The algorithm terminates when all the linear spaces and quadrics in $(L^\alpha_\bullet, Q^\alpha_\bullet)$ are assigned a color by $c^\alpha_\bullet$ (equivalently, after the index pair $(k_h - k_{h-1}, h)$). We call $c^\alpha_\bullet$ the induced coloring.

Let $D^\alpha$ be a sequence of brackets and braces derived from $(D, c_\bullet)$. The algorithm translates to the following. Let $p$ be the position of the largest bracket or brace in $D$ assigned the color 1. Assign the color 1 to the bracket or brace in the largest position less than or equal to $p$ in $D^\alpha$. Proceed to the bracket or brace of color 1 in $D$ in the next largest position. If there isn't one, proceed to the largest bracket or brace in $D$ of color 2. Suppose we have assigned colors until the bracket or brace of color $i$ in $D$ at the $j$-th largest position. Suppose this bracket or brace is at position $p$. Assign the color $i$ to the bracket or brace in $D^\alpha$ at the largest position less than or equal to $p$ that is not already assigned a color. Proceed to the bracket or brace of color $i$ at the $(j+1)$-st largest position or if there are none of color $i$ left, proceed to the bracket or brace of color $i+1$ in $D$ with the largest position. The algorithm terminates when all the brackets and braces in $D^\alpha$ are assigned a color. We will call the resulting coloring the *induced coloring*.

*Example* 7.14. Take the sequence $1]1]0000\}0\}000$ derived from $00]_1 0]_2 00000\}_1 0\}_2 0$. Algorithm 7.13 assigns it the coloring $1]_2 1]_1 0000\}_2 0\}_1 000$. See Examples 7.11 and 7.12 for more illustrations of Algorithm 7.13. Geometrically, the reader should think of the sequence $D^\alpha$ as depicting a potential limit. By semi-continuity, there is a lower bound on the dimension of the intersections of the flag elements $V_i$ with the linear spaces and quadrics depicted by the sequence. Algorithm 7.13 is the way of assigning colors so that these constraints are satisfied. Furthermore, they are the minimal set of constraints implied by semi-continuity.

*A reminder about our notation:* Recall that $\kappa$ denotes the active index of $(L_\bullet, Q_\bullet)$, i.e., the largest index $i$ such that $r_i - r_{i-1} < d_{i-1} - d_i$. Equivalently, $\kappa$ is the largest index in the sequence of brackets and braces among $\{i \mid l(i) \leq \rho(i, i-1)\}$. $x_i$ denotes the number of isotropic subspaces of the sequence contained in the singular locus of $Q^{r_i}_{d_i}$. In particular, if $x_i < s$, then $L_{n_{x_i+1}}$ denotes the smallest dimensional isotropic space in the sequence strictly containing $Q^{r_i, sing}_{d_i}$. Equivalently, $x_i$ is the number of brackets in the sequence whose positions are less than or equal to $l(\leq i)$. $L_{n_{x_i+1}}$, when it exists, is represented by the left most bracket in a position greater than $l(\leq i)$. $y_j$ denotes the index of the largest dimensional quadric containing $L_{n_j}$ in its singular locus or $y_j = k_h - s + 1$ if there are none. Equivalently, $y_j$ is the positive number immediately to the left of $]^j$ or $k_h - s + 1$ if this number is zero.

*Notation* 7.15. Given a sequence $(L_\bullet, Q_\bullet, c_\bullet)$, let $\eta(L_\bullet, Q_\bullet, c_\bullet)$ denote the index of the largest dimensional (equivalently, smallest index) quadric in the sequence for which

$$x_i \leq k - i + 1 - \frac{d_i - r_i}{2} \quad \text{and} \quad d_i + r_i = d_{k_h - s} + r_{k_h - s}.$$

Set $\eta(L_\bullet, Q_\bullet, c_\bullet) = k_h - s + 1$ if there are no indices for which these conditions hold. Equivalently, let $\eta(D, c_\bullet)$ be the index of the brace at the largest position among those braces $\}^i$ that satisfy

$$x_i \leq k - i + 1 - \frac{p(\}^i) - l(\leq i)}{2} \quad \text{and} \quad l(\leq k_h - s) - l(\leq i) = \rho(k_h - s, i).$$

Set $\eta(D, c_\bullet) = k_h - s + 1$ if there are no braces with this property.

Let $x_\kappa < \nu(L_\bullet, Q_\bullet, c_\bullet)$ be the largest index of a linear space $L_{n_j}$ in the sequence such that $n_j - r_\kappa = y_j - \kappa + j - x_\kappa - 1$. If there are no indices that satisfy this equality, set $\nu(L_\bullet, Q_\bullet, c_\bullet) = x_\kappa - 1$. Equivalently, let $\nu(D, c_\bullet)$ be the index of the bracket at the largest position among $]^j$ that satisfy $p(]^j) - l(\leq \kappa) = y_j - \kappa + j - x_\kappa - 1$. If the sequence does not contain any brackets with this property, set $\nu(D, c_\bullet) = x_\kappa - 1$. We preserve the notation from Section §4. With this preparation, we are ready to state the algorithm.

We will first state the algorithm in terms of quadric diagrams. We will then state the same algorithm geometrically.

**Definition 7.16.** *Given a colored quadric diagram $(D, c_\bullet)$, define the following colored sequences of brackets and braces.*

- *Let $(D^a, c^a_\bullet)$ be the sequence obtained from $D$ by changing $(l(\leq \kappa) + 1)$-st integer in the sequence $D$ to $\kappa$. $c^a_\bullet = c_\bullet$ is the induced coloring. (This is the same $D^a$ as in the Grassmannian case.*
- *For $x_\kappa < j \leq \nu(D, c_\bullet)$, let $(D^j, c^j_\bullet)^b$ be the sequence obtained from $(D^a, c^a_\bullet)$ by moving the $j$-th bracket from position $p(]^j) = n_j$ to position $l_{D^a}(\leq \kappa)$. The coloring $c^j_\bullet$ is the coloring induced by $c^a_\bullet$.*
- *For $k_h - s \geq i \geq \max(\kappa, \eta(D, c_\bullet))$, let $(D^i, c^i_\bullet)^e$ be the sequence obtained from $(D^a, c^a_\bullet)$ by moving the $i$-th brace from position $p(\}^i) = d_i$ to position $l_{D^a}(\leq \kappa)$. Subtract 1 from the indices of the braces with index greater than $i$ and from all the integers in the sequence that are greater than $i$. If $i = k_h - s$, change the integers that are equal to $k_h - s$ to zero. The coloring $c^i_\bullet$ is the coloring induced by $c^a_\bullet$.*
- *Finally, let $(D^\#, c^\#_\bullet)$ be the sequence obtained from $(D^a, c^a_\bullet)$ by moving the brace $\}^{\kappa-1}$ from position $p(\}^{\kappa-1}) = d_{\kappa-1}$ to position $l_{D^a}(\leq \kappa)$. Subtract one from all the indices of the braces that have index greater than $\kappa - 1$. Subtract one from every integer in the sequence that is greater than $\kappa - 1$. The coloring $c^\#_\bullet$ is the coloring induced by $c^a_\bullet$.*

Let us give a few examples of these diagrams. Let $D = 22]_1 33]_2 000\}_1 00\}_2 00\}_3 0$ be a quadric diagram for $OF(2, 4, 5; 12)$. Here $\kappa = 1$, so $D^a = 12]_1 33]_2 000\}_1 00\}_2 00\}_3 0$. The diagrams of the form $D^b$ are obtained by moving the brackets to position immediately to the left of the 1. The ones that can be moved are the ones until which the integers are strictly increasing unless they have to repeat because of Condition (D6), i.e. if the sequence looks like $12]_1 2]_2 3455]_3 5]_4 \cdots$, each of these brackets can be moved. If the sequence looks like $12]_1 22]_2 \cdots$ or $12]_1 333]_2 \cdots$, then only the first bracket can be moved. In our case, we get two diagrams of type $D^b$: $1]_1 233]_2 000\}_1 00\}_2 00\}_3 0$ and $1]_2 2]_1 33000\}_1 00\}_2 00\}_3 0$. In this case, there are no sequences of type $D^e$ or $D^\#$. If $D = 00]_1 00\}_2 0$ for $OF(1, 2; 5)$, then $\kappa = 1$. $D^a = 10]_1 00\}_2 0$. We get $D^b = 1]_1 000\}_2 0$. Now, the inequality for $D^e$ is satisfied and we also get $D^e = 0]_2 0]_1 000$ by moving the brace to a bracket immediately after the 1. Finally, if we consider $D = 3000\}_1 0\}_2 0\}_3 0$. Then $\kappa = 2$ and $D^a = 2000\}_1 0\}_2 0\}_3$. In this case, $D^\# = 1]_3 000\}_1 0\}_2 00$ is the diagram obtained by moving $\}^{1=\kappa-1}$ to the immediate left of the 2. Then the integers in the sequence are adjusted by subtracting 1 from all the integers greater than 1. These are the basic sequences used in the algorithm. Note that some of these are not quadric diagrams. We will have to give an algorithm first that turns them into quadric diagrams.

