SYMPLECTIC RESTRICTION VARIETIES AND GEOMETRIC BRANCHING RULES II

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To Joe Harris, in celebration of his sixtieth birthday

Abstract. In this paper, we introduce combinatorially defined subvarieties of symplectic flag varieties called *symplectic restriction varieties*. We study their geometric properties and compute their cohomology classes. In particular, we give a positive, combinatorial, geometric branching rule for computing the map in cohomology induced by the inclusion $i : SF(k_1, \dots, k_h; n) \to F(k_1, \dots, k_h; n)$. These rules have many applications in algebraic geometry, combinatorics, symplectic geometry and representation theory.

Contents

1. Introduction

Let $n = 2m$ be an even integer. Let $0 < k_1 < k_2 < \cdots < k_h \leq m$ be an increasing sequence of positive integers. For ease of notation, set $k_0 = 0$ and $k_{h+1} = n$. Let V be an n-dimensional vector space over $\mathbb{C}$ and let Q be a non-degenerate skew-symetric form on V. The symplectic isotropic partial flag variety $SF(k_1, \dots, k_h; n)$ parameterizes partial flags

$$W_1 \subset W_2 \subset \cdots \subset W_h,$$

where W_i is a k_i-dimensional isotropic subspace of V with respect to Q.

The purpose of this paper is to give a positive, geometric rule for computing the restriction coefficients of $SF(k_1, \dots, k_h; n)$. Since every isotropic linear space is in particular a linear space, there is a natural inclusion

$$i : SF(k_1, \dots, k_h; n) \to F(k_1, \dots, k_h; n)$$

of the isotropic partial flag variety into the flag variety. This inclusion induces a map in cohomology

$$i^* : H^*(F(k_1, \dots, k_h; n), \mathbb{Z}) \to H^*(SF(k_1, \dots, k_h; n), \mathbb{Z}).$$

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The cohomology groups of both $SF(k_1,\dots,k_h;n)$ and $F(k_1,\dots,k_h;n)$ have integral bases consisting of Schubert classes. Hence, the image

$$i^*\sigma_{a,c} = \sum c^{a,c}_{\lambda_i;\mu_i;c_i}\sigma_{\lambda_i;\mu_i;c_i}$$

of a Schubert class $\sigma_{a,c}$ in the cohomology of $F(k_1,\dots,k_h;n)$ is a linear combination of Schubert classes in the cohomology of $SF(k_1,\dots,k_h;n)$ with positive integer coefficients. The coefficients $c^{a,c}_{\lambda_i;\mu_i;c_i}$ are called *symplectic restriction coefficients*. In [C2], we computed the orthogonal restriction coefficients. In [C4], we gave a positive, geometric rule for computing the restriction coefficients for the symplectic isotropic Grassmannian $SG(k,n)$. In this paper, we extend the rule to the setting of partial flag varieties, thereby completing the program of finding positive, geometric rules for the restriction coefficients of all classical flag varieties. The reader is advised to consult [C4] prior to reading this paper. The following theorem is the main theorem of the paper.

Theorem 1.1. *Algorithm 3.43 provides an explicit, geometric, combinatorial, positive rule for computing the symplectic restriction coefficients for $SF(k_1,\dots,k_h;n)$.*

Theorem 1.1 has many applications, most notably to calculating the moment polytopes for the inclusion of $Sp(n)$ in $SU(n)$ and the asymptotic of the restrictions of representations of $SL(n)$ to $Sp(n)$. Let $j : G' \to G$ be an inclusion of complex, reductive, connected Lie groups. Choose Borel subgroups $B' \subset G'$ and $B \subset G$ such that $j(B') \subset B$. Then the inclusion $j : G'/B' \to G/B$ induces a map in cohomology $j^* : H^*(G/B) \to H^*(G'/B')$. The structure coefficients of this map in terms of Schubert bases are called *branching coefficients*. Finding positive rules for calculating branching coefficients is a central problem (see [P] for references and an exposition of the subject). In the case of $Sp(n)$ and $SL(n)$, the map j is given by sending an isotropic flag $F_\bullet$ to the pair $(F_\bullet, F_\bullet^\perp)$. Our theorem calculates all the branching coefficients of $j^* : H^*(F(k_1,\dots,k_h,n-k_h,\dots,n-k_1;n)) \to H^*(SF(k_1,\dots,k_h;n))$ for the classes that are pulled back from $F(k_1,\dots,k_h;n)$ under the natural projection that sends $(F_\bullet, F_\bullet^\perp)$ to $F_\bullet$.

Knowing the set of non-zero branching coefficients has important applications in symplectic geometry and representation theory. Let K and K' be the maximal compact subgroups of G and G', respectively. To each non-vanishing branching coefficient, in [BS], Berenstein and Sjamaar associate an inequality satisfied by the K'-moment polytope of a K-coadjoint orbit. Moreover, the totality of these inequalities gives a sufficient set of inequalities for the moment polytope. Similarly, non-vanishing branching coefficients determine which irreducible representations of G' occur in the restriction of an irreducible representation of G asymptotically (see [BS], [GS], [He], [P]).

More importantly, we will introduce a new set of subvarieties of $SF(k_1,\dots,k_h;n)$ called *symplectic restriction varieties* and compute their cohomology classes in terms of the Schubert basis of $SF(k_1,\dots,k_h;n)$. The analogues of these varieties for orthogonal flag varieties were introduced in [C2]. In many ways, these varieties are more fundamental than Schubert varieties and have applications to many other geometric problems such as problems of smoothability and rigidity (see [C1], [C4], [C5]). Symplectic restriction varieties are defined by imposing rank conditions on a partial flag $(W_1,\dots,W_h)$ with respect to a not-necessarily-isotropic flag. They interpolate between the intersection of $SF(k_1,\dots,k_h;n)$ with a general translate of a Schubert variety in $F(k_1,\dots,k_h;n)$ and a Schubert variety in $SF(k_1,\dots,k_h;n)$. We will discuss their geometric properties in detail in §4.

The beauty of our approach is that, while the combinatorics of branching coefficients can be very complicated (and this is inevitably reflected in the combinatorial formulation of the rule), four basic geometric principles explain all the complexity. Our strategy for calculating the cohomology classes of symplectic restriction varieties is by specialization. We start with a general Schubert variety defined with respect to a flag $G_\bullet$. Since the flag is general, the dimension of the kernel of the restriction of Q to G_i is zero or one depending on whether i is even or odd. On the other hand, a Schubert variety in $SF(k_1, \dots, k_h; n)$ is defined with respect to an isotropic flag $F_\bullet$. The dimension of the kernel of the restriction of Q to F_i is i (respectively, $n - i$) if $i \leq m$ (respectively, if $i > m$). The strategy is to transform the flag $G_\bullet$ to an isotropic flag by increasing the dimension of the kernel of the restriction of Q one flag element at a time. As we specialize the flag, the corresponding restriction variety specializes into a union of restriction varieties defined with respect to the limiting flag, each occurring with multiplicity one. The rule records the outcome of the specialization. We now explain the four basic facts about skew-symmetric forms that govern the order of the specialization and the limits that occur.

Let Q_d^r denote a d-dimensional vector space such that the restriction of Q has corank r. Let $\mathrm{Ker}(Q_d^r)$ denote the kernel of the restriction of Q to Q_d^r. Let L_j denote an isotropic subspace of dimension j with respect to Q. Let $L_j^\perp$ denote the set of $w \in V$ such that $w^T Q v = 0$ for all $v \in L_j$. The reader can easily verify the following four basic facts about skew-symmetric forms.

Evenness of rank. The rank of a non-degenerate skew-symmetric form is even. Hence, $d - r$ is even for Q_d^r. Furthermore, if $d = r$, then Q_d^r is isotropic.

The corank bound. Let $Q_{d_1}^{r_1} \subset Q_{d_2}^{r_2}$ and let $r_2' = \dim(\mathrm{Ker}(Q_{d_2}^{r_2}) \cap Q_{d_1}^{r_1})$. Then $r_1 - r_2' \leq d_2 - d_1$. In particular, $d + r \leq n$ for Q_d^r.

The linear space bound. The dimension of an isotropic subspace of Q_d^r is bounded above by $\lfloor \frac{d+r}{2} \rfloor$. Furthermore, an m-dimensional isotropic subspace L of Q_d^r satisfies $\dim(L \cap \mathrm{Ker}(Q_d^r)) \geq m - \lfloor \frac{d-r}{2} \rfloor$.

The kernel bound. Let L be an $(s+1)$-dimensional isotropic space such that $\dim(L \cap \mathrm{Ker}(Q_d^r)) = s$. If an isotropic linear subspace M of Q_d^r intersects $L - \mathrm{Ker}(Q_d^r)$, then M is contained in $L^\perp$.

Let us explain how these four principles dictate the order of the specialization and determine the limits that occur. Given a flag, we will specialize the smallest dimensional non-isotropic subspace Q_d^r, whose corank can be increased subject to the corank bound, keeping all other flag elements unchanged. We will replace Q_d^r with $\tilde{Q}_d^{r+2}$. The branching rule simply says that under this specialization, the limit $(W_1', \dots, W_h')$ of the one-parameter family of partial flags $(W_1, \dots, W_h)(t)$ satisfy the same rank conditions with the unchanged flag elements and $\dim(W_i' \cap \mathrm{Ker}(\tilde{Q}_d^{r+2})) = \dim(W_i \cap \mathrm{Ker}(Q_d^r))$ for $i < i_0$ and $\dim(W_i' \cap \mathrm{Ker}(\tilde{Q}_d^{r+2})) = \dim(W_i \cap \mathrm{Ker}(Q_d^r)) + 1$ for $i \geq i_0$. Furthermore, all of these cases occur with multiplicity one unless some of these loci lead to smaller dimensional varieties or the linear space bound is violated. These exceptions can be explicitly stated combinatorially (though sometimes resulting in cumbersome statements). See Sections 3 and 5 for an explicit statement of the rule and for examples.

There are other potential methods for computing restriction coefficients. For example, Pragacz gave a positive rule for computing restriction coefficients for Lagrangian Grassmannians [Pr1], [Pr2]. It is also possible to compute restriction coefficients (in a non-positive way) by first computing the pullbacks of the tautological bundles from $F(k_1, \dots, k_h; n)$ to $SF(k_1, \dots, k_h; n)$ and then using localization or the theory of Schubert polynomials to express the Chern classes of these bundles in terms of Schubert classes. To the best of the author's knowledge, Algorithm 3.43 is the first positive, geometric rule for computing the restriction coefficients for all isotropic partial flag varieties $SF(k_1, \dots, k_h; n)$. Moreover, these positive rules are computationally much more efficient than their non-positive counterparts.

The organization of this paper is as follows. In §2, we will recall basic facts concerning the geometry of $SF(k_1, \dots, k_h; n)$. In §3, we will introduce symplectic diagrams, which are the main combinatorial objects of this paper. We will then state the rule combinatorially without reference to geometry. In §4, we will introduce symplectic restriction varieties and discuss their basic geometric properties. In §5, we will interpret the combinatorial rule geometrically and prove that it computes the symplectic restriction coefficients.

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2. Preliminaries

In this section, we recall basic facts concerning the geometry of $SF(k_1, \dots, k_h; n)$.

Let V be an n-dimensional vector space over the complex numbers endowed with a non-degenerate skew-symmetric form Q. Since Q is non-degenerate, n must be even. Set $n = 2m$. A linear space $W \subset V$ is isotropic with respect to Q if for every $w_1, w_2 \in W$, $w_1^T Q w_2 = 0$. Given an isotropic space W, let the orthogonal complement $W^\perp$ be the set of vectors $v \in V$ such that $v^T Q w = 0$ for every $w \in W$. If the dimension of W is k, then the dimension of $W^\perp$ is $n-k$.

The partial flag variety $SF(k_1, \dots, k_h; n)$ parameterizing partial flags $W_1 \subset \cdots \subset W_h$, where W_i is an isotropic subspace of V of dimension k_i, is a homogeneous variety for the symplectic group $Sp(n)$. Let $SG(k,n)$ denote the symplectic isotropic Grassmannian parameterizing k-dimensional isotropic subspaces of V. There is a natural projection map

$$\pi_h : SF(k_1, \dots, k_h; n) \to SG(k_h, n)$$

sending $(W_1, \dots, W_h)$ to W_h. The fiber of π_h over a point W_h is the partial flag variety $F(k_1, \dots, k_{h-1}; k_h)$. The geometric properties of $SF(k_1, \dots, k_h; n)$ can be deduced by studying the geometry of $SG(k_h, n)$ and the projection map π_h. For example, the dimension of $SF(k_1, \dots, k_h; n)$ is

$$\dim(SF(k_1, \dots, k_h; n)) = nk_h - \frac{3k_h^2 - k_h}{2} + \sum_{i=1}^{h-1} k_i(k_{i+1} - k_i),$$

given by the sum of the dimensions of $SG(k_h, n)$ [C4] and $F(k_1, \dots, k_{h-1}; k_h)$.

By Ehresmann's Theorem [E] (see also [Bo, IV, 14.12]), the cohomology of $SF(k_1, \dots, k_h; n)$ is generated by the classes of Schubert varieties. Our indexing for Schubert varieties will take into account the projection map π_h. Let

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s \leq m$$

be a sequence of increasing positive integers. Let

$$m > \mu_{s+1} > \mu_{s+2} > \cdots > \mu_{k_h} \geq 0$$

be a sequence of decreasing non-negative integers such that $\lambda_i \neq \mu_j + 1$ for any $1 \leq i \leq s$ and $s < j \leq k_h$. Then the Schubert varieties in $SG(k_h, n)$ are parameterized by pairs of admissible sequences $(\lambda_\bullet; \mu_\bullet)$ [C4].

Definition 2.1. A *coloring* $c_\bullet$ for a sequence $0 < k_1 < \cdots < k_h \leq m$ is a sequence of integers

$$c_1, c_2, \ldots, c_{k_h}$$

such that $k_i - k_{i-1}$ of the integers are equal to i for $1 \leq i \leq h$. A coloring for $SF(k_1, \ldots, k_h; n)$ or $F(k_1, \ldots, k_h; n)$ is a coloring for the corresponding sequences $k_1 < \cdots < k_h$.

Example 2.2. The sequences $1, 1, 2, 2, 2, 3, 3, 3$ and $1, 3, 3, 2, 3, 2, 1$ are two coloring schemes for $SF(2, 4, 7; n)$. Note that k_i is equal to the number of integers in the sequence that are less than or equal to i.

We parameterize Schubert classes in $SF(k_1, \ldots, k_h; n)$ by colored sequences $(\lambda_\bullet; \mu_\bullet; c_\bullet)$, where $(\lambda_\bullet; \mu_\bullet)$ is an admissible sequence for $SG(k_h, n)$ and $c_\bullet$ is a coloring for $SF(k_1, \ldots, k_h; n)$. Fix an isotropic flag

$$F_\bullet = F_1 \subset F_2 \subset \cdots F_m \subset F_{m-1}^\perp \subset \cdots F_1^\perp \subset V.$$

The Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet;c_\bullet}(F_\bullet)$ is defined as the Zariski closure of the set of partial flags

$$\{(W_1, \ldots, W_h) \in SF(k_1, \ldots, k_h; n) \mid \dim(W_j \cap F_{\lambda_i}) = \#\{c_l \mid l \leq i, c_l \leq j\} \text{ for } 1 \leq i \leq s,$$
$$\dim(W_j \cap F_{\mu_i}^\perp) = \#\{c_l \mid l \leq i, c_l \leq j\} \text{ for } s < i \leq k_h\}.$$

The projection $\pi_h(\Sigma_{\lambda_\bullet;\mu_\bullet;c_\bullet})$ is a Schubert variety in $SG(k_h, n)$ with class $\sigma_{\lambda_\bullet;\mu_\bullet}$. Over a general point W_h in the image of the projection $\pi_h(\Sigma_{\lambda_\bullet;\mu_\bullet;c_\bullet})$, the fiber is a Schubert variety in $F(k_1, \ldots, k_{h-1}; k_h)$ with class determined by the sequence $c_\bullet$ (note that the sequence $c_\bullet$ determines a permutation with $h-1$ descents sending an integer $k_{i-1} < \alpha = k_{i-1} + j \leq k_i$ to the position of the j-th number equal to i in the sequence $c_\bullet$). More explicitly, at a general point W_h of the image, the isotropic flag $F_\bullet$ defines a complete flag $G_\bullet$ on W_h. The Schubert variety in $F(k_1, \ldots, k_{h-1}; k_h)$ is defined by

$$\{(W_1, \ldots, W_{h-1}) \mid \dim(W_i \cap G_j) \geq \#\{c_l \mid l \leq j, c_l \leq i\}\}.$$

Definition 2.3. For $1 \leq u < h$, define $\operatorname{cdim}(u)$, the *codimension of the color* u, by

$$\operatorname{cdim}(u) = \sum_{1 \leq i \leq k_h,\ c_i \leq u} \#\{j > i \mid c_j = u + 1\}.$$

Define $\operatorname{cdim}(c_\bullet)$, the codimension of a coloring, by

$$\operatorname{cdim}(c_\bullet) = \sum_{u=1}^{h-1} \operatorname{cdim}(u).$$

Define the dimension of a coloring $\dim(c_\bullet) = \sum_{u=1}^{h-1} k_u(k_{u+1} - k_u) \ - \operatorname{cdim}(c_\bullet)$.

