The Arithmetic and the Geometry of Kobayashi Hyperbolicity

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ABSTRACT. In this paper we survey recent developments and open problems in the theory of Kobayashi hyperbolic manifolds. We emphasize the relation between the theory of hyperbolicity and questions about rational curves on higher dimensional projective varieties and rational points on varieties defined over number fields.

Curves of genus greater than one exhibit arithmetic and geometric properties very different from curves of genus zero and one. In these notes we will survey a few ways of extrapolating these properties to higher dimensional complex manifolds. We will specifically concentrate on Brody and Kobayashi hyperbolicity, which are generalizations of complex analytic properties of higher genus curves.

The basic metric and geometric properties that distinguish curves of genus two or more from curves of genus zero and one can be listed as follows.

(1) The dimension of the pluricanonical series, $h^0(C, mK)$, grows linearly with m for curves of genus at least two.
(2) The canonical/cotangent bundle of a curve of genus at least two is ample.
(3) A curve of genus at least two admits a hyperbolic metric with constant negative curvature.
(4) Curves of genus at least two are uniformized by the unit disc, hence they do not admit any non-constant holomorphic maps from $\mathbb{C}$.

Each of these properties can be generalized to higher dimensional manifolds, often in many distinct ways. For example, the growth rate of the dimension of sections of the pluricanonical series leads to the concept of Kodaira dimension, a useful and well-studied birational invariant.

In these notes we will discuss generalizations of the last two properties to higher dimensional varieties keeping in mind the Schwarz Lemma and Liouville's Theorem from the theory of complex functions.

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1. Introductory remarks about hyperbolicity

In this section we define Brody and Kobayashi hyperbolicity and state their basic properties. We also give examples of hyperbolic and non-hyperbolic manifolds. The reader can refer to [**La**] and [**Dem**] for more details.[1] Let $\mathbb{C}$ denote the complex plane. We use X and Y to denote complex manifolds. We reserve C for curves.

Observation: If C is a curve of genus at least two, then any holomorphic map $f : \mathbb{C} \to C$ is necessarily constant.

Proof: Any map $f : \mathbb{C} \to C$ factors through the universal cover of C, which is the unit disc. Such a map is constant since by Liouville's Theorem every bounded entire function is constant. $\square$

In contrast, curves of genus zero and one do admit non-constant holomorphic maps from $\mathbb{C}$. This property of higher genus curves can be generalized to higher dimensional manifolds and leads to the concept of Brody hyperbolicity.

DEFINITION 1.1. *A complex manifold X is* **Brody hyperbolic** *if there are no non-constant holomorphic maps from $\mathbb{C}$ to X.*

Examples: • Any variety of the form $\prod_{i=1}^{n} C_i$, where C_i are curves of genus at least two, is Brody hyperbolic. More generally, a finite product of Brody hyperbolic manifolds is Brody hyperbolic.

• If a complex manifold X contains a rational curve or a complex torus, then X is not Brody hyperbolic. If X contains a rational curve, then the inclusion of $\mathbb{C}$ in the rational curve followed by the inclusion of the rational curve in X gives a non-constant holomorphic map from $\mathbb{C}$ to X.

The universal cover of a g-dimensional complex torus is $\mathbb{C}^g$. Taking the image of a general complex line in $\mathbb{C}^g$ under the quotient map gives a non-constant holomorphic map from $\mathbb{C}$ into the complex torus. If X contains a complex torus, composing the previous map with the inclusion of the complex torus in X gives a non-constant holomorphic map from $\mathbb{C}$ into X.

• The blow-up of a variety is not Brody hyperbolic. Consequently, Brody hyperbolicity is not a birational invariant.

• Consider $X = \mathbb{P}^2 - \cup_{i=1}^{4} l_i$ where l_i are general lines. Then X is not Brody hyperbolic since there are $\mathbb{C}^*$'s in X—take the intersection of X with the line passing through $l_1 \cap l_2$ and $l_3 \cap l_4$.