The equivalent definition in geometry is as follows.

**Definition 7.17.** *Let $(L_\bullet, Q_\bullet, c_\bullet)$ be an admissible colored sequence with active index $\kappa$. Let $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ be the sequence obtained by replacing $Q^{r_\kappa}_{d_\kappa}$ in $(L_\bullet, Q_\bullet, c_\bullet)$ with $Q^{r_\kappa+1}_{d_\kappa}$.*

• *For $x_\kappa < j \le \nu(L_\bullet, Q_\bullet, c_\bullet)$, let $(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$ be the sequence obtained from $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ by replacing $L_{n_j}$ with $L_{r'_\kappa}$. The coloring $c^j_\bullet$ is the one induced from $c_\bullet$.*

• *For $k_h - s \ge i \ge \max(\kappa, \eta(L_\bullet, Q_\bullet, c_\bullet))$, let $(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$ be the sequence obtained from $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ by replacing $Q^{r'_i}_{d'_i}$ with $L_{r'_\kappa}$. The coloring $c^i_\bullet$ is the one induced from $c_\bullet$.*

• *Let $(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$ be the sequence obtained from $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ by replacing $Q^{r'_{\kappa-1}}_{d'_{\kappa-1}}$ with $L_{r'_\kappa}$. The coloring $c^\#_\bullet$ is the one induced from $c_\bullet$.*

We first run these sequences through a normalization algorithm to turn them into admissible sequences. This algorithm is slightly easier to express in geometric language, so we first say it in geometric language and then repeat it in terms of sequences of brackets and braces.

*Algorithm* 7.18 (Normalizing a colored sequence). For this algorithm let $(L_\bullet, Q_\bullet, c_\bullet)$ denote one of the sequences $(L^a_\bullet, Q^a_\bullet, c_\bullet)$, $(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$, $(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$ or $(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$. We call the sequences produced by this algorithm the sequences derived from $(L_\bullet, Q_\bullet, c_\bullet)$. Run the following loop on $(L_\bullet, Q_\bullet, c_\bullet)$.

i. If the sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is admissible, output the sequence $(L_\bullet, Q_\bullet, c_\bullet)$. Otherwise, proceed to [ii].

ii. If $d_{k_h-s} = r_{k_h-s} + 2$ (i.e., Condition (7) is violated) in $(L_\bullet, Q_\bullet, c_\bullet)$, then let
$$\eta(L_\bullet, Q_\bullet, c_\bullet) \le l_1 < \cdots < l_u < k_h - s,$$
where $0 \le u \le k_h - s - \eta(L_\bullet, Q_\bullet, c_\bullet)$, be a set of indices such that either $l_u = k_h - s - 1$; or $l_u \ne k_h - s - 1$ and $u$ is even. For each such set of indices, form a pair of sequences
$$(L^{l_1,\dots,l_u,v}_\bullet, Q^{l_1,\dots,l_u,v}_\bullet, c^{l_1,\dots,l_u,v}_\bullet)$$
for $v = 1, 2$ by replacing the quadrics
$$Q^{r_{l_1}}_{d_{l_1}}, \dots, Q^{r_{l_u}}_{d_{l_u}} \quad \text{and} \quad Q^{r_{k_h-s}}_{d_{k_h-s}}$$
in $(L_\bullet, Q_\bullet, c_\bullet)$ with the linear spaces
$$L_{r_{l_1}+1}, \dots, L_{r_{l_u}+1} \quad \text{and} \quad L_{d_{k_h-s}-1},$$
respectively, unless $2(d_{k_h-s} - 1) = n$. If $2(d_{k_h-s} - 1) = n$, then in one of the sequences replace $Q^{r_{k_h-s}}_{d_{k_h-s}}$ with $L_{d_{k_h-s}-1}$ and in the other with $L'_{d_{k_h-s}-1}$ instead. If in addition $2k_h = n$, discard the sequence that parameterizes linear spaces that have the wrong parity for the dimension of intersection with $L_{k_h}$. The coloring $c^{l_1,\dots,l_u,v}_\bullet$ is the one assigned by Algorithm 7.13. Replace $(L_\bullet, Q_\bullet, c_\bullet)$ with the sequences thus formed that have the same dimension as $(L_\bullet, Q_\bullet, c_\bullet)$. For each of the sequences, return to Step [i] and run the Algorithm again setting $(L_\bullet, Q_\bullet, c_\bullet) = (L^{l_1,\dots,l_u,v}_\bullet, Q^{l_1,\dots,l_u,v}_\bullet, c^{l_1,\dots,l_u,v}_\bullet)$. If $d_{k_h-s} \ne r_{k_h-s} + 2$, proceed to [iii].

iii. As long as Condition (9) is violated for a sequence $(L_\bullet, Q_\bullet, c_\bullet)$, let $\mu$ be the largest index for which it is violated. Form a new sequence $(L^\flat_\bullet, Q^\flat_\bullet, c^\flat_\bullet)$ by replacing $Q^{r_\mu}_{d_\mu}$ in $(L_\bullet, Q_\bullet)$ with $Q^{r_\mu+1}_{d_\mu-1}$ unless $Q^{r_\mu+1}_{d_\mu-1}$ is already in the sequence $(L_\bullet, Q_\bullet)$. In the latter case, let $\epsilon$ be the largest integer less than $d_\mu$ that is not the dimension of the ambient space of a quadric in the sequence. Set $\alpha = d_\mu - \epsilon - 1$. Replace $Q^{r_\mu}_{d_\mu}$ by $Q^{r_\mu+\alpha+1}_\epsilon$ (i.e., a quadric of corank one more than the quadric in the linear space of dimension $\epsilon + 1$ in the sequence), instead. The coloring $c^\flat_\bullet$ is the one assigned by Algorithm 7.13. Discard the resulting sequence if its dimension is less than that of $(L_\bullet, Q_\bullet, c_\bullet)$. Repeat the process if Condition (9) is still violated. Otherwise, return to step [i] and run the loop again with $(L_\bullet, Q_\bullet, c_\bullet) = (L^\flat_\bullet, Q^\flat_\bullet, c^\flat_\bullet)$. If at the end of this procedure no sequences remain, the algorithm terminates.

Equivalently, we can state the algorithm for sequences of brackets and braces. If a sequence $(D_\bullet, c_\bullet)$ fails condition (D5), then discard the sequence. No new sequences are derived from such a sequence.

Otherwise, if the sequence fails condition (D4), let

$$\eta(D, c_\bullet) \leq \lambda_1 < \lambda_2 < \cdots < \lambda_u < k_h - s, \text{ with } 0 \leq u \leq k_h - s - \eta(D, c_\bullet)$$

be a set of indices such that either $\lambda_u = k_h - s - 1$ or $\lambda_u \neq k_h - s - 1$ and $u$ is even. For any such set of indices form two new identical sequences $(D^{\lambda_1,\dots,\lambda_u,v}, c_\bullet^{\lambda_1,\dots,\lambda_u,v})$ with $v = 1, 2$, by replacing the braces with indices $\lambda_1, \lambda_2, \dots, \lambda_u$ and $k_h - s$ with brackets at positions $l(\leq \lambda_1) + 1, l(\leq \lambda_2) + 1, \dots, l(\leq \lambda_u) + 1$ and $p(\}^{k_h - s}) - 1$. The coloring is the one induced by $c_\bullet$. Reindex the remaining braces so that they are increasing sequentially from right to left. Replace the integer $i$ with the integer $j$ if $\}^i$ remains in the sequence but its new index is $j$. Replace the integer $i$ with the integer $j$ if $\}^i$ has been replaced by a bracket and $j$ is the new index of the largest brace to the left of $\}^i$ that remains in the sequence. If there are no such braces, replace $i$ with zero. The same caveats as in the Grassmannian case apply when $2(p(\}^{k_h-s}) - 1) = n$. Namely, when $2(p(\}^{k_h-s}) - 1) = n$, in one of the sequences we have to use $]'$ instead. If $2k = 2(p(\}^{k_h-s}) - 1) = n$, we discard the sequence with the wrong parity exactly as in the Grassmannian case.

If the sequence $(D_\bullet, c_\bullet)$ satisfies (D4) but fails (D6) as long as condition (D6) is not satisfied, let $\mu$ be the largest integer for which there exists a bracket with position $l(\leq \mu) + 1$. Replace the integer at the $(l(\leq \mu) + 1)$-st position in the sequence with $\mu$ and move $\}^\mu$ one position to the left unless that position is already occupied by a brace. In the latter case, move $\}^\mu$ to the first position to the left that is not occupied. The coloring is the one induced by $c_\bullet$. Reorder the indices of the braces so that they are increasing sequentially from right to left. Suppose that the new index of the brace we moved is $\epsilon$. Then subtract one from every integer $\mu \leq i < \epsilon$. Change the integer in the $l(\leq \epsilon) + 1$ place in the sequence to $\epsilon$. Repeat the process until (D6) is satisfied. If (D4) is not satisfied for the resulting sequence, return to the previous step and run the algorithm again. In all of these cases discard a sequence if its dimension is less than the dimension of the original sequence.