Remark 2.4. The quantity $\dim(c_\bullet)$ is nothing other than the dimension of the Schubert variety in $F(k_1, \ldots, k_{h-1}; k_h)$ determined by the sequence $c_\bullet$. Combining this discussion with [C4, Proposition 4.21], we conclude that a Schubert variety with class $(\lambda_\bullet, \mu_\bullet, c_\bullet)$ has dimension

$$\sum_{i=1}^{s} (\lambda_i - i) + \sum_{j=s+1}^{k_h} (n - \mu_j + 1 - 2j + \#\{\lambda_i \mid \lambda_i \leq \mu_j\}) + \dim(c_\bullet).$$

We denote Schubert varieties in the ordinary flag variety $F(k_1, \ldots, k_h; n)$ by pairs $(a_\bullet, c_\bullet)$, where $a_\bullet$ is a sequence of increasing positive integers $0 < a_1 < \cdots < a_{k_h} \leq n$ and $c_\bullet$ is a coloring. The corresponding Schubert variety $\Sigma_{a_\bullet, c_\bullet}(F_\bullet)$ is the Zariski closure of the locus

$$\{(W_1, \ldots, W_h) \mid \dim(W_j \cap F_{a_i}) = \#\{c_l \mid l \leq i, c_l \leq j\} \text{ for } 1 \leq j \leq h\}.$$

3. The combinatorial rule

In this section, we define combinatorial objects called colored symplectic diagrams, which represent the main geometric objects, the symplectic restriction varieties, of this paper. We then describe an algorithm for computing their cohomology classes in terms of Schubert cycles. This section makes no reference to geometry. A geometrically minded reader may wish to look ahead at the next two sections.

A colored (admissible) symplectic diagram for $SF(k_1, \dots, k_h; n)$ is the data of an (admissible) symplectic diagram for $SG(k_h, n)$ (defined in [C4]) together with a coloring for $SF(k_1, \dots, k_h; n)$. For the convenience of the reader, we recall the definition of an admissible symplectic diagram for $SG(k_h, n)$.

Let $0 \leq s \leq k_h$ be an integer. A *sequence of n natural numbers of type s* is a sequence of n natural numbers such that every number is less than or equal to $k_h - s$. We write the sequence from left to right with a small gap to the right of each number in the sequence. We refer to the gap after the i-th number in the sequence as the *i-th position.* For example, 1 1 2 0 0 0 0 0 and 3 0 0 2 0 1 0 0 are two sequences of 8 natural numbers of types 1 and 0, respectively, for $k_h = 3$.

A *sequence D of brackets and braces of type s* for $SG(k_h, n)$ consists of a sequence of n natural numbers of type s, s brackets] ordered from left to right and $k_h - s$ braces } ordered from right to left such that:

(1) Every bracket or brace occupies a position and each position is occupied by at most one bracket or brace.
(2) Every bracket is to the left of every brace.
(3) Every positive integer greater than or equal to i is to the left of the i-th brace.
(4) The total number of integers equal to zero or greater than i to the left of the i-th brace is even.

By Definition 2.1, a *coloring* for $SF(k_1, \dots, k_h; n)$ is a sequence of k_h integers $c_1, \dots, c_{k_h}$ such that $1 \leq c_j \leq h$ and the number of integers in the sequence equal to i is $k_i - k_{i-1}$. We then have the following definition of a colored sequence of brackets and braces.

Definition 3.1. A *colored sequence of brackets and braces* $(D, c_\bullet)$ of type s for $SF(k_1, \dots, k_h; n)$ is a sequence of brackets and braces of type s for $SG(k_h, n)$ together with a coloring $c_\bullet$ for $SF(k_1, \dots, k_h; n)$.

Notation 3.2. $11]_1 200\}_2 0\}_2 00$ and $300\}_2 20\}_1 10\}_2 0$ are typical examples of sequences of brackets and braces for $SF(1, 3; 8)$. The colorings in these two examples are $1, 2, 2$ and $2, 1, 2$, respectively. The coloring is recorded in the diagram as subscripts to the brackets or braces and is read from left to right in order. When writing a sequence of brackets and braces, we often omit the gaps not occupied by a bracket or a brace.

Example 3.3. For concreteness, we remark that condition (1) disallows diagrams such as $]_1 0000\}_2$ (the first bracket $]_1$ is not in a position), $00]_1]_2 00$, $0000\}_1\}_2$, $00]_1\}_2 00$ (two brackets, two braces or a bracket and a brace in the same position). Condition (2) disallows diagrams such as $00\}_1 0]_2 000$ (a brace is to the left of a bracket). Condition (3) disallows diagrams such as $00\}_1 20\}_2 10$ (there is a 2 in the sequence to the right of the second brace and a 1 in the sequence to the right of the first brace). Condition (4) disallows diagrams such as $2000\}_1 00\}_2$ (the number of zeros to the left of the second brace is odd).

Notation 3.4. By convention, the brackets are counted from left to right and the braces are counted from right to left. We write $]^i$ and $\}^i$ to denote the i-th bracket and i-th brace, respectively. The index of a bracket or a brace, which is denoted by a superscript, should not be confused with its color, which is denoted by a subscript. The positions of the brackets and braces are denoted by $p(]^i)$ and $p(\}^i)$. The position of a bracket or a brace is equal to the number of integers to its left. Let $l(i)$ denote the number of integers in the sequence that are equal to i. Let r_i denote the number of positive integers less than or equal to i to the left of $\}^i$. For $1 \leq i \leq k_h - s$, let $\rho(i,0) = n - p(\}^i)$ and, for $1 \leq i < j \leq k_h - s$, let $\rho(i,j) = p(\}^i) - p(\}^j)$. Equivalently, $\rho(i,j)$ denotes the number of integers to the right of the i-th brace and to the left of the j-th brace and $\rho(i,0)$ denotes the number of integers to the right of the i-th brace. When discussing several diagrams simultaneously, to avoid confusion, we indicate the diagram with a subscript.

For the sequence of brackets and braces $1]_1 22]_3 00]_3 00\}_2 00\}_1 0$ for $SF(2,3,5;10)$, the positions are $p(]^1) = 1, p(]^2) = 3, p(]^3) = 5, p(\}^2) = 7, p(\}^1) = 9$. The coloring is $1,3,3,2,1$. The numbers of integers in the sequence are $l(1) = 1$ and $l(2) = 2$. Finally, the number of integers between the braces are $\rho(2,1) = 2$ and $\rho(1,0) = 1$.

To give some context to the combinatorial objects, we remark that brackets represent isotropic subspaces and braces represent non-isotropic subspaces. The position of a bracket or a brace is the dimension of the corresponding linear space and $\rho(i,0)$ is its codimension. The quantity r_i is the corank of the restriction of the skew-symmetric form Q to the subspace represented by $\}^i$.

Definition 3.5. Two colored sequences of brackets and braces $(D_1, c^1_\bullet)$ and $(D_2, c^2_\bullet)$ are *equivalent* if $c^1_\bullet = c^2_\bullet$, the length of sequences of integers in D_1 and D_2 are equal, the brackets and braces occur in the same positions and the integers occurring between any two consecutive brackets and/or braces are the same up to reordering.

Example 3.6. $3300\}_2 200\}_1 10\}_2 0$ and $3030\}_2 002\}_1 10\}_2 0$ are equivalent. Note that each equivalence class can be represented by a *canonical representative*, where between any two brackets and/or braces the positive integers are listed in increasing order followed by the zeros. In the example, the first sequence is the canonical representative corresponding to the second sequence. We always represent sequences by their canonical representatives and often blur the distinction between the equivalence class and the canonical representative.

Definition 3.7. A colored sequence of brackets and braces of type s for $SF(k_1, \dots, k_h; n)$ is *saturated* if $\rho(i, i-1) = l(i)$ for every $1 \leq i \leq k_h - s$.

Definition 3.8. A colored sequence of brackets and braces of type s for $SF(k_1, \dots, k_h; n)$ is *in order* if the sequence of numbers consists of a sequence of non-decreasing positive integers followed by zeros except possibly for one i immediately to the right of $\}^{i+1}$ for $1 \leq i < k_h - s$. Otherwise, we say that the sequence is *not in order*. A sequence is *in perfect order* if the sequence of numbers consists of non-decreasing positive integers followed by zeros.

Example 3.9. The diagram $1]_1 333]_2 3]_2 00\}_1 0000\}_1 100\}_2 00$ is in order, but it is not saturated. The diagram $11]_1 22]_2 00\}_1 00\}_2 00$ is in perfect order and saturated. The diagrams $11]_1 00]_2 100\}_1 000$ and $24000000\}_1 0\}_2 1\}_3 0\}_2 0$ are not in order.

Next, we recall the definition of a symplectic diagram for $SG(k_h, n)$ from [C4]. This definition is a technical definition and the reader does not need this definition for running the algorithm. Every sequence of brackets and braces that occurs in the algorithm (with the exception of the intermediate marked diagrams in Algorithm 3.38) is a symplectic diagram. The properties are needed for ensuring that colored symplectic diagrams correspond to subvarieties of $SF(k_1, \dots, k_h; n)$. They also play a role in the dimension counts, so we include them for precision. The reader should refer to these conditions as necessary.

A sequence of brackets and braces D of type s for $SG(k_h, n)$ is a *symplectic diagram*, if satisfies the following conditions:

(S1) $l(i) \leq \rho(i, i-1)$ for $1 \leq i \leq k_h - s$.

(S2) Let τ_i be the sum of $p(]^s)$ and the number of positive integers between $]^s$ and $\}^i$. Then

$$2\tau_i \leq p(\}^i) + r_i.$$

(S3) Either the sequence is in order or there exists at most one integer $1 \leq \eta \leq k_h - s$ such that the sequence of integers is non-decreasing followed by a sequence of zeros except for at most one occurrence of η between $]^s$ and $\}^{\eta+1}$ and at most one occurrence of $i < \eta$ after $\}^{i+1}$.

(S4) Let ξ_j denote the number of positive integers between $\}^j$ and $\}^{j-1}$. If an integer i occurs to the left of all the zeros, then either $i = 1$ and there is a bracket in the position following it, or there exists at most one index j_0 such that $\rho(j, j-1) = l(j)$ for $j_0 \neq j > \min(i, \eta)$ and $\rho(j_0, j_0 - 1) \leq l(j_0) + 2 - \xi_{j_0}$. Moreover, any integer η violating order occurs to the right of $\}^{j_0}$.

Definition 3.10. A colored sequence of brackets and braces for $SF(k_1, \dots, k_h; n)$ is a *colored symplectic diagram* $(D, c_\bullet)$ if D is a symplectic diagram for $SG(k_h, n)$.

Next, we define admissible symplectic diagrams. This definition is crucial for the game and the reader should remember these conditions.

Definition 3.11. A colored symplectic diagram $(D, c_\bullet)$ for $SF(k_1, \dots, k_h; n)$ is called *admissible* if D is admissible for $SG(k_h, n)$, that is if it satisfies the following two conditions:

(A1) The two integers in the sequence to the left of a bracket are equal. If there is only one integer to the left of a bracket and $s < k_h$, then the integer is one.

(A2) Let x_i be the number of brackets $]^l$ such that every integer in the sequence to the left of $]^l$ is positive and less than or equal to i. Then

$$x_i \geq k_h - i + 1 - \frac{p(\}^i) - r_i}{2}.$$

Example 3.12. $11]_1 22]_2 00\}_2 00\}_1 00$ and $22]_3 30000\}_2 0\}_1 100\}_3 0$ are admissible symplectic diagrams in $F(2, 4; 10)$ and $F(1, 2, 4; 12)$, respectively. $2]_1 0000\}_2 10\}_1 0$ and $12]_2 0000\}_2 10\}_1 00$ are not admissible because they violate condition (A1). $22\}_2 00\}_1 00$ and $200\}_1 2\}_2 00$ are not admissible because they violate condition (A2). See §3 of [C4] for more examples.

Definition 3.13. The dimension of a colored symplectic diagram $(D, c_\bullet)$ is defined by

$$\dim(D, c_\bullet) = \sum_{i=1}^{s} (p(]^i) - i) + \sum_{j=1}^{k_h - s} (p(\}^j) - 1 - 2k_h + 2j + x_j) + \dim(c_\bullet).$$

Example 3.14. $11]_1 22]_3 00\}_2 00\}_3 00$ has dimension 9. $00]_3 0000\}_2 2\}_3 000\}_1 00$ has dimension 13.

Remark 3.15. The admissible colored symplectic diagrams are the main combinatorial objects of this paper. They represent symplectic restriction varieties in $SF(k_1, \dots, k_h; n)$. As we will see in the next section, the diagram records a partial flag with respect to a distinguished basis. The brackets represent isotropic subspaces and the braces represent non-isotropic subspaces.

The sequence of integers records the dimension of the kernel of the restriction of the skew-symmetric form Q to the non-isotropic subspaces. The coloring dictates the rank conditions imposed on the flags parameterized by $SF(k_1, \dots, k_h; n)$. The conditions (1)-(4), (S1)-(S4), (A1) and (A2) are translations of the four main properties of skew-symmetric forms described in §1. The dimension of an admissible, colored symplectic diagram is equal to the dimension of the corresponding restriction variety.

Definition 3.16. The symplectic diagram $D(\lambda_\bullet; \mu_\bullet; c_\bullet)$ associated to the Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet;c_\bullet}$ in $SF(k_1, \dots, k_h; n)$ is the saturated, colored symplectic diagram in perfect order with brackets at positions λ_i, braces at positions $n - \mu_i$ and coloring $c_\bullet$.

Lemma 3.17. *The symplectic diagram $D(\lambda_\bullet; \mu_\bullet; c_\bullet)$ associated to the Schubert variety $\Sigma_{\lambda_\bullet;\mu_\bullet;c_\bullet}$ is an admissible colored symplectic diagram.*

Proof. Definition 3.16 agrees with the definition of a symplectic diagram associated to a Schubert variety in $G(k_h, n)$ given in [C4, Definition 3.20]. By [C4, Lemma 3.22], the diagram underlying $D(\lambda_\bullet; \mu_\bullet; c_\bullet)$ is an admissible symplectic diagram for $G(k_h, n)$. Therefore, $D(\lambda_\bullet; \mu_\bullet; c_\bullet)$ is an admissible colored symplectic diagram. □

Definition 3.18. Let $\sigma_{a_\bullet,c_\bullet}$ be a Schubert class in $F(k_1, \dots, k_h; n)$. Let i denote the natural inclusion $i : SF(k_1, \dots, k_h; n) \to F(k_1, \dots, k_h; n)$. If $a_j < 2j - 1$ for some $1 \leq j \leq k_h$, then $i^*\sigma_{a_\bullet,c_\bullet} = 0$ and we do not associate a symplectic diagram to $\sigma_{a_\bullet}$. Suppose that $a_j \geq 2j - 1$ for $1 \leq j \leq k_h$. Let $P(a_\bullet; c_\bullet)$ be the colored symplectic diagram that has a brace at position a_j for $1 \leq j \leq k_h$. The sequence of numbers consists of zeros except for a $k_h - j + 1$ immediately following the brace at position a_{j-1} (or at position 1 if $j = 1$) whenever a_j is odd. The colorings in $\sigma_{a_\bullet,c_\bullet}$ and $P(a_\bullet; c_\bullet)$ are equal.

Example 3.19. The diagram $P(\sigma_{3,5,7;1,1,2})$ in $SF(2, 3; 8)$ is $300\}_1 20\}_1 10\}_2 0$. The diagram

$$P(\sigma_{1,3,6,7,10;1,3,3,2,1}) = 5\}_1 40\}_3 000\}_3 2\}_2 000\}_1$$

in $SF(2, 3, 5; 10)$. Notice that the diagram $P(\sigma_{a_\bullet;c_\bullet})$ does not have to be admissible because it fails to satisfy condition (A2) for braces with $a_j = 2j - 1$.

We will later associate a collection of admissible, colored symplectic diagrams to the diagram $P(\sigma_{a_\bullet;c_\bullet})$. For now, we have the following lemma.

Lemma 3.20. *Let $\sigma_{a_\bullet,c_\bullet}$ be a Schubert class in $F(k_1, \dots, k_h; n)$. If $a_j \geq 2j - 1$ for all $1 \leq j \leq k_h$, then diagram $P(\sigma_{a_\bullet;c_\bullet})$ is a colored symplectic diagram. Furthermore, if $a_j > 2j - 1$ for $1 \leq j \leq k_h$, then $P(\sigma_{a_\bullet;c_\bullet})$ is an admissible, colored symplectic diagram.*

Proof. If $a_j > 2j - 1$ for $1 \leq j \leq k_h$, then, by [C4, Definition 3.34], the underlying diagram is the diagram associated to $\sigma_{a_\bullet}$ in $G(k_h, n)$. By [C4, Lemma 3.37], this is an admissible symplectic diagram for $G(k_h, n)$. Therefore, $P(\sigma_{a_\bullet;c_\bullet})$ is an admissible, colored symplectic diagram.

Even when $a_j = 2j - 1$ for some j, then it is easy to see that $P(\sigma_{a_\bullet;c_\bullet})$ is a symplectic diagram. Briefly, since all the integers in $a_\bullet$ are positive and distinct, the braces are in distinct positions. Hence, condition (1) is satisfied. Since there are no brackets, condition (2) is automatic. Conditions (3) and (4) hold by construction: There is an integer equal to i in the sequence and it occurs between the braces $\}^{i+1}$ and $\}^i$ if the position of $\}^i$ is odd. Otherwise, there are no integers equal to i. Therefore, every integer equal to i occurs to the left of $\}^i$ and the number of integers equal to 0 or greater than i to the left of $\}^i$ is even. For each $1 \leq j \leq k_h$, there is at most one integer equal to j in the sequence and the diagram is in order. Therefore, conditions (S1), (S3) and (S4) evidently hold. The number of positive integers to the left of $\}^j$ is at most

$k_h - j + 1$. Furthermore, $p(\}^j) \geq 2(k_h - j) + 1$ and if equality holds, then $r_j = 1$. We conclude that $2\tau_j \leq 2(k_h - j + 1) \leq p(\}^j) + r_j$. Hence, condition (S2) also holds and $P(\sigma_{a_\bullet; c_\bullet})$ is a colored symplectic diagram. Condition (A1) is also automatic since the diagram does not have any brackets, but condition (A2) may fail. $\square$

After these preliminaries, we are ready to explain the algorithm for computing symplectic restriction coefficients.