• Consider $Y = \mathbb{P}^2 - \cup_{i=1}^{5} l_i$ where l_i are the lines pictured in Figure 1. Y is Brody hyperbolic. Consider the pencil of lines in $\mathbb{P}^2$ based at one of the points where three of the lines l_i intersect. Restricting this pencil to Y, we see that Y is fibered over $\mathbb{P}^1$ punctured at three points with fibers isomorphic to $\mathbb{P}^1$ punctured at three points. Composing any holomorphic map from $\mathbb{C}$ with the map to the base of the fibration, we conclude that the image of the map must lie in a fiber. Since the fibers are Brody hyperbolic, the map must be constant. Using the following

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[1]After these notes were written, I discovered that O. Debarre also has some very nice notes on hyperbolicity [**Deb**]. The reader is encouraged to consult these notes for additional information on hyperbolic varieties, especially about those varieties with ample cotangent bundle.
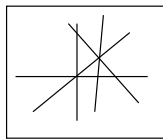
FIGURE 1. A Brody hyperbolic manifold.

easy lemma, which generalizes the idea of this example, one can generate many examples of Brody hyperbolic manifolds.

LEMMA 1.2. *Let $f : X \to Y$ be a smooth morphism of complex manifolds, where Y is Brody hyperbolic and $f^{-1}(y)$ is Brody hyperbolic for every $y \in Y$. Then X is Brody hyperbolic.*

The Schwarz-Pick Lemma states that a holomorphic map between two hyperbolic Riemann surfaces is either a local isometry or distance decreasing. The hyperbolic distance between any two points p and q on a hyperbolic Riemann surface C is equal to the shortest hyperbolic distance between lifts of the points p and q to the universal cover, the unit disc Δ. Any holomorphic map $f : \Delta \to C$ factors through the universal cover. Consequently, the Schwarz-Pick Lemma implies that the infimum of the hyperbolic distances between p' and q' in Δ for which there exists a holomorphic map $f : \Delta \to C$ such that $f(p') = p$ and $f(q') = q$ is achieved when f is the universal covering map. Moreover, this infimum is equal to the hyperbolic distance between p and q. In this form we can generalize the hyperbolic distance to higher dimensional manifolds.

Let Δ_r denote the disc of radius r in the complex plane. We will denote the unit disc by Δ. If X is a complex manifold, then X can be endowed with a pseudo-distance due to Kobayashi.

DEFINITION 1.3. *Given a tangent vector $\xi \in T_{X,x}$ at $x \in X$ we define its Kobayashi pseudo-norm to be*

$$k(\xi) := \inf_{\lambda}\{\lambda : \exists \ f : \Delta \to X, \ f(0) = x, \ \lambda f'(0) = \xi\}.$$

The Kobayashi pseudo-distance d_X is the geodesic pseudo-distance obtained by integrating this pseudo-norm.
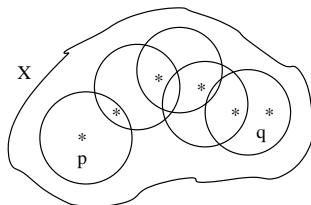

FIGURE 2. Measuring the Kobayashi distance.

More visually, we can describe how to compute the distance between any two points $p, q \in X$ as follows. We find chains of maps $f_i : \Delta \to X$ with two distinguished points p_i, q_i, where $f_1(p_1) = p$, $f_n(q_n) = q$ and $f_i(q_i) = f_{i+1}(p_{i+1})$. The

Kobayashi pseudo-distance is the infimum of the sum of the hyperbolic distances between p_i and q_i over all such chains.

The Schwarz-Pick Lemma generalizes to holomorphic maps between higher dimensional manifolds endowed with the Kobayashi pseudo-distance.

LEMMA 1.4. *If $f : X \to Y$ is a holomorphic map between two complex manifolds, then $d_X(p, q) \geq d_Y(f(p), f(q))$.*

Unfortunately, d_X does not have to be non-degenerate.

Main Counterexample: The Kobayashi pseudo-distance on $\mathbb{C}$ is identically zero. To compute the distance between two points x and y in $\mathbb{C}$, consider for each integer $n > 1$ the functions $f_n(z) = n(y - x)z + x$ from the unit disc to $\mathbb{C}$. The function $f_n(z)$ maps 0 to x and $1/n$ to y. Since the hyperbolic distance between 0 and $1/n$ in the unit disc tends to zero as n tends to infinity, we conclude that the Kobayashi pseudo-distance between x and y in $\mathbb{C}$ is zero.