To make this more concrete, consider the sequence $12]_1 3455]_2 0000\}_1 00\}_2 0\}_3 0\}_4 0\}_5 00$ which fails condition (D6). We replace it with the sequence $11]_1 2345]_2 0000\}_1 0\}_5 0\}_2 0\}_3 0\}_4 000$. This sequence still fails condition (D6), so we repeat the process to obtain $11]_1 2344]_2 500\}_5 0\}_1 00\}_2 0\}_3 0\}_4 000$. Now the condition (D4) is not satisfied, so we replace the sequence with two copies of the sequence $11]_1 2344]_2 00]_5 00\}_1 00\}_2 0\}_3 0\}_4 000$. Similarly, consider $200\}_1 0\}_2$. This sequence does not satisfy condition (D4). We replace it by sequences $00]'_1 00\}_2$ and $0]_2 0]_1 00$. The two other sequences $00]_1 00\}_2$ and $0]_2 0]'_1 00$ parameterize linear spaces in the other connected component, so they are the ones that are discarded. The reader should see Examples 7.11 and 7.12 and the examples below.

It is important to note that no calculation is necessary to decide whether a sequence has smaller dimension while running these algorithms. When we move a bracket $]^j$ or a brace $\}^i$ from position $p_1$ to position $p_2$ during the algorithm, we induce a permutation of the colors. We will prove that the resulting sequence of brackets and braces always has strictly smaller dimension unless the color assigned to $]^j$ or $\}^i$ is strictly larger than the color assigned to every bracket and brace between positions $p_1$ and $p_2$. Equivalently, when a quadric or linear space is replaced by a smaller dimensional one, the quadrics and linear spaces with dimension in between have to have color strictly smaller than the quadric or linear space being replaced. Otherwise, the resulting restriction varieties have smaller dimension. We say that the induced coloring preserves dimension if this property holds.

Note that in all the examples following the definition of the different diagrams, the induced coloring preserved the dimension. If we consider $D = 22]_2 33]_1 000\}_1 00\}_2 00\}_3 0$ instead (the colors of the first two brackets are swapped from the previous example), then only $D^b = 1]_2 233]_1 000\}_1 00\}_2 00\}_3 0$ can have the same dimension. In $D^b = 1]_2 2]_1 33000\}_1 00\}_2 00\}_3 0$, the coloring does not preserve the dimension (moving the bracket from position 4 to 1, crosses a bracket with larger color). The corresponding variety has smaller dimension and will not occur as a component of the support of the limit).

We can now state the main algorithm. We will use geometric language and leave it to the reader to formulate the combinatorial statement by replacing every appearance of $(L_\bullet, Q_\bullet)$ by $D$.

*Algorithm* 7.19. Let $V(L_\bullet, Q_\bullet, c_\bullet)$ be a restriction variety in the orthogonal flag variety.

Step 1. If $V(L_\bullet, Q_\bullet)$ is saturated (i.e., a Schubert variety), then output $V(L_\bullet, Q_\bullet, c_\bullet)$ and stop. The algorithm terminates. Otherwise, proceed to Step 2.

Step 2. Replace $V(L_\bullet, Q_\bullet, c_\bullet)$ by the following restriction varieties depending on the case and stop.

(1) If Condition (8) is not satisfied for $(L^a_\bullet, Q^a_\bullet, c_\bullet)$, replace $V(L_\bullet, Q_\bullet, c_\bullet)$ by the restriction varieties associated to admissible sequences derived from $(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$, for $x_\kappa < j \leq \nu(L_\bullet, Q_\bullet, c_\bullet)$ and $(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$, for $k_h - s \geq i \geq \max(\kappa, \eta(L_\bullet, Q_\bullet, c_\bullet))$, that have the same dimension as $V(L_\bullet, Q_\bullet, c_\bullet)$.

(2) If Condition (8) is satisfied for $(L^a_\bullet, Q^a_\bullet, c_\bullet)$, replace $V(L_\bullet, Q_\bullet, c_\bullet)$ by the restriction varieties associated to admissible sequences derived from $(L^a_\bullet, Q^a_\bullet, c_\bullet)$, $(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$, $(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$, for $x_\kappa < j \leq \nu(L_\bullet, Q_\bullet, c_\bullet)$ and $(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$, for $k_h - s \geq i \geq \max(\kappa, \eta(L_\bullet, Q_\bullet, c_\bullet))$, that have the same dimension as $V(L_\bullet, Q_\bullet, c_\bullet)$.

One can say when a restriction variety produced by the algorithm will have the same dimension as $V(L_\bullet, Q_\bullet, c_\bullet)$ purely in terms of the properties of the sequence $(L_\bullet, Q_\bullet, c_\bullet)$. We refrained from doing this above to avoid further complicating the statement of the algorithm. Although this statement of the algorithm sounds cleaner, in practice it is much harder to compute the dimension of these sequences than to remember when they will have smaller dimension.

Despite initially sounding more complicated, it is simpler in practice to phrase Step 2 of Algorithm 7.22 as follows and be precise as to when the resulting sequences will have the same dimension as $(L_\bullet, Q_\bullet, c_\bullet)$.

Step 2 Replace $V(L_\bullet, Q_\bullet, c_\bullet)$ by the following restriction varieties depending on the case and stop.

(1) If $x_\kappa = s$, $\kappa < k_h - s$ and $2(k_h - s - \kappa) + 3 = d_\kappa - r_\kappa$ in $(L_\bullet, Q_\bullet, c_\bullet)$, then replace $V(L_\bullet, Q_\bullet, c_\bullet)$ by restriction varieties associated to sequences derived from $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ and $(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$ where the induced colorings preserve dimension.

(2) If $x_\kappa = s$ and either $\kappa = k_h - s$ or $2(k_h - s - \kappa) + 3 < d_\kappa - r_\kappa$ in $(L_\bullet, Q_\bullet, c_\bullet)$, then replace $V(L_\bullet, Q_\bullet, c_\bullet)$ by restriction varieties associated to the sequences derived from $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ where the induced coloring preserves dimension.

(3) If $x_\kappa < s$ in $(L_\bullet, Q_\bullet, c_\bullet)$ and $x'_\kappa \geq k_h - \kappa + 1 - \frac{d'_\kappa - r'_\kappa}{2}$ (i.e., Condition (8) is satisfied) for $(L^a_\bullet, Q^a_\bullet, c_\bullet)$, then replace $V(L_\bullet, Q_\bullet, c_\bullet)$ by restriction varieties associated to the sequences derived from $(L^a_\bullet, Q^a_\bullet, c_\bullet)$, $(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$, for $x_\kappa < j \leq \nu(L_\bullet, Q_\bullet, c_\bullet)$, and $(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$, for $k_h - s \geq i \geq \max(\kappa, \eta(L_\bullet, Q_\bullet, c_\bullet))$ and where the induced colorings preserve dimension.

(4) If $x_\kappa < s$ in $(L_\bullet, Q_\bullet, c_\bullet)$ and $x'_\kappa < k_h - \kappa + 1 - \frac{d'_\kappa - r'_\kappa}{2}$ (i.e., Condition (8) fails) for $(L^a_\bullet, Q^a_\bullet, c_\bullet)$, then replace $V(L_\bullet, Q_\bullet, c_\bullet)$ by restriction varieties associated to the sequences derived from $(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$, for $x_\kappa < j \leq \nu(L_\bullet, Q_\bullet, c_\bullet)$, and $(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$, for $k_h - s \geq i \geq \max(\kappa, \eta(L_\bullet, Q_\bullet, c_\bullet))$ where the colorings preserve dimension.

*Remark* 7.20. The geometric meaning of Algorithm 7.19 is clear. If the restriction variety $V(L_\bullet, Q_\bullet, c_\bullet)$ is not a Schubert variety, we apply Degeneration 5.7. The dimension of intersection of the new singular locus of the quadric $Q^{r_\kappa+1}_{d_\kappa}$ with the linear spaces $W_i$ (for $i \geq a$) in the flag may increase by one, provided that the resulting locus has the same dimension as our original variety. Similarly, these dimensions may remain unaltered unless disallowed by Condition (8). The algorithm checks which of these possibilities give varieties of the same dimension as $V(L_\bullet, Q_\bullet, c_\bullet)$.

*Remark* 7.21. The reader can check that in case $h = 1$, Algorithm 7.19 reduces to Algorithm 5.9.

We define degeneration paths for orthogonal flag varieties as in Definition 5.11 except we replace every reference to Algorithm 5.9 with Algorithm 7.19.