Definition 3.21. We say that a sequence of brackets and braces D' of type s' is obtained from a colored sequence $(D, c_\bullet)$ of type s if $s' \geq s$ and the positions p_i, p_i' occupied by brackets or braces in D and D' satisfy $p_i' \leq p_i$ for $1 \leq i \leq k_h$.

The next algorithm associates a coloring $c_\bullet'$ to a sequence D' obtained from a colored sequence $(D, c_\bullet)$. The coloring $c_\bullet'$ is the least restrictive way of assigning a color to D' under the condition that the number of colors less than or equal to i occurring before position p in D' should be greater than or equal to the number of colors less than or equal to i occurring before position p in D for $1 \leq i \leq h$. The next algorithm makes this precise.

Algorithm 3.22. Let $p_{u_1} < \cdots < p_{u_{k_1}}$ be the positions in D occupied by brackets and braces to which the coloring $c_\bullet$ assigns the color 1. Let $p_{u_1'}$ be the largest position such that $p_{u_1'} \leq p_{u_1}$ and $p_{u_1'}$ is occupied by a bracket or a brace in D'. Let $c_\bullet'$ assign the color 1 to the bracket or brace in position $p_{u_1'}$. Continuing this way suppose $c_\bullet'$ assigns the color 1 to brackets and braces in positions $p_{u_1'}, \dots, p_{u_j'}$. Let $p_{u_{j+1}'} \leq p_{u_{j+1}}$ be the largest position occupied by a bracket or brace of D' that has not been assigned a color. Let $c_\bullet'$ assign the color 1 to the bracket or brace in position $p_{u_{j+1}'}$. Inductively, suppose that $c_\bullet'$ has assigned the colors $1 \leq i \leq l$ and has assigned the color $l+1$ to $j < k_{l+1} - k_l$ brackets and braces of D'. Let p be the position of the bracket or brace in D that is assigned the $(j+1)$-st color equal to $l+1$. Let $p' \leq p$ be the largest position in D' that has a bracket or a brace that has not been assigned a color. Let $c_\bullet'$ assign the color $l+1$ to the bracket or brace in position p'. Continuing inductively, we obtain a coloring $c_\bullet'$ for D'.

The reader can easily check that Algorithm 3.22 is well-defined and associates a coloring to every diagram obtained from D. We next give three applications of Algorithm 3.22.

Example 3.23. Let $D = 00]_1 0]_2 000\}_3 00$ and let $D' = 0]0]0]00000$ be a diagram obtained from D. The coloring associated to D' is $3, 1, 2$, so that $D' = 0]_3 0]_1 0]_2 00000$.

Let $D = 0000\}_2 3\}_1 0\}_4 00\}_3 00$ and let $D' = 1]1]00\}00\}0000$ be a diagram obtained from D. The coloring associated to D' is $4, 2, 1, 3$, so that $D' = 1]_4 1]_2 00\}_1 00\}_3 0000$.

Let $D = 00]_1 0]_3 0]_3 0]_1 0]_2 0]_1 0]_2 00000000$ and let $D' = 0]0]0]0]0]0]0]000000000$. Then the coloring associated to D' is $3, 1, 3, 2, 1, 2, 1$ so that $D' = 0]_3 0]_1 0]_3 0]_2 0]_1 0]_2 0]_1 000000000$.

Any time we form a new diagram from an old one, the associated coloring will be the one assigned by Algorithm 3.22. While Algorithm 3.22 is the geometrically and logically correct way of assigning a coloring to a diagram, in Remark 3.45 we will state a short-cut that bypasses this algorithm. However, Algorithm 3.22 will still be useful in the proof of the main theorem.

Next, given an admissible symplectic diagram, we associate several new symplectic diagrams. These might not be admissible. We will then give an algorithm for replacing them with admissible symplectic diagrams. The goal is to transform every admissible symplectic diagram into a collection of saturated symplectic diagrams in perfect order (which correspond to Schubert classes). Initially, we will phrase the rule as replacing a given diagram by a collection of diagrams that have the same dimension as the original diagram. However, we will later reformulate the rule so that it is not necessary to calculate the dimension of the diagrams.

Definition 3.24. Let $(D, c_\bullet)$ be an admissible symplectic diagram of type s for $SF(k_1, \dots, k_h; n)$. For the purposes of this definition, read any mention of $k_h - s + 1$ as 0 and any mention of $\}^{k_h - s+1}$ as $]^s$.

(1) If D is not in order, let η be the integer in condition (S3) violating the order.

(i) If every integer $\eta < i \leq k_h - s$ occurs to the left of η, let ν be the leftmost integer equal to $\eta + 1$ in the sequence of D. Let D^a be the canonical representative of the diagram obtained by interchanging η and ν.

(ii) If an integer $\eta < i \leq k_h - s$ does not occur to the left of η, let ν be the leftmost integer equal to $i + 1$. Let D^a be the canonical representative of the diagram obtained by swapping η with the leftmost 0 not equal to ν to the right of $\}^{i+1}$ and changing ν to i.

(2) If D is in order but is not a saturated admissible diagram in perfect order, let κ be the largest index for which $l(i) < \rho(i, i-1)$.

(i) If $l(\kappa) < \rho(\kappa, \kappa-1) - 1$, let ν be the leftmost digit equal to $\kappa+1$. Let D^a be the canonical representative of the diagram obtained by changing ν and the leftmost 0 not equal to ν to the right of $\}^{\kappa+1}$ to κ.

(ii) If $l(\kappa) = \rho(\kappa, \kappa - 1) - 1$, let η be the integer equal to $\kappa - 1$ immediately to the right of $\}^\kappa$.

(a) If κ occurs to the left of η, let ν be the leftmost integer equal to κ in the sequence of D. Let D^a be the canonical representative of the diagram obtained by changing ν to $\kappa - 1$ and η to zero.

(b) If κ does not occur to the left of η, let ν be the leftmost integer equal to $\kappa + 1$. Let D^a be the diagram obtained by swapping η with the leftmost 0 to the right of $\}^{\kappa+1}$ and changing ν to κ.

In all these cases, the coloring associated to D^a is the one determined by Algorithm 3.22. However, since the formation of D^a does not change the positions of the brackets and braces, this is simply the coloring $c_\bullet$.

Notation 3.25. For an integer α in the sequence of natural numbers of a symplectic diagram, let $\pi(\alpha)$ be the place of α in the sequence. For example, in $1, 4, 2, 3$, $\pi(1) = 1, \pi(2) = 3, \pi(3) = 4$ and $\pi(4) = 2$. For a bracket $]^i$, let y_i denote the smallest positive integer to the left of $]^i$ so that every integer to the left of $]^i$ is positive and less than or equal to y_i. If not all the integers to the left of $]^i$ are positive, set $y_i = k_h - s + 1$.

Definition 3.26. We preserve the notation from Definition 3.24. Suppose that in the diagram $(D, c_\bullet)$ there exists a bracket to the right of ν. For each of the brackets $]^i$ to the right of ν satisfying the equality $p(]^i) - \pi(\nu) = y_i - \nu + i - x_{\nu-1}$ in D, let $(D^b(]^i), c_\bullet^i)$ be the diagram obtained from $(D^a, c_\bullet)$ by moving the bracket $]^i$ to the position immediately following ν. The coloring $c_\bullet^i$ is the one obtained by running Algorithm 3.22. If there are no brackets to the right of ν or none of the brackets satisfy the inequality, there are no diagrams of this form.

Remark 3.27. Conditions (S3), (S4) and (A1) imply that, in an admissible diagram D, a bracket $]^i$ satisfies the equality $p(]^i) - \pi(\nu) = y_i - \nu + i - x_{\nu-1}$ if the number of brackets in positions following an integer j is one less than the number of integers equal to j for every j in the part of the sequence between ν and $]^i$. For example, in the diagram $...33]_1 3]_2 4566]_3 6]_2 6]_1 700]_1 0]_2...$ if $\nu = 3$ is the leftmost 3 in the sequence, then every bracket shown here satisfies the equality.

Whereas, assuming ν is the leftmost 3 in the diagram $...33]_1 33]_2 44]_1 500]_2 ...$, the brackets other than the first bracket do not satisfy the equality.

We now give several examples to illustrate the definition of diagrams D^a and D^b.

Example 3.28. Let $D = 2300\}_1 10\}_2 0\}_3 0$, then $\eta = 1$ violates the order and $\nu = 2$ and 3 occur to the left of it. Hence, we are in case (1)(i) and $D^a = 1300\}_1 20\}_2 0\}_3 0$ is obtained by swapping 1 and 2. Similarly, let $D = 200]_1 200\}_2 00\}_3$, then the second 2 violates the order and $D^a = 220]_1 000\}_2 00\}_3$, $D^b(]^1) = 22]_1 0000\}_2 00\}_3$.

Let $D = 124400\}_1 00\}_2 1\}_2 0\}_1 00$, the 1 in the ninth place violates the order and 3 does not occur to its left, so we are in case (1)(ii) and $D^a = 123400\}_1 10\}_2 0\}_2 0\}_1 00$.

Let $D = 22]_1 00\}_2 00\}_3 00$, then D is in order and $\kappa = 1$. Since $l(1) = 0 < \rho(1,0) - 1$, we are in case (2)(i) and $D^a = 12]_1 00\}_2 10\}_3 00$ and $D^b(]^1) = 1]_1 200\}_2 10\}_3 00$.

Let $D = 3300\}_1 200\}_2 0\}_3$, then D is in order and $\kappa = 3$. Since $l(3) = 2 = \rho(3,2) - 1$, we are in case (2)(ii)(a) and $D^a = 2300\}_1 000\}_2 0\}_3$.

Let $D = 330000\}_1 00\}_2 1\}_3 0$, then D is in order and $\kappa = 2$. Since $l(2) = 0 = \rho(2,1) - 1$ and 2 does not occur in the sequence, we are in case (2)(ii)(b) and $D^a = 230000\}_1 10\}_2 0\}_3 0$.

Let $D = 22]_1 2]_2 344]_3 00\}_1 0000\}_2 0\}_3 000\}_4 00$. Then $D^a = 12]_1 2]_2 344]_3 00\}_1 0000\}_2 0\}_3 100\}_4 00$ and $D^b(]^1) = 1]_1 22]_2 344]_3 00\}_1 0000\}_2 0\}_3 100\}_4 00$, $D^b(]^2) = 1]_2 2]_1 2344]_3 00\}_1 0000\}_2 0\}_3 100\}_4 00$ and $D^b(]^3) = 1]_3 2]_1 2]_2 34400\}_1 0000\}_2 0\}_3 100\}_4 00$.

As can be seen from the examples, the diagrams D^a or D^b may fail to be admissible. D^a may fail to satisfy either condition (A1) or condition (A2). D^b satisfies condition (A2), but may fail to satisfy condition (A1). The next two algorithms transform these diagrams into admissible diagrams.

Algorithm 3.29. Let $(D, c_\bullet)$ be a symplectic diagram arising as $(D^a, c_\bullet)$ or $(D^b, c_\bullet)$ for some admissible symplectic diagram or a marked diagram $(D, c_\bullet, *)$ occurring while running Algorithm 3.38. If $(D, c_\bullet)$ does not satisfy condition (A1) or $(D, c_\bullet, *)$ does not satisfy condition (A1*) (to be defined shortly), apply the following algorithm: Let $]^j$ be the maximal index bracket for which condition (A1) or (A1*) fails. Let i be the integer immediately to the left of $]^j$. Replace i with $i-1$ (or $k_h - s$ if $i = 0$) and move $\}^{i-1}$ ($\}^{k_h - s}$ if $i = 0$) one position to the left unless that position is already occupied. If the position to the left of $\}^{i-1}$ is occupied, let p be the first position to the left of $\}^{i-1}$ which is not occupied and assume that $\}^{i+l}, \dots, \}^i$ are in positions $p+1, \dots, p(\}^{i-1}) - 1$. Move $\}^{i-1}$ to position p, subtract one from all the integers in the sequence equal to $i, i+1, \dots, i+l$ and replace the left most integer equal to $i+l+1$ (or 0 if $i + l = k_h - s$) with $i + l$. The coloring of the new diagram is the one obtained by running Algorithm 3.22. Discard the diagram if its dimension is smaller than the original diagram. Otherwise, repeat this process until condition (A1) or (A1*) is satisfied for the diagram.

Example 3.30. Let $D = 33]_1 00]_2 0000\}_1 00\}_2 00\}_3 00$. Then $D^a = 23]_1 00]_2 0000\}_1 20\}_2 00\}_3 00$, $D^b(]^1) = 2]_1 300]_2 0000\}_1 20\}_2 00\}_3 00$ and $D^b(]^2) = 2]_2 3]_1 000000\}_1 20\}_2 00\}_3 00$ do not satisfy condition (A1). Algorithm 3.29, replaces D^a with $22]_1 00]_2 0000\}_1 2\}_2 000\}_3 00$, which is admissible. Algorithm 3.29, replaces $D^b(]^1)$ with $1]_1 1300]_2 0000\}_1 20\}_2 0\}_3 000$, which is admissible. Algorithm 3.29, replaces $D^b(]^2)$ first with $2]_2 2]_1 000000\}_1 2\}_2 000\}_3 000$, which still violates condition (A1). Then with $1]_2 2]_1 000000\}_1 2\}_2 00\}_3 000$ and finally with $1]_2 1]_1 000000\}_1 2\}_2 0\}_3 0000$, which is admissible.

Let $D = 00]_1 0000\}_2 2\}_3 0\}_4 00$. Then both the diagrams $D^a = 30]_1 2000\}_2 0\}_3 0\}_4 00$ and $D^b(]^1) = 3]_1 20000\}_2 0\}_3 0\}_4 00$ fail condition (A1). Algorithm 3.29 replaces D^a with the admissible diagram $33]_1 200\}_2 00\}_3 0\}_4 00$. Algorithm 3.29 first replaces $D^b(]^1)$ with $2]_1 2300\}_3 0\}_2 00\}_4 00$, which still fails condition (A1), and then with the admissible diagram $1]_1 2300\}_3 0\}_2 0\}_4 000$.

The most significant difference between the rule for $G(k_h, n)$ and $SF(k_1, \ldots, k_h; n)$ is the algorithm for turning a diagram that does not satisfy condition (A2) to an admissible diagram. The reader should think of this algorithm as a special case of a generalized Pieri rule. Since the algorithm is a little involved, we break it into smaller steps. We begin by introducing some definitions.

Definition 3.31. A *marked* diagram $(D, c_\bullet, *)$ is a diagram $(D, c_\bullet)$ where either two equal numbers in the sequence or a number in the sequence and position (-1) are marked by a $*$. Denote by δ and $\pi(\delta)$ the rightmost marked integer and its place in the sequence. Let $\pi(\delta')$ denote the place in the sequence of the leftmost marking. The dimension of a marked diagram $(D, c_\bullet, *)$ is defined to be the dimension of $(D, c_\bullet)$.

Example 3.32. The diagrams $*00\}_1 1^*\}_2 000$ and $1^*2]_1 00]_2 00\}_3 1^*0\}_4 00$ are typical examples of marked diagrams.

We need to slightly modify condition (A1) for marked diagrams to allow for the two integers preceding a bracket not to be equal when one of them is marked. We say that a marked diagram $(D, c_\bullet, *)$ satisfies condition (A1*) if the two integers in the sequence to the left of any bracket $]^j$ omitting the marked integer δ are equal. If there is only one integer to the left of $]^1$ and $s < k_h$, then the integer is one.

Example 3.33. The diagrams $1^*1]_1 00]_2 1^*]_3 00\}_4 000$ and $2^*2]_1 3455]_1 2^*]_1 00\}_2 00\}_2 0\}_1 0\}_2 000\}_1$ satisfy condition (A1*). In both examples, the two integers in the sequence to the left of $]^3$ are not equal, so these diagrams do not satisfy condition (A1). $1^*1]_1 22]_2 1^*]_4 0]_3 000\}_1 00\}_2 000$ does not satisfy condition (A1*) since the two integers preceding $]_3^4$ omitting 1^* are $2, 0$ and not equal. Running Algorithm 3.29 gives $1^*1]_1 22]_2 1^*]_4 2]_3 00\}_1 000\}_2 000$, which satisfies condition (A1*).

Definition 3.34. The *tightening* of a marked symplectic diagram $(D, c_\bullet, *)$ is the diagram defined as follows:

- If the first bracket or brace to the right of δ is the bracket $]^j$, move $]^j$ to position $\pi(\delta)$ immediately to the right of δ.
- If the first bracket or brace to the right of δ is the brace $\}^j$, move $\}^j$ to the position $\pi(\delta)$ immediately to the right of δ. If $j = k_h - s$, replace $\}^j$ with $]^{s+1}$ and replace the integers in the sequence equal to $k_h - s$ with 0. If $j < k_h - s$, replace the numbers in the sequence equal to $j+1$ with j.
- In either case, if there are no brackets or braces between the markings or the markings are adjacent, remove the markings.

Example 3.35. The tightening of $1^*1]_1 1^*0]_2 0]_3 00\}_4 000$ is $1^*1]_1 1^*]_2 00]_3 00\}_4 000$.

The tightening of $1^*2]_1 00]_2 00\}_3 1^*0\}_4 00$ is $1^*1]_1 00]_2 00\}_3 1^*\}_4 000$.

The tightening of $2^*2^*\}_1 00\}_2 00$ is $00]_1 00\}_2 00$.

Finally, the tightening of $1^*1]_1 00]_2 0]_2 1^*00\}_1 0\}_1 000$ is $1^*1]_1 00]_2 0]_2 1^*]_1 000\}_1 000$.