DEFINITION 1.5. *A complex manifold X is called Kobayashi hyperbolic if its Kobayashi pseudo-distance is non-degenerate.*

Kobayashi hyperbolicity implies Brody hyperbolicity. As a consequence of Lemma 1.4 and the above example we see that if a complex manifold X admits a non-constant holomorphic map from $\mathbb{C}$, then X cannot be Kobayashi hyperbolic. In other words, Kobayashi hyperbolicity implies Brody hyperbolicity. For compact complex manifolds the converse of this result is true ([**Br2**]).

THEOREM 1.6 (Brody). *A compact complex manifold X is Kobayashi hyperbolic if and only if it is Brody hyperbolic.*

Sketch of proof: If X is not Kobayashi hyperbolic, then there exists a sequence of maps $f_n : \Delta \to X$ such that $|df_n(0)|$ tends to ∞, where $|\cdot|$ denotes a fixed Hermitian metric on X. By rescaling the map we can assume that we have a sequence of maps $f_n : \Delta_{r_n} \to X$ where $|df_n(0)| = 1$ and the radii r_n tend to ∞.

Let $f : \Delta_r \to X$ be a holomorphic map with $|df(0)| = c$. Let $f_t(z) = f(tz)$. Consider the function
$$g(t) = \sup_{z \in \Delta_r} |df_t(z)|.$$
The function $g(t)$ is increasing for $0 \leq t \leq 1$ and continuous for $0 \leq t < 1$. Moreover, as t tends to 1 from below $g(t)$ tends to $g(1)$. Hence there exists t in the interval $[0, 1]$ and an automorphism h of Δ_r such that
$$\sup_{z \in \Delta_r} |d(f_t \circ h)(z)| = |d(f_t \circ h)(0)| = c.$$
This is known as the Brody reparametrization lemma.

Applying the Brody reparametrization lemma to our family of maps f_n, we obtain a new family of maps $\tilde{f}_n : \Delta_{r_n} \to X$ whose derivatives are bounded in norm by 1 in Δ_{r_n}. Moreover, $|d\tilde{f}_n(0)| = 1$ for every n. If we could select a subsequence of this family that is uniformly convergent on every compact subset of $\mathbb{C}$, then this subsequence would converge to a holomorphic map from $\mathbb{C}$ to X. The limit would be non-constant since the derivatives at 0 all have norm 1. This would violate Brody hyperbolicity.

Using the compactness of X we can check uniform convergence in a neighborhood of every point z_0 in $\mathbb{C}$. Again using the compactness of X (and passing to a

subsequence if necessary) we can assume the family converges at z_0. In this situation the derivative bounds imply that the family forms an equicontinuous family of holomorphic maps around z_0. We can conclude that a subsequence converges uniformly on compact sets by Ascoli's theorem. We thus obtain a holomorphic map from $\mathbb{C}$ to X. $\square$

Note that Brody's theorem does not have to hold for non-compact manifolds (see [**Br1**]). Consider the domain in $\mathbb{C}^2$ given by

$$D := \{(z, w) \mid |z| < 1, \ |zw| < 1 \text{ and } |w| < 1 \text{ if } z = 0\}.$$

The projection of the domain D to the first coordinate gives rise to a family of discs parameterized by the unit disc in the z-plane. Consequently, Lemma 1.2 implies that D is Brody hyperbolic. On the other hand, when z approaches zero, the radius of the disc lying over z tends to infinity. Using this we can show that the Kobayashi pseudo-distance between $p = (0, 0)$ and $q = (0, w_0)$ in D vanishes. We can find two points p' and q' having the same w coordinates as p and q, respectively, in a fiber close to zero. Since p' and q' are in a very large disc, the hyperbolic distance between them is very small. Since the hyperbolic distances between p and p' and q and q' are very small, the sum of the hyperbolic distances can be made arbitrarily small.