**Theorem 7.22.** *Let $V$ be a restriction variety in $OF(k_1, \ldots, k_h; n)$. Then*

$$[V] = \sum [V_i],$$

*where $V_i$ are the restriction varieties obtained from $V$ by applying Algorithm 7.19. In particular, the coefficient $c^\mu_\lambda$ in the expression*

$$[V] = \sum c^\mu_\lambda \sigma^\mu_\lambda$$

*is equal to the number of degeneration paths starting with $V$ and ending with Poincaré dual of the class of $\sigma^\mu_\lambda$. Furthermore, the algorithm respects marking.*

Before the proof, we give three examples of the algorithm. The reader will find it instructive to run the algorithm on these examples. The examples also emphasize the difference between orthogonal Grassmannians and orthogonal flag varieties. The reader might want to run Algorithm 5.9 on the projection of these restriction varieties to $OG(k_h, n)$ and compare the results. Note that the projection to $OG(k_h, n)$ is obtained by simply forgetting the subscripts.

*Example* 7.23. We calculate the class of the restriction variety

$$V(L_2[1] \subset L_3[2] \subset Q^0_8[1] \subset Q^0_9[2])$$

in $OF(2, 4; 9)$.

$$\begin{array}{ccc} 22]_1 2]_2 000\}_1 000\}_2 \leftarrow & 00]_1 0]_2 00000\}_1 0\}_2 & \rightarrow 1]_1 22]_2 000\}_2 00\}_1 0 \\ & \downarrow & \\ & 1]_2 1]_1 2000\}_2 0\}_1 00 & \end{array}$$

We conclude that the class of the variety is

$$\sigma^{3^1,0^2}_{3^1,2^2} + \sigma^{3^2,1^1}_{4^1,3^2} + \sigma^{3^2,2^1}_{4^2,3^1}$$

*Example* 7.24. We calculate the class of the restriction variety

$$V(Q^2_5[1], Q^2_6[2], Q^1_7[3], Q^0_8[4])$$

in $OF(1, 2, 3, 4; 8)$. When we specialize $Q^2_5[1]$ to $Q^3_5[1]$,

$$23000\}_1 0\}_2 0\}_3 0\}_4$$

splits into
$2300]'_1 00\}_2 0\}_3 0\}_4$, $200]_2 0]_1 000\}_3 0\}_4$, $00]_3 0]_2 0]'_1 0000\}_4$, $1]_4 00]_2 0]'_1 000\}_3 0$, $1]_4 1]_3 00]'_1 00\}_2 00$, and $0]_4 0]_3 0]_2 0]_1 0000$. The class of the restriction variety is

$$\sigma^{3^1,2^2,1^3,0^4} + \sigma^{1^3,0^4}_{1^2,0^1} + \sigma^{3^1,0^4}_{2^3,1^2} + \sigma^{3^1,1^3}_{3^4,1^2} + \sigma^{3^1,2^2}_{3^4,2^3} + \sigma_{3^4,2^3,1^2,0^1}.$$

It would also be instructive for the reader to calculate the class of the restriction variety

$$V(Q^0_5[1], Q^0_6[2], Q^0_7[3], Q^0_8[4])$$

in $F(1, 2, 3, 4; 8)$. The answer is

$$\sigma^{3^1,2^2,1^3,0^4} + \sigma^{1^3,0^4}_{1^2,0^1} + \sigma^{2^2,0^4}_{2^3,0^1} + \sigma^{2^2,1^3}_{3^4,0^1} + 2\sigma^{3^1,0^4}_{2^3,1^2} + 2\sigma^{3^1,1^3}_{3^4,1^2} + 2\sigma^{3^1,2^2}_{3^4,2^3} + 4\sigma_{3^4,2^3,1^2,0^1}.$$

*Example* 7.25. As a final example, we calculate the class of

$$V(L_2[1] \subset L_4[2] \subset Q^0_8[1] \subset Q^0_9[2])$$

in $F(2,4;11)$. Since this example is large, we will skip some intermediate steps.

$$
\begin{array}{lll}
\multicolumn{3}{l}{00]_100]_20000\}_10\}_200 \to 1]_1200]_2000\}_20\}_1000 \xrightarrow{\times 2} 1]_1100]_20]_2000\}_1000 \to 1]_111]_200]_2000\}_1000} \\
\downarrow & \quad\quad \downarrow & \\
22]_100]_2000\}_100\}_200 & 1]_11]_212000\}_20\}_1000 & \\
\downarrow \qquad\qquad\qquad \searrow & & \\
1]_1200]_2000\}_100\}_200 & \multicolumn{2}{l}{11]_100]_2000\}_10\}_2000 \xrightarrow{\times 2} 11]_11]_200]_200\}_10000 \to 11]_11]_21]_2000\}_10000} \\
\times 2 \downarrow \qquad\qquad \searrow & & \times 2 \searrow \\
1]_1100]_20]_10000\}_200 & 1]_122]_20000\}_100\}_200 & 11]_100]_20]_1000\}_2000 \to 11]_11]_200]_1000\}_2000 \\
& \quad \downarrow \qquad\qquad\qquad \searrow & \\
& 1]_11]_222000\}_100\}_200 & 1]_111]_22000\}_10\}_2000
\end{array}
$$

We conclude that the class is

$$2\sigma^{3^1}_{5^1,3^2,1^2} + \sigma^{4^2,3^1}_{5^1,4^2} + 2\sigma^{4^1}_{4^1,3^2,2^2} + 2\sigma^{3^2}_{4^1,3^2,1^1} + 2\sigma^{2^2}_{5^1,2^2,1^1} + \sigma^{4^1,2^2}_{5^1,4^2} + \sigma^{4^1,3^2}_{5^1,3^2}.$$

*Proof of Theorem 7.22.* The proof of Theorem 7.22 is very similar to the proof of Theorem 5.12. We first check that Algorithm 7.19 transforms admissible colored sequences to admissible colored sequences. We then interpret replacing $Q^{r_\kappa}_{d_\kappa}$ with $Q^{r_\kappa+1}_{d_\kappa}$ in Step 1 of Algorithm 7.19 as Degeneration 5.7. We show that the flat limit is supported along the restriction varieties described in the algorithm. We conclude the proof by showing that the flat limit is reduced along the generic point of each of these restriction varieties.

If the sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is saturated, then the corresponding variety is a Schubert variety. In this case, the algorithm has achieved its goal. We may, therefore, assume that $(L_\bullet, Q_\bullet, c_\bullet)$ is not saturated. Throughout the proof Step 2 of the Algorithm 7.19 will refer to the more precise formulation and the cases will be the four cases that occur in that formulation.

The sequence $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ may fail to satisfy Condition (8) for the index $\kappa$. During the proof of Theorem 5.12, we showed that then equality must hold in Condition (8) for all indices $i > \kappa$ and if $\kappa = k_h - s$, Condition (7) is satisfied by the sequence. Furthermore, adding an additional linear space (not already contained in the sequence) to the singular locus of a quadric does not effect Condition (8) for that quadric. Consequently, Condition (8) is satisfied for the sequences $(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$ in case (1) of Step 2, $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ in the cases (2),(3) and (4) of Step 2, and for the sequences $(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$ and $(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$ in cases (3) and (4) of Step 2 of Algorithm 7.19. If the left-hand-side of the inequality is at most 1/2 larger than the right-hand-side in Condition (8) for an index $\mu$, then either $d_{\mu-1} > d_\mu + 1$ or $r_{\mu-1} < r_\mu$ in $(L_\bullet, Q_\bullet, c_\bullet)$. Using this observation, it is straightforward to see that both Steps ii and iii of Algorithm 7.18 preserve Condition (8). Hence, every sequence output by the algorithm satisfies Condition (8).

Next we observe that the sequences output by the algorithm preserve Conditions (1)-(6) in Definition 4.2. Conditions (4) and (5) hold by construction. In the proof of Theorem 5.12, we checked that the sequence $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ satisfies Conditions (1)-(6). Replacing a linear space by a smaller dimensional linear space does not affect Conditions (1)-(3) and (6). Hence, Conditions (1)-(6) are satisfied for the sequences $(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$. Replacing a quadric with the linear space $L_{r'_\kappa}$ clearly preserves Conditions (1), (3) and (6). Since the sequence satisfies Condition (8), adding a linear space to the singular locus of a quadric (whenever this linear space is not already in the sequence) does not violate Condition (2). We conclude that the sequences $(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$ in case (1) of Step 2 and $(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$ in cases (3) and (4) of Step 2 of Algorithm 7.19 satisfy Conditions (1)-(6). Step ii of Algorithm 7.18 clearly preserves Conditions (3) and (6). The proof of Condition (1) given during the proof of Theorem 5.12 remains valid. Finally, Condition (2) holds since each time a quadric $Q^{r_i}_{d_i}$ is replaced by a linear space, the linear space is not already contained in the sequence and lies in the singular locus of the quadrics contained in $Q^{r_i}_{d_i}$. Hence Condition (2) is preserved. Finally, it is straightforward to see that Step iii of Algorithm 7.18 preserves

Conditions (1)-(6). We conclude that every sequence output by the algorithm satisfies Conditions (1)-(6).