We now define analogues of diagrams D^a and D^b for marked diagrams. The goal of the algorithm will be to replace each marked diagram with these new diagrams until the tightening is no longer a marked diagram. Geometrically, the marked integer δ represents a vector. To have a restriction variety, we would like this vector to be in position $\pi(\delta')+1$. Informally, δ would like to be next to its marked counterpart δ'. To achieve this, the algorithm moves δ to the left of the next bracket or brace one at a time. In the case of the Grassmannian, there is a unique limit of the corresponding specialization, so it was easy to replace D^a with one diagram. For flag varieties, there may be several limits, so the algorithm is more complicated.

Definition 3.36. Let $(D, c_\bullet, *)$ be a tightened marked symplectic diagram.

• If the bracket or brace at position $\pi(\delta)$ is the bracket $]^{j+1}$, then let ϵ be the integer immediately to the left of $]^j$ (i.e., $\pi(\epsilon) = p(]^j)$). Let $\tilde{D}$ be the canonical representative of the diagram obtained by interchanging ϵ and δ, keeping δ as the marked integer. Define $(D^\alpha, c_\bullet, *)$ to be the tightening of $\tilde{D}$.

Let p be the minimum of $\pi_{\tilde{D}}(\delta)$ in $\tilde{D}$ and the first position to the left of $p(]^j)$ not occupied by a bracket in $\tilde{D}$. If $p \leq p(\delta')$, then $(D^\beta, c_\bullet, *)$ is not defined. Assume $p > p(\delta')$. If there is a bracket between p and $]^j$ in $\tilde{D}$, let q be the position of the rightmost bracket to the left of p. For each $1 \leq t \leq p-q$, let $(D^\beta(t), c^\beta_\bullet, *)$ be the tightening of the diagram obtained from $\tilde{D}$ by moving $]^{j+1}$ to position $q+t$. If there are no brackets between p and $]^j$, for all $t > 0$ such that $p+t \leq p(]^j)$, let $(D^\beta(t), c^\beta_\bullet, *)$ be the tightening of the diagram obtained from $\tilde{D}$ by moving $]^{j+1}$ to position p and $]^j$ to position $p+t$. The coloring c^β is the one obtained by running Algorithm 3.22. We will collectively refer to these diagrams as diagrams of type D^β.

• If the bracket or brace at position $\pi(\delta)$ is the brace $\}^j$, then let ϵ be the leftmost largest integer in the sequence in D between $\}^{j+1}$ and $\}^{j+2}$ (or $]^s$ if $j+1 = k_h - s$). Let $\tilde{D}$ be the diagram obtained from D by moving ϵ to $\pi(\delta)$, sliding all the integers between ϵ and δ one unit to the left and placing δ immediately to the right of $\}^{j+1}$ (keeping δ the marked integer). If $\epsilon = 0$, replace the leftmost integer equal to $j+2$ with $j+1$. If there is no integer equal to $j+2$ in the sequence and the algorithm is being run on a diagram of type $(D^a, c_\bullet)$, replace the leftmost zero with $j+1$. If there is no integer equal to $j+2$ in the sequence and the algorithm is being run on a diagram of type $P(a_\bullet, c_\bullet)$, replace the leftmost zero between $\}^{j+1}$ and $\}^{j+2}$ (or $]^s$ if $j+1 = k_h - s$) with $j+1$. Define $(D^\alpha, c_\bullet, *)$ to be the tightening of the canonical representative of $\tilde{D}$, replacing any integer equal to $j+1$ to the right of $\}^{j+1}$ with 0.

Let p be the first position to the left of $p(\}^{j+1})$ in $\tilde{D}$ not occupied by a brace. Assume that the braces between p and $\}^j$ are $\}^{j+l}, \dots, \}^{j+1}$. Let γ be the leftmost integer in $\tilde{D}$ equal to $j+1$ (or 0 if $j = k_h - s$) and let $]^u,]^{u+1}, \cdots,]^{u+v}$ be the brackets to the right of γ. For $0 \leq t \leq v$, $(D^\beta(]^{u+t}), c^t_\bullet, *)$ be the marked diagram obtained by tightening the canonical representative obtained from $\tilde{D}$ by moving $\}^j$ to position p, subtracting one from the integers $j+l, \dots, j+1$ in the sequence, changing any integer greater than or equal to j to the right of $\}^j$ to zero and moving $]^{u+t}$ to the position $\pi(\gamma)$ immediately to the right of γ. In addition, let $(D^\beta(\}^j), c^{\}^j}_\bullet, *)$ be the diagram obtained by tightening the canonical representative of the diagram obtained by replacing $\}^j$ in $\tilde{D}$ with a bracket at position $\pi(\gamma)$ immediately following γ and subtracting one from the integers greater than or equal to j in the sequence. The colorings are the ones obtained by running Algorithm 3.22. We will collectively refer to these diagrams as diagrams of type D^β.

Example 3.37. Let $D = 1^*1]_1 1^*]_2 0000\}_3 00\}_4 000$, then $\tilde{D} = 1^*1^*]_1 1]_2 0000\}_3 00\}_4 000$. Hence, $D^\alpha = 11]_1 1]_2 0000\}_3 00\}_4 000$. Since $p = p(\delta')$, there are no diagrams of the form D^β in this example.

Let $D = 1^*1]_1 00]_2 00\}_3 1^*\}_4 000$, then $\tilde{D} = 1^*1]_1 20]_2 01^*\}_3 0\}_4 000$. By tightening the canonical representative, we obtain $D^\alpha = 1^*1]_1 00]_2 1^*]_3 00\}_4 000$. We have $D^\beta(]^2_2) = 1^*1]_1 1]_2 00\}_4 1^*\}_3 0000$ and $D^\beta(\}^1_4) = 0^*0]_1 0]_4 0]_2 0^*]_3 00000$.

Let $D = 1^*1]_1 2300]_2 1^*]_3 00\}_1 0\}_2 0\}_3 000$, then $\tilde{D} = 1^*1]_1 1^*230]_2 0]_3 00\}_1 0\}_2 0\}_3 000$. The tightening of the diagram gives $D^\alpha = 1^*1]_1 1^*]_2 2300]_3 00\}_1 0\}_1 0\}_1 000$. We have

$$D^\beta(1) = 1^*1]_1 1^*]_3 2]_2 30000\}_1 0\}_2 0\}_3 000, \quad D^\beta(2) = 1^*1]_1 1^*]_3 23]_2 0000\}_1 0\}_2 0\}_3 000$$

and $D^\beta(3) = 1^*1]_1 1^*]_3 230]_2 000\}_1 0\}_2 0\}_3 000$. Note that the diagrams of the form D^β do not necessarily satisfy condition (A1*). We need to run Algorithm 3.29 on these diagrams. In this example, we obtain the three diagrams $1^*1]_1 1^*]_3 1]_2 2300\}_3 0\}_1 0\}_2 0000$, $1^*1]_1 1^*]_3 22]_2 300\}_2 0\}_1 00\}_3 000$, $1^*1]_1 1^*]_3 233]_2 00\}_1 00\}_2 0\}_3 000$, respectively.

Finally, let $D = 1^*1]_1 2300]_2 0]_3 1^*]_4 000\}_1 0\}_2 0\}_3 000$, then

$$\tilde{D} = D^\alpha = 1^*1]_1 2300]_2 1^*]_3 0]_4 000\}_1 0\}_2 0\}_3 000.$$

The diagrams of type D^β are

$$1^*1]_1 2]_4 300]_2 1^*]_3 0000\}_1 0\}_2 0\}_3 000, \quad 1^*1]_1 23]_4 00]_2 1^*]_3 0000\}_1 0\}_2 0\}_3 000$$

and $1^*1]_1 230]_4 0]_2 1^*]_3 0000\}_1 0\}_2 0\}_3 000$. These diagrams do not satisfy condition (A1*). Running Algorithm 3.29, we replace them with

$$1^*1]_1 1]_4 233]_2 1^*]_3 00\}_3 00\}_1 0\}_2 0000, \quad 1^*1]_1 22]_4 33]_2 1^*]_3 00\}_2 00\}_1 00\}_3 000$$

and $1^*1]_1 233]_4 3]_2 1^*]_3 00\}_1 000\}_2 0\}_3 000$, respectively.

Having discussed the preliminaries, we can now give the algorithm for replacing a diagram that fails condition (A2) with an admissible diagram.

Algorithm 3.38. Let $(D, c_\bullet)$ be a symplectic diagram arising by running the loop defined below on $(D^a, c_\bullet)$ or $P(a_\bullet, c_\bullet)$. If $(D, c_\bullet)$ satisfies condition (A2), stop–the algorithm terminates. If $(D, c_\bullet)$ does not satisfy condition (A2), let $\}^i$ be the brace with the largest index for which condition (A2) is not satisfied. Form the marked diagram $(D, c_\bullet, *)$ by tightening the diagram obtained by either marking the two rightmost integers in the sequence equal to i, or, if there is only one integer equal to i, marking it and position -1. Run the following loop:

As long as a diagram $(D, c_\bullet, *)$ is marked, replace $(D, c_\bullet, *)$ with the diagrams obtained by running Algorithm 3.29 on all the diagrams of type D^α and D^β defined above that have the same dimension as $(D, c_\bullet, *)$. Repeat for each of the diagrams until the diagrams are not marked. Once a diagram is not marked, run Algorithm 3.29 on it and return to the beginning of this algorithm.

We give three examples of Algorithm 3.38.

Example 3.39. Let $D = 00\}_1 00\}_2 00$, then $D^a = 22\}_1 00\}_2 00$ does not satisfy condition (A2). Algorithm 3.38 replaces this diagram with first $2^*2^*\}_1 00\}_2 00$ and then with the admissible diagram $00]_1 00\}_2 00$.

Example 3.40. Let $D = 22]_1 00]_2 00\}_3 00\}_4 00$. Then $D^a = 12]_1 00]_2 00\}_3 10\}_4 00$ fails condition (A2). The tightening of D^a is the diagram $1^*1]_1 00]_2 00\}_3 1^*\}_4 000$. Algorithm 3.38 first replaces this diagram with the three diagrams

$$D^\alpha = 1^*1]_1 00]_2 1^*]_3 00\}_4 000, \;\; D^\beta(]^2) = 1^*1]_1 1]_2 00\}_4 1^*\}_3 0000 \text{ and } D^\beta(\}^1) = 0^*0]_1 0]_4 0]_2 0^*]_3 00000.$$

The third diagram $D^\beta(\}^1)$ is admissible and repeated applications of Algorithm 3.38 results in the diagram $00]_1 0]_4 0]_2 0]_3 00000$ since there can be no diagrams of type D^β. Algorithm 3.38 replaces D^α with the two diagrams

$$11]_1 1]_2 00]_3 00\}_4 000 \;\text{ and }\; 11]_1 1]_3 1]_2 00\}_4 0000,$$

both of which are admissible. Algorithm 3.38 replaces $D^{\beta}(]^2)$ with $11]_1 1]_2 1]_4 00\}_3 0000$ which is also admissible.

Example 3.41. Let $P = {*}00\}_1 00\}_2 1^*\}_3 000$. Then Algorithm first associates the pair of diagrams

$$P_1 = P^{\alpha} = {*}00\}_1 1^*\}_2 00\}_3 000 \quad\text{and}\quad P_2 = P^{\beta} = {*}00\}_1 1\}_3 1^*\}_2 0000.$$

P_1 and P_2 do not satisfy condition (A2). Algorithm 3.38 associates to P_1 the two admissible diagrams

$$D_1 = 1]_1 00\}_2 00\}_3 000 \quad\text{and}\quad D_2 = 1]_2 1]_1 00\}_3 0000.$$

Algorithm 3.38 associates to P_2 the two diagrams

$$P_3 = 1]_1 00\}_3 1\}_2 0000 \quad\text{and}\quad D_3 = 0]_3 0]_1 0]_2 00000.$$

The diagram D_3 is admissible but P_3 violates condition (A2). The algorithm replaces P_3 with $D_4 = 1]_1 1]_3 00\}_2 0000$. Hence, Algorithm 3.38 replaces P with the diagrams D_1, D_2, D_3 and D_4.

Before proceeding, we urge the reader to practice Algorithm 3.38 on several examples. We suggest $0000\}_1 3\}_2 0\}_3 1\}_4 0$ and $12]_1 3400]_2 00\}_3 0\}_4 0\}_5 10\}_6 00$ as instructive examples.

Definition 3.42. When $a_j \geq 2j-1$ for $1 \leq j \leq k_h$, let the diagrams associated to a Schubert class $\sigma_{a_\bullet, c_\bullet}$ in $F(k_1, \ldots, k_h; n)$ be the collection of diagrams derived from $P(a_\bullet; c_\bullet)$ (defined in Definition 3.18) by running Algorithm 3.38.

We are now ready to state the main algorithm that computes the symplectic restriction coefficients.

Algorithm 3.43. Let $(D, c_\bullet)$ be an admissible, colored symplectic diagram. If $(D, c_\bullet)$ is saturated and in perfect order, then return $(D, c_\bullet)$ and stop. Otherwise, replace $(D, c_\bullet)$ with the admissible, colored symplectic diagrams that are derived from the colored diagrams $(D^a, c_\bullet)$ and $(D^b(]^i), c_\bullet^i)$ and that have the same dimension as $(D, c_\bullet)$ by running the Algorithms 3.29 and 3.38.

Remark 3.44. Observe that since D^a and D^{α} always have the same dimension as D, Algorithm 3.43 at each stage replaces a diagram with at least one diagram.

Remark 3.45. Both running Algorithm 3.22 and computing the dimension of a diagram are onerous and unnecessary for running Algorithm 3.43. We now explain when a diagram formed while running Algorithm 3.43 has the same dimension as the original one and its coloring without computing the dimension or running Algorithm 3.22. In a nutshell, if a bracket or brace crosses another bracket or brace of equal or larger color, then the dimension is strictly smaller.

A diagram $(D^b(]^i), c_\bullet^i)$ has the same dimension as $(D, c_\bullet)$ if and only if the color of every bracket between $]^i$ and position $\pi(\nu)$ in $(D, c_\bullet)$ is strictly less than the color of $]^i$. In that case, the new coloring $c_\bullet^i$ is the coloring obtained by moving $]^i$ to position $\pi(\nu)$ preserving its color and the color of every other bracket and brace.

Let $D = 22]_2 2]_1 33]_2 3]_3 00\}_1 000\}_2 000\}_1 00$ and $D^a = 12]_2 2]_1 33]_2 3]_3 00\}_1 000\}_2 100\}_1 00$, then $D^b(]^1) = 1]_2 22]_1 33]_2 3]_3 00\}_1 000\}_2 100\}_1 00$ and $D^b(]^4) = 1]_3 2]_2 2]_1 33]_2 300\}_1 000\}_2 100\}_1 00$ have the same dimension as D. Note that the formation of the diagram in this case amounts to shifting $]_2^1$ or $]_3^4$, respectively, to position 1. On the other hand, $D^b(]^2) = 1]_2 2]_1 233]_2 3]_3 00\}_1 000\}_2 100\}_1 00$ and $D^b(]^3) = 1]_2 2]_2 2]_1 333]_3 00\}_1 000\}_2 100\}_1 00$ have strictly smaller dimension. In particular, one does not need to compute dimensions or run Algorithm 3.22 when forming diagrams of type D^b.

Next, when running Algorithm 3.29, the coloring changes only if we are moving the brace $\}^i$ and there is a brace in $p(\}^i) - 1$. Let p be the first position to the left of $\}^i$ not occupied by a brace. As in the previous case, the diagram obtained has the same dimension as the original diagram if and only if the color of every brace at positions between p and $p(\}^i)$ is strictly less than the color of $\}^i$. The new coloring is the one obtained by moving $\}^i$ to position p preserving its color and the color of every other bracket and brace.

Let $D = 00]_1 0000\}_2 2\}_3 0\}_4 00$ and $D^b = 3]_1 20000\}_2 0\}_3 0\}_4 00$. Running Algorithm 3.29, we get $2]_1 2300\}_3 0\}_2 00\}_4 00$, which has the same dimension as D. In contrast, if $D = 00]_1 0000\}_3 2\}_2 0\}_4 00$ and $D^b = 3]_1 20000\}_3 0\}_2 0\}_4 00$, then $2]_1 2300\}_3 0\}_2 00\}_4 00$ has strictly smaller dimension.

Finally, when running Algorithm 3.38, the diagrams among D^β that have the same dimension as the original diagram are those diagrams that when a bracket or brace $]^j$ or $\}^j$ moves past other brackets or braces, the color of $]^j$ or $\}^j$ is strictly greater than the color of any bracket or brace between the old and new position of $]^j$ or $\}^j$. In that case, the new coloring is the one obtained by moving $]^j$ or $\}^j$ to the new position, preserving the color of all brackets and braces.

Let $D = 1^*1]_1 22]_2 1^*]_1 00\}_2 00\}_1 000$, then $D^\beta = 11]_1 1]_2 1]_1 200\}_2 0\}_1 0000$ has strictly smaller dimension. Similarly, if $D = 1^*1]_1 233]_2 3]_3 1^*]_4 00\}_4 000\}_3 0\}_2 000$, then running Algorithm 3.29 on $D^\beta = 1^*1]_1 2]_4 33]_2 1^*]_3 300\}_4 000\}_3 0\}_2 000$ yields the strictly smaller dimensional diagram $1^*1]_1 1]_4 22]_2 1^*]_3 300\}_4 0\}_3 00\}_2 0000$

Thanks to the previous remark, when running Algorithm 3.43, it is not necessary to compute dimensions of diagrams and to run Algorithm 3.22. We also observe that Algorithm 3.43 reduces to [C4, Algorithm 3.29] in the Grassmannian case. We urge the reader to verify the two previous remarks using the equation in Definition 3.13, which we will do in the proof of Theorems 5.1 and 5.3. In most of our examples above, to show the maximum number of possibilities, we have assigned a coloring that is strictly increasing. We encourage the reader to explore how the possibilities change for different colorings of the same diagrams.