More explicitly, consider the three maps

$$f_1(z) = (z, 0), \ f_2(z) = \left(\frac{1}{n}, nz\right), \ f_3(z) = \left(\frac{1}{n} + \frac{1}{2}z, w_0\right)$$

from the unit disc into D. Note that $f_1(0) = (0, 0)$, $f_1(1/n) = (1/n, 0)$, $f_2(0) = (1/n, 0)$, $f_2(w_0/n) = (1/n, w_0)$ and $f_3(0) = (1/n, w_0)$, $f_3(-2/n) = (0, w_0)$. Hence these maps form a chain of maps from the unit disc to D connecting $(0, 0)$ to $(0, w_0)$. As n tends to infinity, the sum of the hyperbolic distances between the chosen points in the unit disc tends to zero. We conclude that the Kobayashi pseudo-distance between $(0, 0)$ and $(0, w_0)$ is zero. Observe that this example also shows that the analogue of Lemma 1.2 does not hold for Kobayashi hyperbolicity.

Under suitable hypotheses on a subvariety Y of X, we can assert that $X - Y$, the complement of Y in X, is Kobayashi hyperbolic. For example, the following theorem is often useful in proving that the complement of a subvariety is Kobayashi hyperbolic.

THEOREM 1.7. *Let X be a compact variety and let Y be a proper algebraic subset. If Y and $X - Y$ are Brody hyperbolic, then $X - Y$ is Kobayashi hyperbolic.*

Sketch of proof: If $X - Y$ is not Kobayashi hyperbolic, we can get a sequence of maps $f_n : \Delta_{r_n} \to X - Y$ that converges to a holomorphic map g from $\mathbb{C}$ to X. The question is whether the image of g intersects Y. The image cannot be contained in Y since Y is Brody hyperbolic.

If we rule out that the image of g intersects Y, then we obtain a contradiction since $X - Y$ is Brody hyperbolic. One can prove that the intersection points of the image of g with Y have to be isolated. Suppose $g(z_0) \in Y$. Take a circle S around z_0 such that $g(S) \subset X - Y$. The winding number is zero because $f_n(\Delta_{r_n})$ does not meet Y. Hence Y cannot intersect the image of the interior of S, contradicting that $g(z_0) \in Y$. $\square$

We now discuss some examples of Kobayashi hyperbolic manifolds. From now on whenever we say hyperbolic without further qualification, we will always mean

Kobayashi hyperbolic. The following example due to Green ([**Gr**]) gives the first non-trivial examples.

THEOREM 1.8 (Green). *The complement of $2n + 1$ general hyperplanes in $\mathbb{P}^n$ is hyperbolic.*

Generalizing further one can ask whether the complement of a very general irreducible hypersurface in $\mathbb{P}^n$ of large enough degree is hyperbolic. Siu and Yeung answer this question positively in $\mathbb{P}^2$ ([**SY2**]). Here and in the following by 'very general' we will refer to the complement of the union of countably many proper subvarieties.

THEOREM 1.9 (Siu-Yeung). *Let C be a very general curve of sufficiently large degree in $\mathbb{P}^2$. Then $\mathbb{P}^2 - C$ is Kobayashi hyperbolic.*

Note that this theorem can be interpreted as a generalization of Picard's theorem to higher dimensions. Recall that Picard's theorem says that any entire map which omits two values is constant. This theorem implies that any holomorphic map from $\mathbb{C}$ into $\mathbb{P}^n$ which omits a (very general) hypersurface of a large enough degree is constant.

Siu and Yeung also prove the analogue of this result for the complement of very general ample divisors in abelian varieties ([**SY1**] [**SY3**], [**SY4**] [**SY5**]).

THEOREM 1.10 (Siu-Yeung). *Let A be an abelian variety. Let D be a very general ample divisor in A. Then $A - D$ is Kobayashi hyperbolic.*

There is a close relation between the hyperbolicity of complements of divisors in projective space and the hyperbolicity of general hypersurfaces in projective space. We have the following theorem due to Siu ([**Siu1**], [**Siu2**]).