In Step 2 of Algorithm 7.19 we have seen that the sequences $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$ and $(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$ satisfy Conditions (1)-(6) and (8). Since $\kappa < k_h - s$, Condition (7) is satisfied for both sequences. Condition (9) is clearly satisfied for $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$. Condition (9) is also satisfied for $(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$. This needs to be checked only for the new linear space. It holds since the quadric with index $\kappa - 1$ in $(L_\bullet, Q_\bullet)$ has been removed, the singular locus of $Q^{r^\#_{\kappa-1}}_{d^\#_{\kappa-1}}$ is at least codimension two in $L_{r^\#_\kappa}$ by Condition (6) for $(L_\bullet, Q_\bullet, c_\bullet)$. We conclude that both sequences are admissible.

In Step 2 of Algorithm 7.19, if $\kappa > k_h - s$; or if $\kappa = k_h - s$ and $d_\kappa - r_\kappa < 3$ in $(L_\bullet, Q_\bullet, c_\bullet)$, then $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$ is admissible. However, if $\kappa = k_h - s$ and $d_\kappa - r_\kappa = 3$, then Condition (7) fails for $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$. After applying Step ii of Algorithm 7.18, all the sequences output satisfy Condition (7). However, they may fail to satisfy Condition (9) for some indices $i < \eta(L^a_\bullet, Q^a_\bullet, c_\bullet)$. Sequences output by Step iii satisfy Condition (9) but may fail to satisfy Condition (7) again. Note that each time we apply Condition (7) the number of quadrics in the sequence strictly decreases. Since there are finitely many quadrics in the sequence, the process must stop leading to admissible sequences.

In cases (3) and (4) of Algorithm 7.19, the sequences $(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$ and $(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$ satisfy Conditions (1)-(8), but may fail to satisfy Condition (9) for $i \leq \kappa$. It is easy to see that after running Step iii, Condition (9) is also satisfied and the sequences output are admissible. Finally, the sequence $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ in case (3) of Step 2 may fail Condition (9) for the index $\kappa$. If $\kappa < k_h - s$, then running Step iii of Algorithm 7.18 outputs an admissible sequence. If $\kappa = k_h - s$, then running Step iii of Algorithm 7.18 may output a sequence that fails Condition (7). As in the discussion of case 2 of Step 2, repeated applications of Steps ii and iii of Algorithm 7.18 result in admissible sequences. We conclude that Algorithm 7.19 replaces admissible colored sequences with admissible colored sequences. We can, therefore, apply the algorithm to each of the resulting sequences. Since at each stage either the number of quadrics decreases by at least one or the corank of at least one quadric strictly increases, eventually the sequences must become saturated. We conclude that repeated application of the algorithm results in sequences associated to Schubert varieties.

We interpret replacing $Q^{r_\kappa}_{d_\kappa}$ with $Q^{r_\kappa+1}_{d_\kappa}$ in Step 1 of Algorithm 7.19 as Degeneration 5.7 and show that the algorithm describes the components of the support of the flat limit and that the flat limit is reduced at the generic point of each of these components. We combine the analysis in the proof of Theorem 5.9 with a study of the fiber dimension of the morphism

$$\pi : OF(k_1, \dots, k_h; n) \to OG(k_h, n).$$

Now Observation 5.13 has to hold for each vector space $V_u$, for $1 \leq u \leq h$.

*Observation* 7.26. The linear spaces $V_u$ parameterized by the flat limit $V(L_\bullet, Q_\bullet, c_\bullet)(0)$ have to intersect the linear spaces $L_{n_j}[c_j](0)$ in a subspace of dimension at least $\#\{l \leq j | c_l \leq u\}$ and the quadrics $Q^{r_i}_{d_i}[c_{k_h-i+1}](0)$ in a subspace of dimension at least $\#\{l \leq k_h - i + 1 | c_l \leq u\}$. Furthermore, they intersect $Q^{r_i,sing}_{d_i}[c_{k_h-i+1}](0)$ in a subspace of dimension at least $\#\{l \leq x_i | c_l \leq u\}$.

Let $Y$ be an irreducible component of the support of the flat limit. As in the case of Grassmannians, Observation 7.26 allows us to build a minimal sequence $(\tilde{L}_\bullet, \tilde{Q}_\bullet, \tilde{c}_\bullet)$ such that the closure of the locus of linear spaces satisfying the rank conditions imposed by this sequence contains $Y$. We complete the sequence $(L^a_\bullet, Q^a_\bullet)$ to a sequence of isotropic linear spaces and quadrics of consecutive dimensions satisfying Conditions (4) and (5) of Definition 4.2. We then select the linear spaces and quadrics in our sequence where the dimension of intersection with the linear space $W_h$ parameterized by a general point of $Y$ jumps. At each jump we specify the smallest linear space among $W_1, \dots, W_h$ for which the jump occurs. We thus obtain a colored sequence. Observation 7.26 translates to the statement that the $j$-th linear space or quadric of color at most $u$ (counting in increasing dimension) in the new sequence has dimension at most that of the $j$-th linear space or quadric of color at most $u$ in $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ for every $1 \leq u \leq h$. The fiber dimension of the projection $\pi$ restricted to the locus imposed by the sequence is governed by the coloring $\tilde{c}_\bullet$. The expression $\dim(\tilde{c}_\bullet)$ gives the generic fiber dimension of $\pi$ on this locus. Accordingly, if the colors in two consecutive positions are swapped, the dimension of the fiber increases by one when a larger color is associated to the smaller member in the sequence. For example, the color sequence $1,2,3,1,2$ has fiber dimension 2 less than the color sequence $1,3,2,2,1$. By Observation 7.26, there can only be a color swap between colors $c_i = u < c_{i+j} = v$ if the dimension of the subspace of $W_v$ intersecting the $i$-th constraint in the sequence is at least one larger than before. Correspondingly, the $(i+j)$-th linear space or quadric has to shrink to the position of that of at least one less than the position of the $i$-th linear space or quadric. The fiber dimension increases by at most $j$ and the increase is precisely $j$ only when every color $c_i, \dots, c_{i+j-1}$ is smaller than $c_{i+j}$. Now we combine this observation with the dimension counts in the proof of Theorem 5.12 for the image of the projection $\pi$. We use the same notation.

- When we replace the $(i+j)$-th linear space in $(L^a_\bullet, Q^a_\bullet)$ with a linear space in the $i$-th position, we saw that the dimension of the image of $\pi$ decreases by $n'_{i+j} - n^*_i + j + y'_{i+j} - y^*_i$. By Conditions (6) and (9) for $(L_\bullet, Q_\bullet)$ this is at least $j$, with equality when $n^*_i = r_\alpha$ for some $\alpha$ and $n'_{i+j} - r_\alpha = y'_{i+j} - \alpha + j$ in $(L^a_\bullet, Q^a_\bullet)$. On the other hand, the fiber dimension can increase by at most $j$, with equality if the color of $L^a_{n_{i+j}}$ is larger than the color of every linear space $L^a_{n_t}$ for $i \le t < i+j$. We conclude that replacing a linear space by a smaller dimensional linear space either strictly decreases the dimension or may keep it the same in the case just described.
- When we replace the quadric with index $i$ with a quadric in the $(i+j)$-th position, the dimension of the image of $\pi$ changes by $d'_i - d^*_{i+j} + x'_i - x^*_{i+j}$. The decrease in the image of $\pi$ is at least $j$ with equality only if every time the quadric is shrunk by one without coinciding with a quadric already contained in the sequence, the number of linear spaces of the sequence contained in its singular locus also increases by one. The fiber dimension can increase by at most $j$, with equality precisely when all the quadrics of index $i \le t < i+j$ in $(L^a_\bullet, Q^a_\bullet)$ have color strictly less than the color of $Q^{r'_{i+j}}_{d'_{i+j}}$. Hence, replacing a quadric with a smaller dimensional one either strictly decreases the dimension or may keep it the same in the case described.
- When we replace the quadric with index $i$ with a linear space in the $j$-th position, the dimension of the image of $\pi$ changes by
$$(k_h - i + 1 - \lfloor \frac{d'_i - r'_i}{2} \rfloor - x'_i) + (n^*_j - \lceil \frac{d'_i + r'_i}{2} \rceil + k_h - s - y^*_j) + (-k_h + s + i).$$
The first sum is strictly negative unless Condition (8) is violated or there is equality in Condition (8) for the index $i$, in which case it is zero. The second term is less than or equal to $j - s - 1$ with equality only if either $d'_i + r'_i = d'_{k_h - s} + r'_{k_h - s}$ and equality holds in Condition (8) for the index $i$; or $d'_i + r'_i = d'_{k_h - s} + r'_{k_h - s} + 1$, $x_i = s$, $n_j* > n_s$ and equality holds in Condition (8) for the index $i$. Hence, the dimension of the image decreases by at least $k_h - i - j + 1$ with equality only if we have one of the cases described. The fiber dimension increases by at most $k_h - i - j + 1$ with equality when all the linear spaces and quadrics between $Q^{r'_i}_{d'_i}$ and the new linear space have color strictly less than the color of $Q^{r'_i}_{d'_i}$.