Definition 3.46. A *degeneration path* is a sequence of admissible colored symplectic diagrams

$$(D_1, c^1_\bullet) \to (D_2, c^2_\bullet) \to \cdots \to (D_r, c^r_\bullet)$$

such that $(D_{i+1}, c^{i+1}_\bullet)$ is one of the admissible colored symplectic diagrams output by running Algorithm 3.43 on $(D_i, c^i_\bullet)$ for $1 \leq i < r$.

The main combinatorial theorem of this paper is the following.

Theorem 3.47. *Let $(D, c_\bullet)$ be an admissible colored symplectic diagram for $SF(k_1, \dots, k_h; n)$. Let $V(D, c_\bullet)$ be the symplectic restriction variety associated to $(D, c_\bullet)$. Then, in terms of the Schubert basis of $SF(k_1, \dots. k_h; n)$, the cohomology class $[V(D, c_\bullet)]$ can be expressed as*

$$[V(D, c_\bullet)] = \sum \alpha_{\lambda;\mu;c_\bullet} \sigma_{\lambda;\mu;c_\bullet},$$

where $\alpha_{\lambda;\mu;c_\bullet}$ is the number of degeneration paths starting with $(D, c_\bullet)$ and ending with the symplectic diagram $D(\lambda; \mu; c_\bullet)$.

A more precise version of Theorem 1.1 is the following corollary of Theorem 3.47.

Corollary 3.48. *Let $\sigma_{a_\bullet, c_\bullet}$ be a Schubert class in $SF(k_1, \dots, k_h; n)$. If $a_j < 2j - 1$ for some $1 \leq j \leq k_h$, then $i^* \sigma_{a_\bullet, c_\bullet} = 0$. Otherwise, let $D(\sigma_{a^i_\bullet, c^i_\bullet})$ be the diagrams associated to the Schubert class $\sigma_{a_\bullet, c_\bullet}$ (obtained by running Algorithm 3.38 on $P(a_\bullet, c_\bullet)$). Express*

$$i^* \sigma_{a_\bullet, c_\bullet} = \sum c_{\lambda;\mu;c_j} \sigma_{\lambda;\mu;c_j}$$

in terms of the Schubert basis of $SF(k_1, \dots, k_h; n)$. Then $c_{\lambda;\mu,c_i}$ is the number of degeneration paths starting with one of the diagrams $D(\sigma_{a^i_\bullet, c^i_\bullet})$ and ending with the symplectic diagram $D(\sigma_{\lambda;\mu;c_j})$.

Proof. In the next two sections, in Lemma 4.18 and Theorem 5.3, we will prove that the intersection of a general Schubert variety $\Sigma_{a_\bullet,c_\bullet}$ with $SF(k_1,\dots,k_h;n)$ has class equal to the sum of the classes of admissible restriction varieties associated to $P(a_\bullet,c_\bullet)$ by Algorithm 3.38. The corollary is then an immediate consequence of Theorem 3.47. $\square$

We now give three examples of Algorithm 3.43 to illustrate the algorithm and urge the reader to carry out similar calculations for themselves. We note that the algorithm is very efficient and it is no trouble at all to carry out calculations for n as large as 20 easily by hand.

Example 3.49. The first example is a computation in $SF(1,2,3;8)$.

$$\begin{array}{ccccccccc}
300\}_1 20\}_2 10\}_3 0 & \to & 200\}_1 00\}_2 10\}_3 0 & \to & 200]_1 00\}_2 10\}_3 0 & \to & 1]_1 0000\}_2 00\}_3 0 & \to & 1]_1 2200\}_2 00\}_3 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 100]_1 00\}_2 00\}_3 0 & \to & 11]_1 200\}_2 0\}_3 00 \\
 & & & & & & \downarrow & & \\
 & & & & & & 100]_1 0]_2 000\}_3 0 & &
\end{array}$$

The calculation shows that
$$i^*\sigma^{1,2,3}_{3,5,7} = \sigma^{1,2,3}_{1;3,1} + \sigma^{1,2,3}_{2;3,2} + \sigma^{1,2,3}_{3,4;1}.$$

Example 3.50. The second example calculates $i^*\sigma^{2,1,2,1}_{2,3,5,8}$ in $SF(2,4;10)$.

$$\begin{array}{ccccccc}
*00\}_2 3^*\}_1 20\}_2 000\}_1 00 & \to & *1]_2 00\}_1 2^*\}_2 000\}_1 000 & \to & 1]_2 1]_2 1]_1 00\}_1 00000 & \to & 0]_2 0]_2 0]_1 00]_1 00000 \\
 & & \downarrow & & & & \\
 & & 1]_2 1]_1 00\}_2 00\}_1 0000 & \to & 1]_2 1]_1 00]_2 00\}_1 0000 & \to & 0]_2 0]_1 00]_2 0]_1 00000 \\
 & & & & \downarrow & & \\
 & & & & 1]_2 1]_1 1]_2 100\}_1 0000 & &
\end{array}$$

We conclude that $i^*\sigma^{2,1,2,1}_{2,3,5,8} = \sigma^{2,2,1,1}_{1,2,3,5;} + \sigma^{2,1,2,1}_{1,2,4,5;} + \sigma^{2,1,2,1}_{1,2,3;4}$.

Example 3.51. As a final more complicated example, we compute $i^*\sigma^{1,2,3,2}_{2,3,6,9}$ in $SF(1,3,4;10)$.

$$\begin{array}{ccccccc}
*00\}_1 3^*\}_2 000\}_3 100\}_2 0 & \to & 1]_2 1]_1 00\}_3 100\}_2 000 & \to & 1]_2 1]_1 00]_3 100\}_2 000 & \to & 1]_2 1]_1 11]_3 00\}_2 0000 \\
\downarrow & & & & \downarrow & & \\
1]_1 00\}_2 00\}_3 100\}_2 00 & & & & 1]_2 1]_1 1]_3 0000\}_2 000 & &
\end{array}$$

$$\begin{array}{ccccccc}
1]_1 00\}_2 00\}_3 100\}_2 00 & & & & & & \\
\downarrow & & & & & & \\
1]_1 00]_2 00\}_3 100\}_2 00 & \to & 1]_1 1]_2 200\}_3 10\}_2 000 & \to & 1]_1 1]_2 100\}_3 00\}_2 000 & \to & 1]_1 1]_2 100]_3 00\}_2 000 \\
\downarrow & & & & & & \\
1]_1 00]_2 0]_3 1000\}_2 00 & \to & 1]_1 11]_2 1]_3 00\}_2 0000 & & & &
\end{array}$$

$$\begin{array}{ccc}
 & 1]_1 00]_2 0]_3 1000\}_2 00 & \\
\swarrow & & \searrow \\
1]_1 1]_2 00]_3 0000\}_2 00 & & 1]_1 1]_3 1]_2 0000\}_2 000
\end{array}$$

We conclude that $i^*\sigma^{1,2,3,2}_{2,3,6,9} = \sigma^{2,1,3,2}_{1,2,4;4} + \sigma^{2,1,3,2}_{1,2,3;3} + \sigma^{1,2,3,2}_{1,2,5;3} + \sigma^{1,2,3,2}_{1,3,4;4} + \sigma^{1,3,2,2}_{1,2,3;3} + \sigma^{1,2,3,2}_{1,2,4;2}$.

We conclude this section by proving that Algorithm 3.43 is well-defined and terminates. The proof of Theorem 3.47 is geometric and will be taken up in the next two sections.

Proposition 3.52. *Let $(D,c_\bullet)$ be an admissible colored symplectic diagram for $SF(k_1,\dots,k_h;n)$. Algorithm 3.43 replaces $(D,c_\bullet)$ with admissible, colored, symplectic diagrams. Furthermore, the algorithm terminates after finitely many steps.*

Proof. The formation of D^a from D is defined exactly as in [C4]. Therefore, by [C4, Proposition 3.39], D^a is a (not necessarily admissible) symplectic diagram. Since D satisfies condition (A1) and $\nu \neq 1$, there cannot be a bracket in the position immediately following ν in D. The diagrams of the type D^b are all formed by moving a bracket to the right of $\pi(\nu)$ to position $p = \pi(\nu)$. Moving a bracket to the left does not effect conditions (3) and (4) and can only improve condition (2). Since the position to which we are moving the bracket is not occupied, condition (1) holds. Conditions (S1), (S3) and (S4) are unaffected and the inequality can only improve in condition (S2) when we move a bracket to the left. We conclude that both D^a and D^b are (not necessarily admissible) symplectic diagrams.

Since Algorithms 3.29 and 3.38 differ from the corresponding algorithms in [C4], we need to check that the outputs of these algorithms are still symplectic diagrams. The diagram D^a may fail to satisfy condition (A2). The formation of D^a does not change the positions of the brackets or braces. Hence, the quantities $p(\}^j)$ and x_j remain constant. In cases (1)(i) and (2)(ii)(a), the quantities r_j also remain constant. Hence, in these cases D^a satisfies condition (A2). In cases (1)(ii), (2)(i) and (2)(ii)(b), these quantities remain unchanged except r_i increases by two in (1)(ii) and r_κ increases by two in cases (2)(i) and (2)(ii)(b). We conclude that D^a can violate condition (A2) only by one for one index when the equality $x_j = k - j + 1 - \frac{p(\}^j)-r_j}{2}$ holds for $j = i$ in case (1)(ii) and $j = \kappa$ in cases (2)(i) and (2)(ii)(b). For future reference, observe that if equality holds in the inequality in condition (A2) for an index j' in an admissible diagram D, then equality holds for every index $j \geq j'$. Notice that since x_i (respectively, x_κ) increases by one in case (1)(ii) (respectively, (2)(i) and (2)(ii)(b)) in the diagrams of the type D^b, we conclude that D^b always satisfies condition (A2). The initial diagram P fails to satisfy condition (A2) for indices j such that $a_j = 2j - 1$.

We next check that Algorithm 3.29 preserves condition (A2) and the fact that the diagrams are symplectic diagrams. This has been checked in the proof of [C4, Proposition 3.39] except in the case when position $p(\}^i) - 1$ is occupied by a brace. In that case, we move $\}^i$ to the first unoccupied position to the left of $p(\}^i)$ and change the integers in the sequence as specified by Algorithm 3.29. The conditions (1)-(4) do not change for any of the braces or brackets except for the brace that we move. Since we are moving the brace to an unoccupied position, condition (1) holds. Since the original diagram is admissible, there is an unoccupied position between the brackets and braces, so condition (2) holds. Conditions (3) and (4) hold by construction. Observe that by changing one of the integers from $i + l + 1$ to $i + l$, we guarantee condition (4). Similarly, (S1) holds for $\}^{i+l}$ and $\}^{i+l+1}$ by construction. If the diagram is in order, applying Algorithm 3.29 preserves order. If η violates order, then η remains the only integer violating the order and conditions (S3) and (S4) are preserved. Finally, condition (S2) must hold for the new $\}^{i+l}$ (the only brace for which the quantities change) since compared to the $\}^{i+l}$ before applying the algorithm τ does not change, $p(\}^{i+l})$ decreases by one and r_{i+l} increases by one. Hence, the inequality remains the same. There is one possible exception: if $i + l = k_h - s$, then τ_{i+l} may increase by one provided that the leftmost zero is to right of all the brackets. However, in that case, condition (S2) holds automatically. We conclude that Algorithm 3.29 preserves symplectic diagrams. Similarly, the operation preserves the inequality in condition (A2). If no two braces occupy the same position, then x_{i-1} and r_{i-1} both increase by one and $p(\}^{i-1})$ decreases by one, preserving the inequality in condition (A2). If two braces occupy the same position when running Algorithm 3.29 on $(D, c_\bullet)$, then the diagram must look like $\cdots\ i]i+1 \cdots i+l \cdots \}^{i+l}0\}^{i+l-1} \cdots \}^i 0 \}^{i-1} \cdots$. Hence, if after applying the algorithm condition (A2) is violated by 1 for one of the braces, condition (A2) must violated for $\}^{i-1}$ by one in $(D, c_\bullet)$. We conclude that Algorithm 3.29 preserves condition (A2).

We next check that Algorithm 3.38 replaces marked diagrams with admissible symplectic diagrams. Note that diagrams of the type D^{β} may have more braces that fail to satisfy condition (A2) when running the algorithm on initial diagrams $P(a_{\bullet}, c_{\bullet})$. However, after every run of the loop in Algorithm 3.38, there is at least one more bracket and one fewer brace. Since the total number of braces is bounded, Algorithm 3.38 terminates.

By construction, it is easy to see that the diagrams D^{α} and D^{β} satisfy conditions (1)-(4), (S1) and (S2). When running Algorithm 3.38 on a diagram derived from D^a, the only possibly out of order integer is the marked integer δ. The diagrams underlying the marked diagrams occurring while running the algorithm may fail to satisfy (S3) or (S4) because δ may be to the left of $]^s$. Nevertheless, the loop terminates when there are no brackets or braces between the two markings. At that point, the diagram is in order. Hence, (S3) and (S4) hold for the resulting diagrams. The fact that the resulting diagrams satisfy condition (A1) is built into Algorithm 3.38. When running the Algorithm 3.38 on a diagram D^a, the formation of D^{β} does not create any new braces failing condition (A2). When running Algorithm 3.38 on an initial diagram P, the diagrams may have one more integer other than δ out of order. In that case, the integer is $\cdots \}^{j+1} j-1 \}^{j} \delta^* \}^{j-1} \cdots$ and at the next stage of the algorithm, we swap δ and $j-1$ to obtain a diagram where again only δ (possibly) violates the order. The fact that the diagrams D^{β} resulting after the swap satisfy condition (3) has been explicitly built into the construction. We conclude that every diagram resulting by running Algorithm 3.29 and 3.38 are admissible symplectic diagrams. Hence, Algorithm 3.43 replaces admissible symplectic diagrams or an initial diagram with a collection of admissible symplectic diagrams.

The termination of the algorithm is clear. In case (2)(i), the formation of D^a from D increases the number of positive integers in the sequence. In cases (1)(i), (1)(ii), (2)(ii)(a) and (2)(ii)(b), the formation of D^a from D either increases the number of positive integers in the initial part of the sequence or decreases at least one of the positive integers in the initial part of the sequence. The formation of D^b shifts one bracket to the left. Similarly, Algorithm 3.29 decreases at least one positive integer in the initial part of the sequence or increases the number of positive integers in the sequence and shifts at least one brace to the left. Similarly, Algorithm 3.38 decreases the number of braces and increases the number of brackets. Since the total number of brackets and braces is fixed at k_h, no new braces are formed during the algorithm and no bracket or brace ever moves to the right, each of these steps can only be repeated finitely many times until the resulting diagram is saturated and in order.

$\square$

Remark 3.53. It can be useful to estimate the sizes of restriction coefficients. We note that at each stage of the algorithm, a diagram is replaced by at most $h+1$ new diagrams. Here a stage of the algorithm should be interpreted as each replacement of $(D, c_{\bullet})$ with diagrams of type D^a or D^b or each run of the loop in Algorithm 3.38 replacing a marked diagram with diagrams of the type D^{α} or D^{β}.

4. The symplectic restriction varieties

In this section, we define symplectic restriction varieties in $SF(k_1, \ldots, k_h; n)$ and show that they can be represented by admissible colored symplectic diagrams. We develop the basic geometric properties of these varieties. The translation between the combinatorics and the geometry is almost identical to the Grassmannian case discussed in [C4]. For the convenience of the reader, we recall the main definitions from [C4].

Let Q denote a non-degenerate skew-symmetric form on an n-dimensional vector space V. Let L_{n_j} denote an isotropic subspace of Q of dimension n_j. Let $Q^{r_i}_{d_i}$ denote a linear space of dimension d_i such that the restriction of Q to it has corank r_i. Let K_i denote the kernel of the restriction of Q to $Q^{r_i}_{d_i}$.

A sequence $(L_\bullet, Q_\bullet)$ for $SG(k_h, n)$ is a partial flag of linear spaces

$$L_{n_1} \subset \cdots \subset L_{n_s} \subset Q^{r_{k_h-s}}_{d_{k_h-s}} \subset \cdots \subset Q^{r_1}_{d_1} \text{ such that}$$

- $\dim(K_i \cap K_l) \geq r_i - 1$ for $l > i$,
- $\dim(L_{n_j} \cap K_i) \geq \min(n_j, \dim(K_i \cap Q^{r_{k_h-s}}_{d_{k_h-s}}) - 1)$ for every $1 \leq j \leq s$ and $1 \leq i \leq k_h - s$.

Definition 4.1. A *colored sequence* $(L_\bullet, Q_\bullet, c_\bullet)$ for $SF(k_1, \dots, k_h; n)$ is a sequence $(L_\bullet, Q_\bullet)$ for $SG(k_h, n)$ together with a coloring $c_\bullet$ for $SF(k_1, \dots, k_h; n)$.

The main geometric objects of this paper are colored sequences satisfying further properties. A colored sequence for $SF(k_1, \dots, k_h; n)$ is *in order* if the underlying sequence $(L_\bullet, Q_\bullet)$ satisfies:

- $K_i \cap K_l = K_i \cap K_{i+1}$, for all $l > i$ and $1 \leq i \leq k_h - s$, and
- $\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q^{r_{k_h-s}}_{d_{k_h-s}}))$, for $1 \leq j \leq s$ and $1 \leq i < k_h - s$.

A colored sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is *in perfect order* if

- $K_i \subseteq K_{i+1}$, for $1 \leq i < k_h - s$, and
- $\dim(L_{n_j} \cap K_i) = \min(n_j, r_i)$ for all i and j.

Definition 4.2. A colored sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is called *saturated* if $d_i + r_i = n$, for $1 \leq i \leq k_h - s$.

Definition 4.3. A colored sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is called a *symplectic sequence* if it satisfies the following properties.