THEOREM 1.11 (Siu). *A very general surface of sufficiently high degree in $\mathbb{P}^3$ is Kobayashi hyperbolic.*

In fact, Siu recently has generalized Theorems 1.9 and 1.11 to hypersurfaces in $\mathbb{P}^n$. Not all the proofs have appeared in print.

THEOREM 1.12 (Siu). *Let X be a very general hypersurface of sufficiently large degree in $\mathbb{P}^n$. Then $\mathbb{P}^n - X$ is Kobayashi hyperbolic.*

THEOREM 1.13 (Siu). *A very general hypersurface of sufficiently high degree in $\mathbb{P}^n$ is Kobayashi hyperbolic.*

For surfaces in $\mathbb{P}^3$ the term 'sufficiently high degree' can be taken to mean at least 11. Conjecturally a very general hypersurface of degree larger than $2n$ in $\mathbb{P}^n$ is expected to be hyperbolic. However, the current known bounds are much larger than the expected bounds.

Finally, we observe that some properties of the tangent or cotangent bundle of a complex manifold forces the manifold to be hyperbolic. For example, if X admits a metric with negative sectional curvature; or if X has ample cotangent bundle; or if X is the quotient of a bounded domain in $\mathbb{C}^n$ by a free group action, then X is Kobayashi hyperbolic.

As an amusing corollary we note that M_g^0, the moduli space of automorphism-free, smooth curves of genus $g > 2$, is Kobayashi hyperbolic since the Weil-Petersson metric is a metric on M_g^0 with negative sectional curvature. Consequently, M_g^0 does not contain any complete rational or elliptic curves. It would be interesting to give lower bounds on the genus of complete curves contained in M_g^0.

2. The geometry of Kobayashi hyperbolicity

In this section we discuss some proven and conjectural geometric characterizations of hyperbolicity. Hyperbolicity imposes strong restrictions on the geometry of a variety. In particular it constrains the type of subvarieties the variety can have.

PROPOSITION 2.1. *Let X be a compact complex hyperbolic manifold with Hermitian metric ω. Then $\exists\, \epsilon > 0$ such that for any reduced, irreducible curve $C \subset X$, the genus g of the normalization satisfies*

$$g \geq \epsilon \deg_\omega(C)$$

Sketch of proof: If X is hyperbolic, then there exists a constant such that $d_X(\xi) \geq \epsilon_0 ||\xi||_\omega$ for every tangent vector ξ. Let $\nu : C^\nu \to C$ be the normalization of a curve $C \subset X$. If we denote the hyperbolic metric on C^ν by k_{C^ν}, then the Gauss-Bonnet formula implies that

$$-\frac{1}{4}\int_{C^\nu} \mathrm{curv}(k_{C^\nu}) = -\frac{\pi}{2}\chi(C^\nu).$$

There is a natural holomorphic map $i \circ \nu : C^\nu \to X$ obtained by composing the normalization map ν from C^ν to C by the inclusion i of C in X. Since the Kobayashi distance can only decrease under compositions of holomorphic maps, we conclude that

$$k_{C^\nu}(\xi) \geq \epsilon_0 ||(i \circ \nu)_*(\xi)||_\omega$$

for any tangent vector $\xi \in T_{C^\nu}$. Integrating both sides of the inequality yields the proposition. $\square$

Lang has conjectured that the converse also holds. Let X be a compact, complex manifold endowed with a Hermitian metric. Lang conjectures that if there exists a positive constant ϵ such that for every reduced, irreducible curve C in X the ratio of the genus of the normalization of C to the degree of C (with respect to the Hermitian metric) is bounded below by ϵ, then X is hyperbolic.

If the geometric genus of every curve in a variety X is bounded below by some fixed positive multiple of their degree, then X does not admit any non-constant holomorphic maps from any abelian variety. More generally, Lang has conjectured that a projective variety X is hyperbolic if and only if it does not admit any holomorphic maps from an abelian variety. The latter conjecture, of course, implies the former one for projective varieties.

One can ask for the relation between varieties of general type and hyperbolic varieties. Since varieties of general type can contain rationally connected subvarieties or abelian subvarieties, an arbitrary variety of general type cannot be hyperbolic. However, Lang has conjectured that the existence of subvarieties which are not of general type accounts for the failure of hyperbolicity.