We conclude that the increase in the fiber dimension of $\pi$ when restricted to one of the restriction varieties we constructed can equal at most the decrease in the dimension of the image of $\pi$. Hence, the irreducible component $Y$ of the support of the flat limit has to be a component of one of these loci associated to a sequence.

Note that we have limited the possible irreducible components of the support of the flat limit of Degeneration 5.7 to a small list. However, not all these possibilities occur as limits. For instance, after we apply the degeneration to $24000\}_1 0\}_2 0\}_3 0\}_4$, according to Step 2, only the last two of the following four cycles

$$1]_4 1]_3 000\}_1 0\}_2 00, \quad 1]_4 2000\}_1 0\}_2 0\}_3 0, \quad 22]_3 000\}_1 0\}_2 00\}_4, \quad 23000\}_1 0\}_2 0\}_3 0\}_4$$

that have the same dimension occur in the limit. In Steps 4 and 5 of Algorithm 7.19, the same phenomenon can occur. For example, consider the cycle $1344]_2 000\}_1 00\}_1 0\}_2 0\}_1 0$. Applying the algorithm results in the cycle $1244]_2000\}_100\}_10\}_20\}_10$. Even though, the cycle $1]_2244000\}_100\}_10\}_20\}_10$ has the same dimension, it does not occur in the limit. Hence, we require a more subtle analysis to further restrict the possible limits. The argument is identical to the one given in the Grassmannian case.

*Observation* 7.27. At a general point of $V(L_\bullet, Q_\bullet, c_\bullet)$, the linear space everywhere tangent to $Q^{r_i}_{d_i}$ along $W_h \cap Q^{r_i}_{d_i}$ intersects the linear space $L_{n_j}$ with $n_j \geq x_i$ in a subspace of dimension $j - x_i + r_i$. By semi-continuity, in the limit, the linear space everywhere tangent to $Q^{r_i}_{d_i}(0)$ along $W_h(0) \cap Q^{r_i}_{d_i}(0)$ intersects $L_{n_j}(0)$ in a subspace of dimension at least $j - x_i + r_i$.

Similarly, at a general point of $V(L_\bullet, Q_\bullet, c_\bullet)$, the linear space tangent to $Q^{r_i}_{d_i}$ along the linear space $W_h \cap Q^{r_i}_{d_i}$ intersects $Q^{r_l}_{d_l}$ for $l > i$ in a subspace of dimension $s + k_h - l + 1 + r_i - x_i$. By semi-continuity, in the limit, the linear space tangent to $Q^{r_i}_{d_i}(0)$ along $W_h(0) \cap Q^{r_i}_{d_i}(0)$ intersects $Q^{r_l}_{d_l}(0)$ in a subspace of dimension at least $s + k_h - l + 1 + r_i - x_i$.

*Observation* 7.28. Suppose that $x_\kappa = s$, $\kappa < k_h - s$ and $2(k_h - s - \kappa) = d_\kappa - r_\kappa$ in $(L_\bullet, Q_\bullet, c_\bullet)$. Let $i \leq \kappa - 1$ be the index of a quadric for which equality holds in Condition (8) and $d_i + r_i = d_{k_h-s} + r_{k_h-s} + 1$. Then each irreducible component of the space of $(k_h - i)$-dimensional linear subspaces of $W_h$ contained in $Q^{r_{i+1}}_{d_{i+1}}$ determines an irreducible component of the space of $(k_h - i + 1)$-dimensional subspaces of $W_h$ contained in $Q^{r_i}_{d_i}$. The component parameterized by $V(L_\bullet, Q_\bullet, c_\bullet)$ is characterized by the following: if

$$\dim((W_h \cap Q^{r_{i+1}}_{d_{i+1}}) \cap Q^{r_{i+1}, sing}_{d_{i+1}}) = x_{i+1},$$

then $\dim((W_h \cap Q^{r_i}_{d_i}) \cap Q^{r_{i+1}, sing}_{d_{i+1}}) = x_{i+1}$. The other component has $\dim((W_h \cap Q^{r_i}_{d_i}) \cap Q^{r_{i+1}, sing}_{d_{i+1}}) = x_{i+1} + 1$.

Finally, in Step ii of Algorithm 7.18, the Condition that if $l_u \neq k_h - s - 1$, then $u$ has to be even does not yet follow from our dimension count (see Example 7.12). We need the following observation.

*Observation* 7.29. If $\kappa = k_h - s$, $x_{k_h-s} =$ and $d_{k_h-s-1} + r_{k_h-s-1} = d_{k_h-s} + r_{k_h-s}$ in $V(L_\bullet, Q_\bullet, c_\bullet)$, then the linear subspaces of dimension $k_h - s + 2$ contained in $Q^{r_{k_h-s-1}}_{d_{k_h-s-1}}$ have two irreducible components (see Proposition 2.2). These components are distinguished by their parity of intersection with linear spaces in the half-dimensional components in $Q^{r_{k_h-s-1}}_{d_{k_h-s-1}}$. In the flat limit, these linear spaces have to continue to lie in the same irreducible component. We conclude that in Step ii of Algorithm 7.18, general linear spaces parameterized by restriction varieties with $l_u \neq k_h - s - 1$ and $u$ odd cannot occur in the flat limit.

The dimension counts together with Observations 7.27, 7.28 and 7.29 imply that $Y$ has to be an irreducible component of the restriction varieties that are output by Algorithm 7.19. We now check this claim and show that each of the varieties in Algorithm 7.19 occur with multiplicity one in the limit. The proof will be identical to the Grassmannian case. For each irreducible component of a potential limit, we exhibit a cycle that intersects $V(L_\bullet, Q_\bullet, c_\bullet)$ and that irreducible component in a reduced point and that does not intersect any of the other irreducible components of the potential limits. The calculations are almost identical to the Grassmannian case. This will conclude the proof.

**Case 1:** If $x_\kappa = s$ and $2(k_h - s - \kappa) + 3 < d_\kappa - r_\kappa$ in $V(L_\bullet, Q_\bullet, c_\bullet)$, then by Condition (7) for $V(L_\bullet, Q_\bullet, c_\bullet)$, for all the quadrics in $(L^a_\bullet, Q^a_\bullet)$ there is strict inequality in Condition (8). Furthermore, by Conditions (6) and (9) for $V(L_\bullet, Q_\bullet, c_\bullet)$, $n_j - r_i > y_j - i + j - x_i$ for every linear space $n_j > x_i$. Hence, by our dimension calculations, replacing a linear space with a smaller dimensional linear space or a quadric by a linear space or smaller dimensional quadric results in a strictly smaller dimensional variety. We conclude that $Y$ has to be a component of $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$. To show that the multiplicity is one we may, without loss of generality, assume that $\kappa = 1$ and $x_\kappa = 0$ and $n = d_\kappa + r_\kappa + 1$. Let $\beta_i = n - d_i + 1$. Consider the Schubert variety $S$ defined with respect to the sequence

$$L_{\beta_\kappa}[c_\kappa] \subset L_{\beta_{\kappa+1}}[c_{\kappa+1}] \subset \cdots \subset L_{\beta_{k_h-s}}[c_{k_h-s}].$$

Then it is immediate that $S$ intersects both $V(L_\bullet, Q_\bullet, c_\bullet)$ and $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$ in a unique reduced point. Hence, $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$ occurs in the limit with multiplicity one.

**Case 2:** If $\kappa = k_h - s$, $x_{k_h-s} = s$ and $r_{k_h-s} + 3 = d_{k_h-s}$ in $V(L_\bullet, Q_\bullet, c_\bullet)$, then the quadric $Q_{d'_\kappa}^{r'_\kappa}$ in $(L^a_\bullet, Q^a_\bullet)$ is necessarily reducible consisting of two linear spaces of dimension $d'_\kappa - 1$. If $2(d'_\kappa - 1) = n$, then the two linear spaces belong to distinct connected components of half-dimensional linear spaces. By Conditions (6) and (9) for $V(L_\bullet, Q_\bullet, c_\bullet)$, replacing a linear space with a smaller dimensional one leads to a strictly smaller dimensional variety. By our dimension counts, replacing a quadric $Q_{d'_i}^{r'_i}$ with a linear space $L_{n_j}$ may lead to a variety of the same dimension provided $n_j \geq r_i + 1$. However, if $n_j > r_{i+1}$, then Condition (9) must be violated for one of the quadrics of index larger than $i$. "The variation of tangent spaces" forces us to replace that quadric with a linear space of dimension smaller than $n_j$, hence leading to a smaller dimensional locus. Combining this discussion with Observation 7.29, we conclude that $Y$ has to be an irreducible component of a restriction variety described by Step ii of Algorithm 7.18. Now we check that they each occur with multiplicity one. Suppose replacing the quadrics with indices

$$\eta(L_\bullet, Q_\bullet, c_\bullet) \leq l_1 < \cdots < l_u < k_h - s$$

with linear spaces $L_{r_{l_1}+1}, \ldots, L_{r_{l_u}+1}, L_{d_{k_h-s}-1}$ leads to a locus of the same dimension as $V(L_\bullet, Q_\bullet, c_\bullet)$. For the local calculation we may assume that $\eta(L_\bullet, Q_\bullet, c_\bullet) = 1$, $x_{\eta(L_\bullet, Q_\bullet, c_\bullet)} = 0$ and $n = d_{k_h-s} + r_{k_h-s} + 1$. Then each of the restriction varieties

$$V(L_\bullet^{l_1,\ldots,l_u,v}, Q_\bullet^{l_1,\ldots,l_u,v}, c_\bullet^{l_1,\ldots,l_u,v})$$

are Schubert varieties. It is straightforward to see that their duals intersect $V(L_\bullet, Q_\bullet, c_\bullet)$ in a unique, reduced point. When we intersect the dual with $V(L_\bullet, Q_\bullet, c_\bullet)$, the linear spaces and the quadrics with index not equal to one of the $l_1, \ldots, l_u$ in the sequence $V(L_\bullet, Q_\bullet, c_\bullet)$ determine unique points that need to be contained in $W_c$ if the color of the linear space or quadric is $c$. The other linear spaces are then determined by "the variation of tangent spaces". The assumption that if $l_u \neq k_h - s - 1$, then $u$ is even guarantees that the linear space thus determined lies in the correct connected component. It follows that each of these varieties occur with multiplicity one in the limit.