(GS1) The sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is either in order or there exists at most one integer $1 \leq \eta \leq k_h - s$ such that

$$K_i \subseteq K_l \quad \text{for} \quad l > i > \eta \quad \text{and} \quad K_i \cap K_l = K_i \cap K_{i+1} \quad \text{for} \quad i < \eta \quad \text{and} \quad l > i.$$

Furthermore, if $K_\eta \subseteq K_{k_h-s}$, then

$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q^{r_{k_h-s}}_{d_{k_h-s}})) \quad \text{for} \quad i < \eta \quad \text{and}$$

$$\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q^{r_{k_h-s}}_{d_{k_h-s}}) - 1) \quad \text{for} \quad i \geq \eta.$$

If $K_\eta \not\subseteq K_{k_h-s}$, then $\dim(L_{n_j} \cap K_i) = \min(n_j, \dim(K_i \cap Q^{r_{k_h-s}}_{d_{k_h-s}}))$ for all i.

(GS2) If $\alpha = \dim(K_i \cap Q^{r_{k_h-s}}_{d_{k_h-s}}) > 0$, then either $i = 1$ and $n_\alpha = \alpha$ or there exists at most one j_0 such that, for $j_0 \neq j > \min(i, \eta)$, $r_j - r_{j-1} = d_{j-1} - d_j$. Furthermore,

$$d_{j_0-1} - d_{j_0} \leq r_{j_0} - r_{j_0-1} + 2 - \dim(K_{j_0-1}) + \dim(K_{j_0-1} \cap Q^{r_{j_0}}_{d_{j_0}})$$

and $K_\eta \not\subset Q^{r_{j_0}}_{d_{j_0}}$.

Remark 4.4. Given a sequence $(L_\bullet, Q_\bullet, c_\bullet)$, the basic principles about skew-symmetric forms imply inequalities among the invariants of a sequence. The evenness of rank implies that $d_i - r_i$ is even for every $1 \leq i \leq k_h - s$. The corank bound implies that $r_i - \dim(Q^{r_i}_{d_i} \cap K_{i-1}) \leq d_{i-1} - d_i$. The linear space bound implies that $2(n_s + r_i - \dim(K_i \cap L_{n_s})) \leq r_i + d_i$ for every $1 \leq i \leq k_h - s$. These inequalities are implicit in the sequence $(L_\bullet, Q_\bullet, c_\bullet)$.

Remark 4.5. For a symplectic sequence $(L_\bullet, Q_\bullet, c_\bullet)$, the coloring $c_\bullet$, the invariants n_j, r_i, d_i and the dimensions $\dim(L_{n_j}, K_i)$ and $\dim(Q^{r_h}_{d_h} \cap K_i)$ determine the sequence $(L_\bullet, Q_\bullet, c_\bullet)$ up to the action of the symplectic group. This will become clear when we construct these sequences by choosing bases.

Definition 4.6. A colored symplectic sequence $(L_\bullet, Q_\bullet, c_\bullet)$ is *admissible* if it satisfies the following additional conditions:

(GA1) $n_j \neq \dim(L_{n_j} \cap K_i) + 1$ for any $1 \leq j \leq s$ and $1 \leq i \leq k_h - s$.
(GA2) Let x_i denote the number of isotropic subspaces L_{n_j} that are contained in K_i. Then

$$x_i \geq k_h - i + 1 - \frac{d_i - r_i}{2}.$$

The symplectic restriction varieties will be defined in terms of colored admissible sequences.

The translation between sequences and symplectic diagrams. Colored symplectic sequences can be represented by colored symplectic diagrams introduced in §3. An isotropic linear space L_{n_j} is represented by a bracket] in position n_j. A linear space $Q^{r_i}_{d_i}$ is represented by a brace } in position d_i such that there are exactly r_i positive integers less than or equal to i to the left of the i-th brace. Finally, $\dim(L_{n_j} \cap K_i)$ and $\dim(Q^{r_l}_{d_l} \cap K_i)$, $l > i$, are recorded by the number of positive integers less than or equal to i to the left of $]^j$ and $\}^l$, respectively. The colorings associated to the sequence and the diagram are the same.

More explicitly, given a colored symplectic sequence $(L_\bullet, Q_\bullet, c_\bullet)$, the corresponding symplectic diagram $D(L_\bullet, Q_\bullet, c_\bullet)$ is determined as follows: The sequence of integers begins with $\dim(L_{n_1} \cap K_1)$ integers equal to 1, followed by $\dim(L_{n_1} \cap K_i) - \dim(L_{n_1} \cap K_{i-1})$ integers equal to i, for $2 \leq i \leq k_h - s$, in increasing order, followed by $n_1 - \dim(L_{n_1} \cap K_{k_h - s})$ integers equal to 0. The sequence then continues with $\dim(L_{n_j} \cap K_1) - \dim(L_{n_{j-1}} \cap K_1)$ integers equal to 1, followed by $\dim(L_{n_j} \cap K_i) - \max(\dim(L_{n_{j-1}} \cap K_i), \dim(L_{n_j} \cap K_{i-1}))$ integers equal to i in increasing order, followed by $n_j - \max(n_{j-1}, \dim(L_{n_j} \cap K_{k_h - s}))$ zeros for $j = 2, \dots, s$ in increasing order. The sequence then continues with $\dim(Q^{r_{k_h-s}}_{d_{k_h-s}} \cap K_1) - \dim(L_{n_s} \cap K_1)$ integers equal to 1, followed by $\dim(Q^{r_{k_h-s}}_{d_{k_h-s}} \cap K_i) - \max(\dim(Q^{r_{k_h-s}}_{d_{k_h-s}} \cap K_{i-1}), \dim(L_{n_s} \cap K_i))$ integers equal to i in increasing order, followed by zeros until position d_{k_h-s}. Between positions d_i and d_{i-1} $(i > k_h - s)$, the sequence has $\dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_1) - \dim(Q^{r_i}_{d_i} \cap K_1)$ integers equal to 1, followed by $\dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_l) - \max(\dim(Q^{r_i}_{d_i} \cap K_l), \dim(Q^{r_{i-1}}_{d_{i-1}} \cap K_{l-1}))$ integers equal to l in increasing order, for $l \leq i - 1$, followed by zeros until position d_{i-1}. Finally, the sequence ends with $n - d_1$ zeros. The brackets occur at positions n_j and the braces occur at positions d_i. The colorings are the same.

Example 4.7. Given a sequence $L_1(1) \subset L_3(3) \subset L_5(2) \subset L_7(1) \subset Q^6_{10}(2) \subset Q^3_{13}(3) \subset Q^2_{16}(1)$ for $SF(3,5,7;18)$, where the numbers in parentheses denote the coloring, and the relations $K_2 \subset K_3$, $\dim(K_1 \cap Q^3_{13}) = 1$, $\dim(K_1, L_1) = 1$, $L_3 = K^3_{13}$ and $L_5 = K^6_{16} \cap L_7$, the corresponding diagram is $1]_1 22]_3 33]_2 00]_1 300\}_2 000\}_3 100\}_1 00$. The dimensions of the isotropic spaces are $1, 3, 5, 7$, hence the brackets occur at these positions. The dimensions of the non-isotropic spaces are $10, 13, 16$, so the braces occur at these positions. The dimensions of the kernels are $6, 3, 2$, so the sequence has 2 integers equal to 1. Since $\dim(K_1 \cap Q^3_{13}) = 1$ and $\dim(K_1, L_1) = 1$, one of these occur at position 1 and one at position 14. The sequence then has 2 integers equal to 2 and since $L_3 = K^3_{13}$, these occur at positions 2 and 3. Finally, the sequence has 3 integers equal to 3. Since $L_5 = K^6_{16} \cap L_7$, these have to occur at positions 4,5 and 8.

Proposition 4.8. *The diagram $D(L_\bullet, Q_\bullet, c_\bullet)$ is a symplectic diagram for $SF(k_1, \dots, k_h; n)$. Furthermore, if $(L_\bullet, Q_\bullet, c_\bullet)$ is admissible, then $D(L_\bullet, Q_\bullet, c_\bullet)$ is admissible.*

Proof. The construction of $D(L_\bullet, Q_\bullet, c_\bullet)$ from $(L_\bullet, Q_\bullet, c_\bullet)$ for $SF(k_1, \dots, k_h; n)$ is identical to the construction of $D(L_\bullet, Q_\bullet)$ from $(L_\bullet, Q_\bullet)$ for $G(k_h, n)$ except for the data of the coloring. By [C4, Proposition 4.9], $D(L_\bullet, Q_\bullet)$ is a symplectic diagram for $G(k_h, n)$, which is admissible if $(L_\bullet, Q_\bullet)$ is admissible. We conclude that $D(L_\bullet, Q_\bullet, c_\bullet)$ is a symplectic diagram for $SF(k_1, \dots, k_h; n)$ which is admissible if $(L_\bullet, Q_\bullet, c_\bullet)$ is admissible. $\square$

Remark 4.9. Proposition 4.8 explains the conditions in the definition of a symplectic diagram in geometric terms. Condition (1) holds since every linear space in the flag is a distinct vector space and its dimension corresponds to the position of the corresponding bracket or brace. Condition (2) reflects the fact that isotropic subspaces precede the non-isotropic ones in the sequence. Integers in the sequence equal to i represent vectors in the kernel of the restriction of Q to $Q^{r_i}_{d_i}$. Hence, condition (3) holds. Condition (4) of Definition 3.1 is implied by the evenness of rank and simply states that $d_i - r_i$ is even. Condition (S1) is a translation of the corank bound saying that the codimension of $K_{i-1} \cap K_i$ in K_i is bounded by the codimension of $Q^{r_i}_{d_i}$ in $Q^{r_{i-1}}_{d_{i-1}}$. Condition (S2) is a consequence of the linear space bound since the linear space $Q^{r_i}_{d_i}$ contains a linear space of dimension at least τ_i.

Conversely, we can associate an admissible sequence to every admissible symplectic diagram $(D, c_\bullet)$ for $SF(k_1, \dots, k_h; n)$. By Darboux's Theorem, we can take the skew-symmetric form to be defined by $\sum_{i=1}^m x_i \wedge y_i$. Let the dual basis for x_i, y_i be e_i, f_i such that $x_i(e_j) = \delta_i^j$, $y_i(f_j) = \delta_i^j$ and $x_i(f_j) = y_i(e_j) = 0$. Given an admissible symplectic diagram, we associate $e_1, \dots, e_{p(]^s)}$ to the integers to the left of $]^s$ in order. We then associate $e_{p(]^s)+1}, \dots, e_{r'}$ to the positive integers to the right of $]^s$ and left of $\}^{k_h - s}$ in order. Let $e_{i_1}, \dots, e_{i_l}$ be vectors that have so far been associated to zeros. Then associate $f_{i_1}, \dots, f_{i_l}$ to the remaining zeros to the left of $\}^{k_h-s}$ in order. If there are any zeros to the left of $\}^{k_h-s}$ that have not been assigned a basis vector, assign them $e_{r'+1}, f_{r'+1}, \dots, e_{r''}, f_{r''}$ in pairs in order. Continuing this way, if there is a positive integer between $\}^{i+1}$ and $\}^i$ associate to it the smallest index basis element e_α that has not yet been assigned. Assume that the integers equal to $i+1$ have been assigned the vectors $e_{j_1}, \dots, e_{j_l}$. Assign to the zeros between $\}^{i+1}$ and $\}^i$, the vectors $f_{j_1}, \dots, f_{j_l}$. If there are any zeros between $\}^{i+1}$ and $\}^i$ that have not been assigned a vector, assign them $e_{\alpha+1}, f_{\alpha+1}, \dots, e_\beta, f_\beta$ in pairs until the zeros are exhausted. Let L_{n_j} be the span of the basis elements associated to the integers to the left of $]^j$. Let $Q^{r_i}_{d_i}$ be the span of the basis elements associated to the integers to the left of $\}^i$. We thus obtain a sequence $(L_\bullet, Q_\bullet, c_\bullet)$ whose associated symplectic diagram is $(D, c_\bullet)$. The vectors associated to the integers in a marked diagram occurring in Algorithm 3.38 is the same as the sequence associated to the underlying diagram $(D, c_\bullet)$.

Example 4.10. Let $D = 11]_3 233]_1 00]_2 30000\}_2 200000\}_3 100\}_1 000$ be a diagram for $SF(2, 4, 6; 24)$. To this diagram we associate the vectors

$$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, f_6, f_7, e_9, f_9, e_{10}, f_4, f_5, f_8, e_{11}, f_{11}, e_{12}, f_3, f_{10}, f_1, f_2, f_{12}$$

in order. Then $L_2(3)$ is the span of e_1 and e_2. $L_5(1)$ is the span of e_i for $1 \le i \le 5$. $L_7(2)$ is the span of e_i for $1 \le i \le 7$. $Q^6_{12}(3)$ is the span of e_i, $1 \le i \le 9$ and f_6, f_7, f_9. $Q^4_{18}(3)$ is the span of $Q^6_{12}(3)$ and the vectors $e_{10}, e_{11}, f_4, f_5, f_8, f_{11}$. Finally, $Q^3_{21}(1)$ is the span of $Q^4_{18}(3)$ and the vectors e_{12}, f_3, f_{10}.

Remark 4.11. As observed in [C4, Remark 4.13], the construction of $(L_\bullet, Q_\bullet, c_\bullet)$ from $(D, c_\bullet)$ is well-defined thanks to conditions (4), (S1), (S2) and (S3). The formation of a sequence

$(L_\bullet, Q_\bullet, c_\bullet)$ from an admissible diagram $(D, c_\bullet)$ and the formation of an admissible symplectic diagram $(D, c_\bullet)$ from an admissible sequence are inverse constructions. Further note that equivalent symplectic diagrams correspond to permutations of the basis elements that do not change the vector spaces in $(L_\bullet, Q_\bullet, c_\bullet)$.

We are now ready to define symplectic restriction varieties.

Definition 4.12. Let $(L_\bullet, Q_\bullet, c_\bullet)$ be an admissible sequence for $SF(k_1, \ldots, k_h; n)$. Then the *symplectic restriction variety* $V(L_\bullet, Q_\bullet, c_\bullet)$ is the Zariski closure of the locus in $SF(k_1, \ldots, k_h; n)$ parameterizing

$$\{(W_1, \ldots, W_h \in SF(k_1, \ldots, k_h; n) \mid \dim(W_u \cap L_{n_j}) = \#\{c_l \mid l \leq j, c_l \leq u\} \text{ for } 1 \leq j \leq s,$$
$$\dim(W_u \cap Q_{d_i}^{r_i}) = \#\{c_l \mid l \leq k_h - i + 1, c_l \leq u\} \text{ and}$$
$$\dim(W_u \cap K_i)) = \#\{c_l \mid l \leq x_i, c_l \leq u\} \text{ for } 1 \leq i \leq k_h - s\}.$$

Remark 4.13. The geometric reasons for imposing conditions (A1) and (A2) in Definition 3.11 are now clear. Condition (A1) is an immediate consequence of the kernel bound. If $\dim(L_{n_j} \cap K_i) = n_j - 1$ and a linear space of dimension $k_h - i + 1$ intersects n_j in dimension j and K_i in dimension $j-1$, then the linear space is contained in $L_{n_j}^{\perp}$. Hence, we need to impose condition (A1).

The inequality
$$x_i \geq k_h - i + 1 - \frac{d_i - r_i}{2}$$
is an immediate consequence of the linear space bound. We require the k_h-dimensional isotropic subspaces to intersect $Q_{d_i}^{r_i}$ in a subspace of dimension $k_h - i + 1$ and to intersect the singular locus of $Q_{d_i}^{r_i}$ in a subspace of dimension x_i. By the linear space bound, any linear space of dimension $k_h - i + 1$ has to intersect the singular locus in a subspace of dimension at least $k_h - i + 1 - \frac{d_i - r_i}{2}$, hence the inequality in condition (A2).

Example 4.14. The two most basic examples of symplectic restriction varieties are:

(1) A Schubert variety $\Sigma_{\lambda;\mu;c_\bullet}$ in $SF(k_1, \ldots, k_h; n)$, which is the restriction variety associated to a symplectic diagram $D(\sigma_{\lambda;\mu;c})$, and
(2) The intersection $\Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \ldots, k_h; n)$ of a general Schubert variety in $F(k_1, \ldots, k_h; n)$ satisfying $a_j > 2j - 1$ for all $1 \leq j \leq k_h$ with $SF(k_1, \ldots, k_h; n)$, which is the restriction variety associated to $D(a_\bullet, c_\bullet)$.

In general, symplectic restriction varieties interpolate between these two examples. Unlike the case of Grassmannians, the intersection $\Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \ldots, k_h; n)$ when $a_j = 2j - 1$ for some j need not be a symplectic restriction variety. However, it can be degenerated into a union of symplectic restriction varieties as we will see in the next section.

Lemma 4.15. *A symplectic restriction variety corresponding to a saturated and perfectly ordered admissible sequence is a Schubert variety in $SF(k_1, \ldots, k_h; n)$. Conversely, every Schubert variety in $SF(k_1, \ldots, k_h; n)$ can be represented by such a sequence.*

Proof. The projection $\pi_h : SF(k_1, \ldots, k_h; n) \to SG(k_h, n)$ maps a symplectic restriction variety corresponding to a saturated and perfectly ordered admissible sequence to the same type of restriction variety in $SG(k_h, n)$. By [C4, Lemma 4.17], the latter variety is a Schubert variety Σ in $SG(k_h, n)$. Therefore, the original symplectic restriction variety is a Schubert variety in $SF(k_1, \ldots, k_h; n)$. $\square$

The next proposition shows that symplectic restriction varieties are irreducible and calculates their dimension. Note that the dimension of a symplectic diagram introduced in Definition 3.13 is equal to the dimension of the corresponding symplectic restriction variety.