CONJECTURE 2.2. *A projective algebraic variety X is hyperbolic if and only if every subvariety of X is of general type.*

Ein ([**Ein**]) and later Voisin ([**Voi**]) have shown that any subvariety of a very general hypersurface of degree at least $2n + 1$ in $\mathbb{P}^n$ is of general type. Combining their theorem with this conjecture one obtains a conjectural sharp form of Siu's Theorem 1.13.

Proving that a projective variety X is hyperbolic usually has two components. One has to show that there are no non-algebraic maps of $\mathbb{C}$ into X and that there

are no rational or elliptic curves in X. When the problem is broken into these two components, then some progress can be made on each component of the problem under some geometric restrictions. We now survey some of the results on these questions.

One of the first people to make important progress on hyperbolicity questions was the French mathematician André Bloch, even though at the time hyperbolicity was not defined (see [**Bl**]). Bloch begins by determining the Zariski closure of an entire map into a complex torus.

THEOREM 2.3. *The Zariski closure of a holomorphic map $\mathbb{C} \to T$ to a complex torus T is the translate of a subtorus of T.*

This theorem leads to a fairly complete understanding of the hyperbolic subvarieties of complex tori.

COROLLARY 2.4. *A subvariety X of an abelian variety A which does not contain any translates of subtori of A is hyperbolic.*

Originally Bloch used these ideas to prove the following theorem often referred to as Bloch's Theorem ([**Bl**]).

THEOREM 2.5 (Bloch's Theorem). *Any holomorphic map of $\mathbb{C}$ into a smooth, compact Kähler variety X whose irregularity ($h^0(X, \Omega_X^1)$) is bigger than its dimension is analytically degenerate, i.e. its image lies in a proper analytic subvariety.*

Proof: Recall that given a smooth, compact Kähler variety X, we can associate to it a complex torus $\text{Alb}(X)$ of dimension $h^0(X, \Omega_X^1)$ and a map $a : X \to \text{Alb}(X)$ with the following universal property: for any complex torus T and any morphism $f : X \to T$, there exists a unique morphism $g : \text{Alb}(X) \to T$ such that $f = g \circ a$. The complex torus $\text{Alb}(X)$ is referred to as the *Albanese* variety of X and the map a is called the *Albanese map*. $\text{Alb}(X)$ is unique up to isomorphism.

Bloch's theorem follows by considering the Albanese map. Let $f : \mathbb{C} \to X$ be a holomorphic map. Consider $a \circ f : \mathbb{C} \to \text{Alb}(X)$. Since the irregularity of X is larger than the dimension of X, the image of X under the Albanese map lies in a proper subvariety of $\text{Alb}(X)$. By the universality of the Albanese variety and its uniqueness, the Albanese image of X cannot be the translate of a subtorus. Hence the image of $\mathbb{C}$ under the map $a \circ f$ in the Albanese variety has to be analytically degenerate in the image of X. It follows that the image of the original map $f : \mathbb{C} \to X$ has to be analytically degenerate. □

Using these ideas one also obtains information about the hyperbolicity of complements of ample divisors in abelian varieties.

THEOREM 2.6. *Let A be an abelian variety. Let D be an ample divisor that does not contain any translates of abelian subvarieties. Then $A - D$ is hyperbolic.*

Recall that a holomorphic map from $\mathbb{C}$ into an algebraic variety is called algebraically degenerate if the Zariski closure of the image lies in a proper algebraic subvariety. More recently McQuillan ([**McQ**]) has proved the algebraic degeneracy of entire maps into surfaces of general type with $c_1^2 > c_2$. More precisely,

THEOREM 2.7 (McQuillan). *If X is a surface of general type satisfying $c_1^2 > c_2$, then all entire curves on X are algebraically degenerate. In particular, if X does not contain any rational or elliptic curves, then X is hyperbolic.*

McQuillan obtains this result by first proving a result about the algebraic degeneracy of leaves of certain foliations on surfaces of general type, then showing that under the assumptions on X there always is a foliation which contains the image of any entire map in one of its leaves.