**Case 3:** If $\kappa < k_h - s$, $x_\kappa = s$ and $2(k_h - s - \kappa) + 3 = d_\kappa + r_\kappa$ for $(L_\bullet, Q_\bullet, c_\bullet)$, then by Conditions (6) and (9) replacing a linear space by a smaller dimensional one leads to strictly smaller dimensional loci. Replacing a quadric with index $i$ for which equality holds in Condition (8) with $L_{r_{i+1}}$ may lead to the same dimensional loci, but by Observation 7.28 except when $i = \kappa$, the loci that can occur as the flat limit are proper subvarieties of every irreducible component. We conclude that $Y$ has to be a component of $V(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$ or $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$. For the local calculation to show that every component occurs with multiplicity one, we may assume that $\kappa - 1 = 1$, $n = d_\kappa + r_\kappa + 1$ and $x_\kappa = s = 0$. Let $S$ (respectively, $S'$) be defined exactly as in Case 1 (respectively, by replacing $L_{k_h-s}$ with $L'_{k_h-s}$). It is clear that $S$ and $S'$ intersect the two components of $V(L_\bullet, Q_\bullet, c_\bullet)$ and $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$ in a single, reduced point and do not intersect $V(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$. If $V(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$ has the same dimension as $V(L_\bullet, Q_\bullet, c_\bullet)$, let $\beta_i = n - d_{i+1} + 1$ for $i = 1, \ldots, k_h - 1$. Let $\alpha_1 = n - r'_\kappa + 1$. Let $T$ be the Schubert variety defined with respect to the following sequence

$$L_{\beta_1}[c_{k_h-1}] \subset \cdots \subset L_{\beta_{k_h-1}}[c_2] \subset Q^{n-\alpha_1}_{\alpha_1}[c_1]$$

Note that the coloring is the reverse of the coloring $c^\#_\bullet$. If the largest dimensional linear space has dimension $n/2$, let $T'$ be the Schubert variety defined by replacing the half-dimensional linear space by one from the other connected component. It is immediate that $T$ (and when appropriate $T'$) intersect $V(L_\bullet, Q_\bullet, c_\bullet)$ and $V(L^\#_\bullet, Q^\#_\bullet, c^\#_\bullet)$ in a unique point and they do not intersect $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$. It follows that the restriction varieties listed in Step 2 of the algorithm occur with multiplicity one.

**Case 4:** If $x_\kappa < s$, then the dimension counts and Observation 7.27 imply that $Y$ has to be an irreducible component of one of the loci defined by the sequences $(L^a_\bullet, Q^a_\bullet, c_\bullet), (L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$ or $(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$. In addition, if the quadric with index $\kappa$ fails to satisfy Condition (8) in $(L^a_\bullet, Q^a_\bullet, c_\bullet)$, then by the "linear space bound", every linear space of dimension $k_h - \kappa + 1$ intersects the singular locus of the quadric in dimension $x_\kappa + 1$. Hence, in that case every sequence we construct must intersect the singular locus in a larger dimensional subspace than allowed by $(L^a_\bullet, Q^a_\bullet, c_\bullet)$. Hence, we omit the sequence from the list as in Step 5 of Algorithm 7.19. Now we show that when these varieties have the same dimension as $V(L_\bullet, Q_\bullet, c_\bullet)$, then every component occurs in the limit with multiplicity one. We will assume that if $\kappa = k_h - s$, then Condition (7) is satisfied after running Step iii of Algorithm 7.18 on $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$. Otherwise, the argument for the multiplicity of the sequences derived from $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$ is identical to Case 2. We leave this case to the reader. We will also assume that Condition (8) holds for $(L^a_\bullet, Q^a_\bullet, c_\bullet)$. Otherwise, the same argument works, but every reference to $(L^a_\bullet, Q^a_\bullet, c_\bullet)$ should be removed since that sequence does not occur on our list. For the local calculation we may assume that $\kappa = 1$, $x_\kappa = 0$ and $n = d_\kappa + r_\kappa + 1$. Let $i_0$ denote the smallest index for which equality holds in Condition (8) in $(L_\bullet, Q_\bullet, c_\bullet)$. For $n_j \leq r_{i_0}$, let $\alpha_j = n - n_j + 1$. For $n_j > r_{i_0}$, set $\alpha_j = n - n_{j-1}$. Next, for each index $i < i_0$, let $l_i$ be the largest positive integer such that $r_i + l_i + 1 = n_{x_i + l_i}$. If there does not exist such an integer $l_i$, set $l_i = 0$. Let $\beta_i = n - d_i + l_i$ for $i < i_0$ and let $\beta_i = n - d_i + 1$ for $i \geq i_0$. Let $S$ be the Schubert variety defined by the sequence

$$L_{\beta_1}[c_{k_h}] \subset \cdots \subset L_{\beta_{k_h - s}}[c_{s+1}] \subset Q^{n-\alpha_s}_{\alpha_s}[c_s] \subset \cdots \subset Q^{n-\alpha_1}_{\alpha_1}[c_1].$$

Note that $S$ intersects both $V(L_\bullet, Q_\bullet, c_\bullet)$ and $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$ in a unique reduced point. Also note that since $Q^{n-\alpha_1}_{\alpha_1} \cap L_{r'_\kappa} = \emptyset$, $S$ has empty intersection with all the other $V(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$ or $V(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$. For each sequence $(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$, let $\alpha_1 = n - r'_\kappa + 1$. Let $\alpha_l = n - n_{l-1} + 1$ for $n_l \leq \min(j, r_{i_0})$ and $l > 1$ and $\alpha_l = n - n_l + 1$ for $n_l \leq r_{i_0}$ and $l > j$ and $\alpha_l = n - n_{l-1}$ for $n_l > r_{i_0}$. For each index $i < i_0$, let $l_i$ be the largest positive integer such that $r_i + l_i + 1 = n_{x_i + l_i + 1}$ in the sequence $(L^j_\bullet, Q^j_\bullet)$. If there does not exist such an integer $l_i$, set $l_i = 0$. Let $\beta_i = n - d_i + l_i$ for $i < i_0$ and let $\beta_i = n - d_i + 1$ for $i \geq i_0$. Let $T_j$ be the Schubert variety defined with respect to the following sequence

$$L_{\beta_1}[c_{k_h}] \subset \cdots \subset L_{\beta_{k_h - s}}[c_{s+1}] \subset \cdots \subset Q^{n-\alpha_{j+1}}_{\alpha_{j+1}}[c_{j+1}] \subset Q^{n-\alpha_j}_{\alpha_j}[c_{j-1}] \subset \cdots \subset Q^{n-\alpha_2}_{\alpha_2}[c_1] \subset Q^{n-\alpha_1}_{\alpha_1}[c_j].$$

Note that the coloring is the reverse of $c^j_\bullet$. Similarly, for each sequence $V(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$, let $\alpha_1 = n - r'_\kappa + 1$. Let $\alpha_l = n - n_{l-1} + 1$ for $n_l \leq r_{i_0}$ and let $\alpha_l = n - n_{l-2}$ for $n_l > r_{i_0}$. For each index $m < i_0$, let $l_m$ be the largest positive integer such that $r_m + l_m + 1 = n_{x_m + l_m + 1}$ in the sequence $(L^j_\bullet, Q^j_\bullet)$. If there does not exist such an integer $l_m$, set $l_m = 0$. Let $\beta_{m-1} = n - d_m + l_m$ for $m < i$, let $\beta_m = n - d_m + l_m$ for $i < m < i_0$, and let $\beta_m = n - d_m + 1$ for $m \geq i_0$. Let $U_i$ be the Schubert variety defined with respect to the following sequence

$$L_{\beta_1}[c_{k_h}] \subset \cdots \subset L_{\beta_{i-1}}[c_{k_h - i + 2}] \subset L_{\beta_i}[c_{k_h - i}] \subset \cdots \subset L_{\beta_{k_h - s - 1}}[c_{s+1}] \subset \cdots \subset Q^{n-\alpha_2}_{\alpha_2}[c_1] \subset Q^{n-\alpha_1}_{\alpha_1}[c_{k_h - i + 1}].$$