Proposition 4.16. *Let $(L_\bullet, Q_\bullet, c_\bullet)$ be an admissible colored sequence. Then the symplectic restriction variety $V(L_\bullet, Q_\bullet, c_\bullet)$ is an irreducible subvariety of $SF(k_1, \dots, k_h; n)$ of dimension*

$$\dim(V(L_\bullet, Q_\bullet, c_\bullet)) = \sum_{j=1}^{s}(n_j - j) + \sum_{i=1}^{k_h - s}(d_i - 1 - 2k_h + 2i + x_i) + \dim(c_\bullet). \tag{1}$$

Proof. The projection $\pi_h(V(L_\bullet, Q_\bullet, c_\bullet))$ is the symplectic restriction variety $V(L_\bullet, Q_\bullet)$ in $SG(k_h, n)$. By [C4, Proposition 4.21], the latter variety is irreducible of dimension

$$\sum_{j=1}^{s}(n_j - j) + \sum_{i=1}^{k_h - s}(d_i - 1 - 2k_h + 2i + x_i).$$

The fibers of the projection π_h over the Zariski open set defining $V(L_\bullet, Q_\bullet, c_\bullet)$ are open subsets of the Schubert variety corresponding to the class $c_\bullet$ in $F(k_1, \dots, k_{h-1}; k_h)$. By the Theorem on the Dimension of Fibers [S, I.6.7], $V(L_\bullet, Q_\bullet, c_\bullet)$ is irreducible of the claimed dimension. □

Next, we show that the intersection of a general Schubert variety $\Sigma_{a_\bullet, c_\bullet}$ with $SF(k_1, \dots, k_h; n)$ is non-empty if and only if $a_j \geq 2j - 1$ for $1 \leq j \leq k_h$. Furthermore, the intersection is a symplectic restriction variety if $a_j > 2j-1$ for $1 \leq j \leq k_h$. Otherwise, the class of the intersection is the sum of the classes of the restriction varieties derived from $D(a_\bullet, c_\bullet)$ by Algorithm 3.38. We postpone the proof of the last statement to the next section.

Lemma 4.17. *Let $\Sigma_{a_\bullet, c_\bullet}$ be a Schubert variety defined with respect to a general partial flag $F_{a_1} \subset \dots \subset F_{a_{k_h}}$. Then $\Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \dots, k_h; n) \neq \emptyset$ if and only if $a_j \geq 2j - 1$ for $1 \leq j \leq k_h$.*

Proof. Suppose $a_j < 2j - 1$ for some j. If $[W_1 \subset \dots \subset W_h] \in \Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \dots, k_h; n)$, then $W_h \cap F_{a_j}$ is an isotropic subspace of $Q \cap F_{a_j}$ of dimension at least j. Since F_{a_j} is general, the corank of $Q \cap F_{a_j}$ is 0 or 1 depending on whether a_i is even or odd. By the linear space bound, the largest dimensional isotropic subspace of $Q \cap F_{a_j}$ has dimension less than or equal to $j - 1$. Therefore, W_h cannot exist and $\Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \dots, k_h; n) = \emptyset$.

Conversely, let $a_j = 2j - 1$ for every j. Then $G_1 = F_1$ is isotropic, $G_2 = F_1^\perp$ in F_3 is the unique two-dimensional isotropic subspace of $Q \cap F_3$ containing G_1. By induction, we see that $G_j = G_{j-1}^\perp$ is the unique subspace of dimension j isotropic with respect to $Q \cap F_{2i-1}$ that contains G_{j-1}. Continuing this way, we construct a unique isotropic subspace W_h of dimension k_h contained in $\Sigma_{a_\bullet, c_\bullet} \cap SG(k_h, n)$. The flag $F_\bullet$ induces a complete flag $G_\bullet$ on W_h and the intersection $\Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \dots, k_h; n)$ is the Schubert variety Ω in $F(k_1, \dots, k_{h-1}; k_h)$ corresponding to $c_\bullet$. In particular, the intersection is non-empty. If $a_j \geq 2j - 1$, Ω is still contained in $\Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \dots, k_h; n)$, hence this intersection is non-empty. □

Lemma 4.18. *Let $\Sigma_{a_\bullet, c_\bullet}$ be a Schubert variety defined with respect to a general partial flag $F_{a_1} \subset \dots \subset F_{a_{k_h}}$ such that $a_j \geq 2j - 1$ for $1 \leq j \leq k_h$. Then $\Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \dots, k_h; n) = V(P(a_\bullet, c_\bullet))$ and is irreducible. Furthermore, if $a_j > 2j - 1$ for $1 \leq j \leq k_h$, then $\Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \dots, k_h; n) = V(D(a_\bullet, c_\bullet))$.*

Proof. The Schubert variety $\Sigma_{a_\bullet, c_\bullet}$ is irreducible. The complement of the open cell in $\Sigma_{a_\bullet, c_\bullet}$ is a finite union of lower dimensional Schubert varieties. By Kleiman's Transversality Theorem, their intersection with $SF(k_1, \dots, k_h; n)$ have lower dimension and cannot form components of the intersection. We may, therefore, restrict our attention to the open cell U. The lemma follows by induction on h and k_h. If $h = 1$, the irreducibility is clear by induction on k_h and has been observed in [C4, Lemma 4.18]. The fibers of the restriction of the projection $\pi_h : SF(k_1, \dots, k_h; n) \to SG(k_h, n)$ to $U \cap SF(k_1, \dots, k_h; n)$ are open subsets of a Schubert variety in $F(k_1, \dots, k_{h-1}; k_h)$. Therefore, the intersection $\Sigma_{a_\bullet, c_\bullet} \cap SF(k_1, \dots, k_h; n)$ is irreducible and when $a_j > 2j - 1$ for all j equal to $V(D(a_\bullet, c_\bullet))$. □

Similarly, we can associate a variety to a marked diagram $(D, c_\bullet, *)$ occurring while running Algorithm 3.38. Let e_δ be the vector corresponding to the rightmost marking δ and let F' be the span of the vectors up to and including the leftmost marking δ'. Let F be the span of F' and e_δ. Assume that the number of brackets to the left of position $\pi(\delta')$ inclusive is m. Then the variety corresponding to the marked diagram $(D, c_\bullet, *)$ is defined by

$$\{(W_1, \dots, W_h \in SF(k_1, \dots, k_h; n) \mid \dim(W_u \cap L_{n_j}) = \#\{c_l \mid l \leq j, c_l \leq u\} \text{ for } 1 \leq j \leq s,$$
$$\dim(W_u \cap Q_{d_i}^{r_i}) = \#\{c_l \mid l \leq k_h - i + 1, c_l \leq u\}$$
$$\dim(W_u \cap K_i)) = \#\{c_l \mid l \leq x_i, c_l \leq u\} \text{ for } 1 \leq i \leq k_h - s \text{ and } \dim(W_h \cap F) = m + 1\}.$$

As in the proof of Lemmas 4.16 and 4.18, induction on h and [C4, Lemma 4.18] imply that the variety associated to a marked diagram $(D, c_\bullet, *)$ is irreducible and has dimension equal to the dimension of the marked diagram.

5. The geometric explanation of the rule

In this section, we interpret the combinatorial rule introduced in §3 as a specialization of the corresponding symplectic restriction variety. We will analyze this specialization and show that the diagrams that replace $(D, c_\bullet)$ in Algorithm 3.43 parameterize the irreducible components of the flat limit and each of these components are generically reduced.

The main specialization. The specialization we use is identical to the specialization in the Grassmannian case introduced in [C4]. Of course, the flat limits of restriction varieties will typically have more irreducible components in the case of flag varieties. For the convenience of the reader, we recall the specialization. There are several cases depending on whether $(D, c_\bullet)$ is in order and whether $l(\kappa) < \rho(\kappa, \kappa - 1) - 1$ or not.

In the previous section, given an admissible symplectic diagram $(D, c_\bullet)$, we associated an admissible sequence by defining each of the vector spaces $(L_\bullet, Q_\bullet)$ as a union of basis elements that diagonalize the skew-symmetric form Q. All our specializations will replace exactly one of the basis elements $v = e_u$ or $v = f_u$ for some $1 \leq u \leq m$ in a one-parameter family $v(t) = e_u(t)$ or $v(t) = f_u(t)$. For $t \neq 0$, the resulting set of vectors will be a new basis for V, but when $t = 0$ two of the basis elements will become equal. Correspondingly, we get a one-parameter family of vector spaces $(L_\bullet(t), Q_\bullet(t))$ by changing every occurrence of the vector v to $v(t)$. We thus get a flat family of symplectic restriction varieties $V(D(t))$, where the restriction variety at $t \neq 0$ is defined with respect to the linear spaces $(L_\bullet(t), Q_\bullet(t))$. We now explicitly describe the specialization.

In case (1)(i), D is not in order, η is the unique integer violating the order, and ν is the leftmost integer equal to $\eta + 1$. Suppose that under the translation between symplectic diagrams and sequences of vector spaces, e_u is the vector associated to η and e_v is the vector associated to ν. Then consider the one-parameter family obtained by changing e_v to $e_v(t) = te_v + (1-t)e_u$ and keeping every other vector fixed. When the set of basis elements spanning a vector space L_{n_j} or $Q_{d_i}^{r_i}$ contains e_v, $L_{n_j}(t)$ or $Q_{d_i}^{r_i}(t)$ is obtained by replacing e_v with $e_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q_{d_i}^{r_i}(t) = Q_{d_i}^{r_i}$.

In case (1)(ii), D is not in order, η is the unique integer violating the order, $i > \eta$ does not occur in the sequence to the left of η and ν is the leftmost integer equal to $i+1$. Let e_u be the vector associated to η and let e_v be the vector associated to ν. Consider the one-parameter family obtained by changing f_v to $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains f_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

In case (2)(i), D is in order and $l(\kappa) < \rho(\kappa, \kappa-1) - 1$. Suppose that e_v is the vector associated to ν, the leftmost $\kappa+1$. Let e_u and f_u be two vectors associated to the zeros between $\}^{\kappa}$ and $\}^{\kappa-1}$. These exist since $l(\kappa) < \rho(\kappa, \kappa-1) - 1$. Consider the one-parameter specialization replacing f_v with $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains f_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

In case (2)(ii)(a), D is in order and $l(\kappa) = \rho(\kappa, \kappa-1) - 1$. Let ν be the leftmost integer equal to κ and suppose that e_v is the vector associated to ν. Let e_u be the vector associated to the $\kappa - 1$ following $\}^{\kappa}$. Then let $e_v(t) = te_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains e_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is obtained by replacing e_v with $e_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

Finally, in case (2)(ii)(b), D is in order, $l(\kappa) = \rho(\kappa, \kappa-1)-1$ and there does not exist an integer equal to κ to the left of κ. Let e_v be the vector associated to ν, the leftmost integer equal to $\kappa+1$ and let e_u be the vector associated to $\kappa - 1$ to the right of $\}^{\kappa}$. Then let $f_v(t) = tf_v + (1-t)e_u$. When the set of basis elements spanning a vector space L_{n_j} or $Q^{r_i}_{d_i}$ contains f_v, $L_{n_j}(t)$ or $Q^{r_i}_{d_i}(t)$ is obtained by replacing f_v with $f_v(t)$. Otherwise, $L_{n_j}(t) = L_{n_j}$ or $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$.

The flat limit of the vector spaces is easy to describe. If L_{n_j} or $Q^{r_i}_{d_i}$ does not contain the vector v, then $L_{n_j}(t) = L_{n_j}$ and $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$ for all t. Similarly, if L_{n_j} or $Q^{r_i}_{d_i}$ contains both of the basis elements spanning $v(t)$, then $L_{n_j}(t) = L_{n_j}$ and $Q^{r_i}_{d_i}(t) = Q^{r_i}_{d_i}$ for all $t \neq 0$. Then in the limit $L_{n_j}(0) = L_{n_j}$ and $Q^{r_i}_{d_i}(0) = Q^{r_i}_{d_i}$. A vector space changes under the specialization only when it contains one and not the other of the two basis elements spanning $v(t)$. In that case, in the limit $t = 0$ one swaps the basis vector with coefficient t with the basis vector with coefficient $(1-t)$.

In each of these cases, the set of limiting vector spaces is depicted by the symplectic diagram D^a. In case (1)(i), the degeneration swaps e_v and e_u. The effect on symplectic diagrams is simply switching η and ν. In case (1)(ii), the degeneration switches e_u and f_v. As a result, the restriction of Q to the linear space $Q^{r_i}_{d_i}$ has e_v in its kernel. The resulting set of linear spaces is depicted by switching η and the zero corresponding to f_v and replacing the integer corresponding to e_v with i. In case (2)(i), as a result of the specialization, the restriction of Q to $Q^{r_\kappa}_{d_\kappa}$ contains e_u and e_v in its kernel. The resulting set of vector spaces is denoted by switching the zero corresponding to e_u and ν to κ. The cases (2)(ii)(a) and (2)(ii)(b) are analogous to the cases (1)(i) and (1)(ii), respectively.

Correspondingly, we get a family of restriction varieties $V(D(t))$, where $V(D(t))$ is the symplectic restriction variety defined with respect to the linear spaces $L_\bullet(t)$ and $Q_\bullet(t)$. As long as $t \neq 0$, the corresponding varieties are projectively equivalent, hence form a flat family. The first theorem of this section, which is the main geometric theorem of this paper, describes the flat limit when $t = 0$.

Theorem 5.1. *Let $(D, c_\bullet)$ be an admissible colored symplectic diagram. Let $V(D(t))$ be the main specialization described in this section and let $V(D(0))$ be the flat limit at $t = 0$. Then*

$V(D(0))$ is supported along the union $\bigcup V(D_i)$ of restriction varieties, where D_i range over the symplectic restriction varieties associated to the diagrams replacing D in Algorithm 3.43. Furthermore, the flat limit is generically reduced along each $V(D_i)$. In particular, we have the equality of cohomology classes

$$[V(D)] = \sum [V(D_i)].$$

Proof. The proof has two steps. We first show that the flat limit is supported along the union of the varieties $V(D_i)$, where D_i ranges over the diagrams replacing D in Algorithm 3.43. We then do a local calculation to show that the flat limit is generically reduced along each D_i. The second step is a straightforward local calculation. The first step is the step that requires more care.

We determine the components of the support of the flat limit using a dimension count. The following observation puts very strong restrictions on the support of the flat limit.

Observation 5.2. Let $c_\bullet : c_1, \dots, c_{k_h}$ be the coloring. If $j \leq s$, let $\gamma_l(j)$ denote the number of integers among $c_1, \dots, c_j$ less than or equal to l. Then the linear spaces $W_l(t)$ parameterized by $V(D(t))$ intersect the linear spaces $L_{n_j}(t)$ in a subspace of dimension at least $\gamma_l(j)$. Similarly, let $\gamma_l(k_h - i + 1)$ denote the number of integers less than or equal to l among $c_1, \dots, c_{k_h - i+1}$. Then the linear spaces $W_l(t)$ parameterized by $V(D(t))$ intersect the linear spaces $Q_{d_i}^{r_i}(t)$ in a subspace of dimension at least $\gamma_l(k_h - i + 1)$. Since intersecting a proper variety in a subspace of dimension greater than or equal to a given integer is a closed condition, the flag elements W_l parameterized by the flat limit $V(D(0))$ have to intersect the linear spaces $L_{n_j}(0)$ in a subspace of dimension at least $\gamma_l(j)$ and the linear spaces $Q_{d_i}^{r_i}(0)$ in a subspace of dimension at least $\gamma_l(k_h - i + 1)$.

We use Observation 5.2 repeatedly to obtain strong restrictions on the flat limit. Let Y be an irreducible component of the support of $V(D(0))$. Since the family is flat, $\dim(Y) = \dim(V(D))$. Consider the complete flag (not necessarily isotropic) determined by the vectors ordered from left to right depicted by D^a. We claim that the rank conditions that the generic element of Y satisfies with respect to this flag determines Y.

Build a diagram $D(Y)$ that depicts the flag elements for which the dimension of the intersection of the flag elements W_h parameterized by the generic element in Y jumps. By Observation 5.2, each W_h has to intersect the flag elements $L_{n_j}(0)$ in dimension at least n_j and the flag elements $Q_{d_i}^{r_i}(0)$ in dimension at least $k_h - i + 1$. Consequently, the diagram $D(Y)$ is a diagram obtained from D^a. Assign $D(Y)$ the coloring obtained by Algorithm 3.22. A priori, the flag elements W_l for $l < h$ may intersect the flag elements more specially. However, by Observation 5.2, they have to satisfy the rank conditions imposed by $D(Y)$ assigned the coloring by Algorithm 3.22.

We now compare the dimension of linear spaces satisfying the rank conditions specified by $D(Y)$ to the dimension of the diagram D and determine under which conditions the two dimensions can be the same. This almost determines the set of diagrams occurring in Algorithm 3.43. The change in dimension is determined by Equation (1).

- If we replace a linear space $L^a_{n_{i+j}}$ of dimension n_{i+j} in the $(i+j)$-th position in $(L^a_\bullet, Q^a_\bullet)$ with a linear space F_{u_i} not contained in $(L^a_\bullet, Q^a_\bullet)$, then according to Equation (1) the dimension changes as follows. Let y^a_{i+j} be the index of the smallest index linear space $Q^{r_l}_{d_l}$ such that $L^a_{n_{i+j}} \subset K_l$. Similarly, let y^u_i be the smallest l such that $F_{u_i} \subset K_l$. The left sum in Equation (1) decreases by $n^a_{i+j} - u_i$. The quantities x_l increase by one for

$y_i^u \leq l < y_{i+j}^a$. Hence, the dimension decreases by $n_{i+j}^a - n_i^u + y_{i+j}^a - y_i^u$. By condition (S4) for D^a and condition (A1) for D, in D^a, there is at most one missing integer among the positive integers to the left of the brackets and the two integers preceding all but at most one of the brackets are equal. Therefore, $n_{i+j}^a - n_i^u + y_{i+j}^a - y_i^u \geq j$ with equality only if the index $i = x_{\nu-1}+1$ and in D the equality $p(]^{x_{\nu-1}+1}) - \pi(\nu) - 1 = y_{x_{\nu-1}+1} - \nu$ holds. Let the color of $L_{n_{i+j}}^a$ be c_{i+j}. Then the change in $\dim(c_\bullet)$ between the original coloring and the coloring assigned by Algorithm 3.22 can be calculated as follows. Let $i \leq u_1 < u_2 < \cdots < u_z = i+j$ be the indices with $c_{u_1} > c_{u_2} > \cdots > c_{u_z} = c_{i+j}$ and there does not exist and index between u_q and $i+j$ of color larger than or equal to c_{u_q}. Then $\dim(c_\bullet)$ increases by $(u_1 - i) + \#\{i \leq l < u_1 \mid c_l = c_{u_1}\} + \sum_{q=2}^{z}(u_q - u_{q-1} - 1) - \#\{u_{q-1} < l < u_q \mid c_l = c_{u_q}\})$. In particular, the increase is at most j with equality if and only if the color of every linear space in D^a with index $i, \dots, i+j-1$ is strictly less than c_{i+j}.

- If we replace the linear space $Q_{d_i}^{r_i,a}$ of dimension d_i^a in $(L_\bullet^a, Q_\bullet^a)$ with a non-isotropic linear space $F_{u_{k-i-j+1}}$ of dimension d_{i+j}^u, then, by Equation (1), the dimension changes as follows. Let x_{i+j}^u be the number of linear spaces that are contained in the kernel of the restriction of Q to $F_{u_{k-i-j+1}}$. Then the dimension decreases by $d_i^a - d_{i+j}^u + x_i^a - x_{i+j}^u$. This decrease is at least j, with strict inequality unless Condition (A1) fails for an integer equal to i. As in the previous case, the increase in $\dim(c_\bullet)$ is at most j with equality only if the color of every linear space with index between $i+j$ and i in D^a is strictly less than the color of $Q_{d_i}^{r_i,a}$.
- Finally, if we replace a linear space $Q_{d_{k-s}}^{r_{k-s}}$ of dimension d_{k-s}^a in $(L_\bullet^a, Q_\bullet^a)$ with an isotropic linear space $F_{u_{s+1}}$, then the first sum in Equation (1) changes by $u_{s+1} - s - 1$. The second sum changes by $-d_{k-s}^a + y_{s+1}^u - x_{k-s}^a + (2s+1)$, where y_{s+1}^u denotes the number of non-isotropic subspaces containing $F_{u_{s+1}}$ in the kernel of the restriction of Q. Hence, the total change is
$$-d_{k-s}^a + u_{s+1} - x_{k-s}^a - y_{s+1}^u + s.$$
Since the dimension of an isotropic linear space is bounded by $(d_{k-s} + r_{k-s})/2$, we conclude that this quantity is less than or equal to zero, with strict inequality unless $x_{k-s} = s$, $r_{k-s} = d_{k-s}$ and $d_{k-s} = u_{s+1}$. In this case, since we are not changing the ordering of the linear spaces, the coloring does not change.

We will now conclude from the dimension calculation that $D(Y)$ can only be one of the diagrams D^a or D^b replacing D in Algorithm 3.43. We need one further observation from the proof of Theorem 5.2 in [C4]. In the first case, if u_i is less than $\pi(\nu)$, by semi-continuity of the dimension of intersection of $W_h^\perp$ with $K_\nu \cap L_{x_{\nu-1}+1}$, not every linear space parameterized by $D(Y)$ can be in the flat limit. Hence, Y yielding such a $D(Y)$ has strictly smaller dimension.

Now the proof of the first statement of the theorem is immediate. If D^a is admissible and the generic linear space parameterized by Y does not satisfy more special rank conditions, then Y is contained in $V(D^a)$. Since the two varieties are irreducible of the same dimension, they are equal. If D^a violates condition (A1), then as argued in the proof of Theorem 5.2 in [C4], Y is the variety associated to the diagram obtained by applying Algorithm 3.29 to D^a. We will analyze the case when D^a fails condition (A2) in the next theorem. In any case, if D^a does not satisfy condition (A2), then, by the same argument, Y is equal to the variety obtained by tightening the marked diagram corresponding to D^a.

If $D(Y)$ is obtained by moving a bracket $]^i$ from D^a to a position p, then by the first dimension count and the requirement that $p \geq \pi(\nu)$, we can move at most one bracket and $p = \pi(\nu)$.

Furthermore, the bracket has to satisfy the equality $p(]^i) - \pi(\nu) = y_i - \nu + i - x_{\nu-1}$ in D and every bracket between $p(]^i)$ and $\pi(\nu)$ must have color strictly less than the color of $]^i$. If the resulting diagram is admissible, then by the second and third dimension counts, we cannot move a brace or replace a brace with a bracket without obtaining a strictly smaller dimensional locus. We conclude that Y has to be contained in $D^b(]^i)$. Since they have the same dimension, Y is equal to $D^b(]^i)$. If the resulting diagram is not admissible, then it fails condition (A1). By the kernel bound, Y has to be contained in the variety associated to the result of applying Algorithm 3.29 to the diagram $D^b(]^i)$. By the second dimension count, this has the same dimension if and only if either two braces never occupy the same position while running Algorithm 3.29 or if two braces occupy the same position, there are no braces of equal or larger color until the first unoccupied position to the left. We thus conclude that the flat support of the flat limit is contained in the union of the varieties $V(D^a)$ and $V(D^b)$ described in Algorithm 3.43 and Remark 3.45.

Finally, we need to check that each of the components are generically reduced. This is a routine local calculation, which is almost identical to the calculation in [C2] or [C4] for orthogonal flag varieties or symplectic Grassmannians. As a sample, we check the case (2)(i) and leave the modifications in other cases to the reader. For each D^a and $D^b(]^i)$, we exhibit a variety that has intersection number one with $V(D)$ and that $V(D^a)$ or $V(D^b(]^i))$ and zero with the others. It follows that each of the components occur and are generically reduced. Since this is a local calculation, we may assume that $\kappa = 1, d_\kappa + r_\kappa = n - 2$ and $x_\kappa = 0$. For each $D^b(]^i)$ and D^a, there is a dual Schubert variety obtained as follows. Apply Algorithm 3.43 to $D^b(]^i)$ or D^a taking the branch of type D^a at each step. (Notice that the coefficient of the resulting Schubert class is one in the class of $V(D)$ according to Algorithm 3.43.) Let Σ be the variety dual to this Schubert variety. By the same argument as in [C2] or [C4], Kleiman's Transversality Theorem implies that Σ intersects $V(D^b(]^i))$ and $V(D)$ in one point and does not intersect the other $V(D^b(]^l)$ for $l \neq i$ and $V(D^a)$. We conclude that the flat limit is generically reduced along each component. $\square$

The specialization for Algorithm 3.38. We now need to do a careful analysis of Algorithm 3.38. We can interpret Algorithm 3.38 also as a sequence of specializations. We begin by describing this specialization.

Let $(D, c_\bullet, *)$ be a marked diagram. Recall that the vectors associated to a marked diagram are the same as the vectors associated to the underlying diagram $(D, c_\bullet)$. Let e_δ be the vector corresponding to δ. If ϵ is positive or to the left of a bracket, let e_ϵ be the vector corresponding to ϵ. If ϵ is zero and does not have a bracket to its right, then let f_ϵ be the vector of least index associated to the zeros between the two braces and/or brackets bounding ϵ. To unify the notation, let v_ϵ denote this vector in either case. Consider the specialization that replaces v_ϵ by $tv_\epsilon + (1-t)e_\delta$. When $t = 1$, we have the original set of vector spaces. When $t = 0$, the specialization replaces v_ϵ with e_δ. If a vector space in the sequence contains both v_ϵ and e_δ or if it does not contain v_ϵ, then the specialization leaves the vector space fixed. Otherwise, it interchanges v_ϵ for e_δ in the vector space.

The diagram $\tilde{D}$ encodes the flat limit of the sequence of vector spaces. If ϵ is positive or to the left of a bracket, then the limit of the linear spaces that contain e_ϵ but not e_δ is obtained by swapping e_ϵ and e_δ. These precisely correspond to swapping ϵ and δ in the diagram. If ϵ is zero and to the left of all the brackets, then δ is between $\}^j$ and $\}^{j+1}$ for some j. In this case, ϵ corresponds to a vector f_ϵ. If $(D, c_\bullet, *)$ is obtained by repeatedly running the loop in Algorithm 3.38 on a diagram of type D^a, then e_ϵ is the leftmost zero in the sequence. Hence, the specialization swaps ϵ and δ and changes the leftmost $j+2$ (or if $j+2$ does not exist in the sequence, the leftmost zero) to $j+1$. On the other hand, if $(D, c_\bullet, *)$ is obtained by repeatedly running the loop in Algorithm 3.38 on a diagram of type P, then e_ϵ is a vector contained in $Q^{r_{j+2}}_{d_{j+2}}$, but not in any vector space in the sequence of smaller dimension. Hence, the flat limit of the linear spaces is depicted by the diagram changing ϵ and δ and changing the leftmost $j+2$, or if there does not exist an integer equal to $j+2$, the leftmost zero between $\}^{j+1}$ and $\}^{j+2}$ to $j+1$. Hence, the difference between running the algorithm on an initial diagram P or a diagram of type D^a is explained by the location of e_δ in these diagrams.

The next theorem determines the flat limit of the symplectic restriction varieties under this specialization and shows that Algorithm 3.38 replaces a diagram with a collection of diagrams whose cohomology classes sum to the cohomology class of the original diagram.

Theorem 5.3. *Let $(D, c_\bullet, *)$ be a marked symplectic diagram that arises while running Algorithm 3.38 on $(D^\alpha, c_\bullet)$ or $P(a_\bullet, c_\bullet)$. Then the flat limit of the specialization applied to $V(D, c^\bullet, *)$ is supported along the varieties associated to the diagrams D^α and D^β described by Algorithm 3.38. Moreover, the flat limit is generically reduced along each component. In particular, the class of $V(D, c_\bullet, *)$ is equal to the sum of the classes of admissible restriction varieties associated to $(D, c_\bullet, *)$ by Algorithm 3.38.*

Proof. As in the previous theorem, the proof has two steps. We interpret Algorithm 3.38 as the specialization described above. We first show that the flat limit of the specialization is supported along the $V(D_i, c^i_\bullet, *)$, where D_i are the marked diagrams replacing D in Algorithm 3.38. We then check that the flat limit is generically reduced along each of the components. The proofs of both steps are almost identical to the proofs in the previous theorem.

First, using the dimension count in the proof of Theorem 5.1, we determine the support of the flat limit. In an initial diagram P, for each j such that $a_j = 2j-1$, the linear space Q^1_{2j-1} contains a j-dimensional isotropic subspace. Hence, by the linear space bound, this j-dimensional subspace must contain the kernel of the restriction of Q to Q^1_{2j-1}. The kernel for the smallest j for which $a_j = 2j-1$ is denoted by the marked integer δ. Similarly, if D^a fails condition (A2) for the linear space $Q^{r_i}_{d_i}$, then by the linear space bound, the $(k_h - i + 1)$-dimensional subspace intersects the kernel of the restriction of Q in a subspace of dimension $x_i + 1$. This kernel is denoted by the span of the vectors corresponding to δ and the integers up to and including δ'. Hence, the markings denote the additional intersection with K_i.

More generally, observe that when D^α does not satisfy condition (A2) for $\}^i$, then the linear spaces are required to intersect $Q^{r_i}_{d_i}$ in a subspace of dimension $k_h - i + 1$. By the linear space bound, we conclude that the linear spaces have to intersect K_i in a subspace of dimension x_i+1. Consequently, the $(k_h - i + 1)$-dimensional subspace contained in $Q^{r_i}_{d_i}$ is contained in the span of $Q^{r_{i+1}}_{d_{i+1}}$ and e_δ. Hence, we can replace $Q^{r_i}_{d_i}$ by the span of $Q^{r_{i+1}}_{d_{i+1}}$ and e_δ in the diagram D^α. Notice that these vector spaces are depicted by the diagram obtained by tightening D^α.

If δ is between $\}^{j+1}$ and $\}^j$, let $F = Q^{r_{j+1}}_{d_{j+1}}$ and if δ is between $]^j$ and $]^{j+1}$, let $F = L_{n_j}$. In the first case, let $m_j = k_h - d_j + 1$. In the latter case, let $m_j = n_j$. The flag elements W_h parameterized by $(D, c_\bullet, *)$ intersect F in a subspace of dimension m_j and intersect the span of F with e_δ in a subspace of dimension $m_j + 1$. When we make the specialization, δ becomes a vector of the flat limit of F. There are two possibilities. Either the flag elements W_h parameterized by a component of the flat limit intersect F in a subspace of dimension m_j or m_j+1 or greater. In the former case, the intersection of the linear spaces with F have to be contained in the span of the one smaller linear space in the sequence and e_δ. Hence, this component is the variety corresponding to D^α, the tightening of $\tilde{D}$. Notice that by the dimension count in the proof of Theorem 5.1, there cannot be any components where the linear spaces are more special. Such a component would be contained in a variety corresponding to a diagram where one or more of the brackets or braces are shifted to the left. By the dimension counts in the proof of Theorem 5.1, shifting any of the brackets or braces would produce a locus of strictly smaller dimension. We conclude that such a locus cannot support a component of the flat limit.

Else, W_h intersects F in a subspace of dimension m_j+1 (or possibly greater). In this case, we have to determine the possible limits. Let Y be an irreducible component of the support of the flat limit. Associate to it the diagram $D(Y)$ depicting the rank conditions satisfied by a linear space corresponding to a general point of Y. By Observation 5.2, $D(Y)$ is a diagram obtained from the diagram one gets from $\tilde{D}$ by moving the brace $\}^j$ or $]^{j+1}$, depending on the case, to the first position not occupied by a bracket or brace to its left. This is because by assumption there is a $(k_h - j + 2)$-dimensional subspace contained in the span of the vectors up to $\}^{j+1}$ or $]^j$. We have to check whether it is possible to obtain any components of the same dimension by moving bracket and/or braces to the left.

Now we can go through the possibilities quickly. If δ is between $\}^{j+1}$ and $\}^j$, then by the linear space bound, the linear spaces have to intersect the kernel of the restriction of Q to $Q_{d_{j+1}}^{r_{j+1}}$ in dimension $x_{j+1}+1$. By the first dimension count, we can move at most one of the brackets and the bracket we move cannot cross any brackets of equal or larger color. We recover the diagrams $D^\beta(]^i)$. In this case, since $\}^j$ is moved to the first empty spot, note that there is one more possibility. We can move $\}^j$ provided that the color of all the brackets and braces in between is strictly less than the color of $\}^j$. We recover the diagram $D^\beta(\}^j)$. By the dimension count in the proof of Theorem 5.1, the locus of linear spaces satisfying more special rank conditions has strictly smaller dimension, hence cannot be a component of the support of the flat limit. We also remark that when running Algorithm 3.38, the equality in Definition 3.26 automatically holds for all the brackets. If the diagram is obtained from repeated applications of the loop on an initial diagram P, there are no brackets that lead to diagrams of the type $D^\beta(]^i)$. In this case, the remark is vacuous. If the diagram is obtained by repeated applications of the loop on a diagram of type D^a, then D^a fails condition (A2) for an index i. Hence, there must be equality for all indices $j \geq i$ in D in condition (A2). This implies that all brackets of D to the right of $\pi(\delta')$ satisfy the equality in Definition 3.26. This is the reason we do not need to specify that the brackets satisfy this equality in Algorithm 3.38. If some of these diagrams do not satisfy condition (A1*), then the list of possibilities is analogous to the diagrams that do not satisfy condition (A1) in Theorem 5.1 and are obtained by running Algorithm 3.29.

If δ is between $]^j$ and $]^{j+1}$ and there is no bracket between position p and $]^j$, we move the bracket $]^{j+1}$ to position p since there is a j-dimensional subspace contained in the span of $L_{n_{j-1}}$ and e_δ and a $(j+1)$-dimensional subspace contained in L_{n_j}. This has the same dimension if $c_j < c_{j+1}$. Otherwise, it has strictly smaller dimension. By the dimension count in the proof of Theorem 5.1, moving $]^j$ to any position $p+t \leq p(]^j)$ preserves the dimension and moving any other bracket or brace to the left strictly decreases the dimension. If any of the resulting diagrams fail condition (A1*), then the component Y is contained in the locus corresponding to the diagram obtained by running Algorithm 3.29. We conclude that Y is one of the diagrams of type D^β produced by running the loop in Algorithm 3.38.

If δ is between $]^j$ and $]^{j+1}$ and there are brackets between position p and $]^j$, we move the bracket $]^{j+1}$ to a position to the right of p since there is a $(j+1)$-dimensional subspace contained in L_{n_j}. By the first dimension count in the proof of Theorem 5.1, moving $]^{j+1}$ to any of the positions between the first bracket to the left of p and p preserves the dimension provided that the brackets between p and $]^j$ all have color less than c_{j+1}. Moving any other bracket or brace strictly decreases the dimension. If any of the resulting diagrams fail condition (A1*), then the component Y is contained in the locus corresponding to the diagram obtained by running Algorithm 3.29. We conclude that Y is one of the diagrams of type D^β produced by running the loop in Algorithm 3.38. We conclude that the flat limit is supported along the union of the varieties associated to the diagrams assigned to $(D, c_\bullet, *)$ by Algorithm 3.38.

The multiplicity calculation is identical to the calculation in the proof of the previous theorem, so we leave it to the reader. This concludes the proof of the theorem. $\square$

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