Note that the coloring is the reverse of that of $c^i_\bullet$. For each of these Schubert varieties if the largest linear space occurring in their definition has dimension $n/2$, then define a corresponding Schubert variety (denoted by $S', T'_j$ and $U'_i$) by replacing the half-dimensional linear space with one in the other connected component. The following observations are straightforward. $T_j$ intersects $V(L_\bullet, Q_\bullet, c_\bullet)$ and $V(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$ in a unique reduced point. Its intersection with $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$, or $V(L^{j_0}_\bullet, Q^{j_0}_\bullet, c^{j_0}_\bullet)^b$ for $j_0 \neq j$, or $V(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$ is empty. Similarly, $U_i$ intersects $V(L_\bullet, Q_\bullet, c_\bullet)$ and $V(L^i_\bullet, Q^i_\bullet, c^i_\bullet)^e$ in a unique reduced point. Its intersection with $V(L^a_\bullet, Q^a_\bullet, c_\bullet)$, or $V(L^j_\bullet, Q^j_\bullet, c^j_\bullet)^b$, or $V(L^{i_0}_\bullet, Q^{i_0}_\bullet, c^{i_0}_\bullet)^e$ for $i_0 \neq i$ is empty. Finally, repeating the calculation with $S', T'_j$ and $U'_i$ when these varieties are reducible, we see that each component occurs with multiplicity one and the algorithm preserves marking. This concludes the proof of the theorem. $\square$

## 8. Applications of Algorithm 7.19

In this section, we will give a geometric algorithm for multiplying two Schubert cycles in the cohomology ring of orthogonal flag varieties when $n$ is odd. When $n$ is even, the same argument gives a method of multiplying cycles in the subring invariant under the involution interchanging the half-dimensional linear spaces on $Q$. The discussion for the orthogonal Grassmannians holds with little change. We will use the notation in [C3] to denote Schubert varieties in flag varieties.

The pull-back of a Schubert class under the inclusion $j : OF(k_1, \ldots, k_h; n) \to F(k_1, \ldots, k_h; n)$ can be expressed as a sum of the classes of restriction varieties. Let $\Sigma^{c_1, \ldots, c_{k_h}}_{\lambda_1, \ldots, \lambda_{k_h}}$ denote a Schubert cycle in the flag variety $F(k_1,\dots,k_h;n)$ (see [C3] for detailed information about Schubert cycles in flag varieties). The following proposition is almost identical to the Grassmannian case.

**Proposition 8.1.** *Let $\sigma^{c_1,\dots,c_{k_h}}_{\lambda_1,\dots,\lambda_{k_h}}$ be a Schubert cycle in $F(k_1,\dots,k_h;n)$. Let $j : OF(k_1,\dots,k_h;n) \to F(k_1,\dots,k_h;n)$ be the natural inclusion. Then*

(1) *$j^*\sigma^{c_1,\dots,c_{k_h}}_{\lambda_1,\dots,\lambda_{k_h}} = 0$ unless $n-k_h-i \geq \lambda_i$ for every $1 \leq i \leq k_h$.*
(2) *Suppose that $n-k_h-i = \lambda_i$ for $i=1,\dots,\alpha$ and $n-k-i > \lambda_i$ for $i=\alpha+1$. Further suppose that if $2k_h = n$, then $\alpha \neq k_h$. Let $(L_\bullet, Q_\bullet, c_\bullet)$ be the admissible sequence*

$$L_1[c_1] \subset \cdots \subset L_\alpha[c_\alpha] \subset Q^\alpha_{n-k_h+\alpha+1-\lambda_{\alpha+1}}[c_{\alpha+1}] \subset \cdots \subset Q^\alpha_{n-\lambda_{k_h}}[c_{k_h}].$$

*Then $j^*\sigma^{c_1,\cdots,c_{k_h}}_{\lambda_1,\dots,\lambda_{k_h}} = 2^\alpha[V(L_\bullet,Q_\bullet,c_\bullet)]$, where $[V(L_\bullet,Q_\bullet,c_\bullet)]$ denotes the cohomology class of the restriction variety $V(L_\bullet,Q_\bullet,c_\bullet)$. If $2\alpha = 2k_h = n$, then the class is $2^{\alpha-1}$ times $[V(L_\bullet,c_\bullet)]$, where the sequence $(L_\bullet)$ defines the point class in $OG(k_h;n)$.*

*Algorithm* 8.2 (Reversing Algorithm 5.9). Let $V(L_\bullet,Q_\bullet,c_\bullet)$ be a restriction variety in $OF(k_1,\dots,k_h;n)$ with $n$ odd.

(1) If the class of $V(L_\bullet,Q_\bullet,c_\bullet)$ is a fraction of a restriction of a Schubert cycle in $F(k_1,\dots,k_h;n)$ (In Proposition 8.1, we determined that this happens precisely when $r_i = n_{x_1} = x_1 = s$ for all $1 \leq i \leq k_h - s$), let $(Q'^\alpha_\bullet$ be the sequence consisting of the linear sections defining the corresponding Schubert variety in $F(k_1,\dots,k_h;n)$ with the same coloring.
(2) If in the sequence $(L_\bullet,Q_\bullet,c_\bullet)$, $r_i = n_{x_1} = x_1$ for all $i$, but $s \neq x_1$, then let $\alpha$ be the largest non-negative integer with $n_{s-\alpha} = n_s - \alpha$. Let $(L^\alpha_\bullet, Q^\alpha_\bullet, c_\bullet)$ be the sequence obtained from $(L_\bullet,Q_\bullet,c_\bullet)$ by replacing $L_{n_s}$ with $Q^{n_s-\alpha-2}_{n_s+\alpha+1}$.
(3) If in the sequence $(L_\bullet,Q_\bullet,c_\bullet)$, $r_i \neq x_1$ for some $i$, let $i$ be the largest index for which $r_i > x_1$ and there does not exist a smaller index $l$ such that $r_l = r_{l-1} > x_1$. If $r_i \neq n_j$ for any $j$, let $(L^\alpha_\bullet, Q^\alpha_\bullet, c_\bullet)$ be the sequence obtained from $(L_\bullet,Q_\bullet,c_\bullet)$ by replacing $r_i$ with $r_i - 1$. If $r_i = n_j$ for some $j$, let $\alpha$ be the largest non-negative integer for which $n_{j-\alpha} = n_j - \alpha$. Let $(L^\alpha_\bullet, Q^\alpha_\bullet, c_\bullet)$ be the sequence obtained from $(L_\bullet,Q_\bullet,c_\bullet)$ by replacing $Q^{r_i}_{d_i}$ with $Q^{r_i-\alpha-2}_{d_i+\alpha+1}$.

In Case (1), $[V(L_\bullet,Q_\bullet,c_\bullet)]$ is already a fraction of the restriction of a Schubert cycle in $F(k_1,\dots,k_h;n)$. There is nothing further to do.

In Case (2), by the Algorithm 7.19 we can express

$$[V(L^\alpha_\bullet, Q^\alpha_\bullet, c_\bullet)] = 2[V(L_\bullet,Q_\bullet,c_\bullet)] + \text{other terms}.$$

In Case (3), by the Algorithm 7.19 we can express

$$[V(L^\alpha_\bullet, Q^\alpha_\bullet, c_\bullet)] = [V(L_\bullet,Q_\bullet,c_\bullet)] + \text{other terms}.$$

In both Cases (2) and (3), the other terms have the property that either the sum of the dimension of the linear spaces is strictly smaller (as is the sum of the ranks of the quadrics) than those in $(L_\bullet,Q_\bullet,c_\bullet)$ or the coloring has strictly larger dimension (and the projection to $OG(k_h,n)$ has strictly smaller dimension). On the other hand, $(L^\alpha_\bullet, Q^\alpha_\bullet, c_\bullet)$ has the property that either it has fewer linear spaces than $(L_\bullet,Q_\bullet,c_\bullet)$ or the sum of the ranks of the quadrics is smaller than those of $(L_\bullet,Q_\bullet,c_\bullet)$. We can solve for the class $[V(L_\bullet,Q_\bullet,c_\bullet)]$, then apply the algorithm to each of the terms. It is clear that this eventually terminates expressing the class as a linear combination of the classes of restriction of Schubert varieties. We thus obtain a geometric formula for expressing Schubert varieties in $OF(k_1,\dots,k_h;n)$ in terms of restrictions of Schubert varieties in $F(k_1,\dots,k_h;n)$. We thus reduce any multiplication in the orthogonal flag variety to a multiplication in the ordinary flag variety and Algorithm 7.19.

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University of Illinois at Chicago, Department of Mathematics, Statistics and Computer Science, Chicago, IL 60607

*E-mail address*: coskun@math.uic.edu