Most ways of proving the algebraic degeneracy of entire maps into complex projective manifolds depend on producing enough differential relations on the manifold that every entire map has to satisfy. One then shows that these relations have a small base locus. This forces the holomorphic map to lie in this base locus. One often formalizes these ideas in terms of jet bundles (see [**DEG**], [**Dem**] or any of Siu's papers cited above).

In the other direction, there has been extensive work on showing that certain varieties do not contain any rational or elliptic curves. Here we mention the work of G. Xu bounding below the geometric genus of any curve on a very general surface of degree at least 5 in $\mathbb{P}^3$ ([**Xu**], see also Clemens' paper [**Cl**]).

THEOREM 2.8 (Xu). *On a very general surface of degree d in $\mathbb{P}^3$, the geometric genus of any curve is greater than or equal to $d(d-3)/2 - 2$. Tritangent plane sections achieve this bound. For $d \geq 6$ the tritangent plane sections are the only curves that achieve the bound.*

A very simple consequence of the theorem is that a general hypersurface of degree at least 5 in $\mathbb{P}^3$ contains no rational or elliptic curves. Thus the problem of showing the hyperbolicity of a very general hypersurface in $\mathbb{P}^3$ of degree at least 5 reduces to showing that any entire map into such a surface is algebraically degenerate.

Motivated by our discussion one can ask the following question:

Question: Can the closure of the image of a holomorphic map $\mathbb{C} \to \mathbb{P}^n$ be a variety of general type?

A negative answer to this question would imply one direction of Conjecture 2.2. If all the subvarieties of a compact variety are of general type, then the variety would have to be hyperbolic. At present this question seems very hard to answer.

We close this section with a discussion of how hyperbolicity varies in families. Let X be a compact C^∞ manifold with Hermitian metric $|\ |$. Brody proved that if one considers the various complex structures that one can put on X, the set of those that are hyperbolic is open in the analytic topology. More precisely, one can consider the function

$$D(s) = \sup_{f \in \mathrm{Hol}(\Delta, X_s)} |f'(0)|$$

on the moduli spaces of complex structures parameterized by S. Brody proves

THEOREM 2.9. *$D(s)$ is a continuous function. Since the hyperbolic complex structures correspond to those for which $D(s)$ is finite, the hyperbolic complex structures form an open set in the analytic topology.*

Note that if one considers an algebraic family of quasiprojective smooth varieties, then Brody's theorem may fail. In fact in such a family (at least if we do not fix the C^∞ type) the set of fibers that are Brody hyperbolic can be Zariski locally closed. Varying two of the lines that meet a third at the same point in Figure 1 provides such an example.

Brody's theorem naturally raises the following question.

Question: Is hyperbolicity Zariski open in algebraic families of projective varieties?

This question, to the best of my knowledge, is still open. A positive answer would have interesting geometric implications. For example, it is not known whether the space of high degree surfaces in $\mathbb{P}^3$ that contain rational and elliptic curves form an algebraic variety or a countable union of subvarieties of the space of surfaces. A positive answer to the Zariski openness of hyperbolicity would settle this and similar questions.

3. The arithmetic of Kobayashi hyperbolicity

In this section we summarize some conjectures, mainly due to Lang, about rational points on hyperbolic varieties. We also give some surprising consequences of the conjectures about the distribution of rational points on curves. The main references for this section are [**La**], [**Cap**], [**CHM1**] and [**CHM2**].

The arithmetic of curves of genus two or more and those of genus one and zero exhibit drastically different properties. If C is a curve of genus zero or one defined over a number field K, then after passing to a finite field extension L there are infinitely many L-rational points on C. In fact, L can be chosen so that these points are dense in the analytic topology. The Mordell-Faltings Theorem stands in sharp contrast to this result.

THEOREM 3.1 (Mordell-Faltings). *Let C be a smooth curve of genus $g \geq 2$ defined over a number field K. Then C has only finitely many rational points over any finite field extension L of K.*

DEFINITION 3.2. *We say a variety V defined over a number field K is of Mordell type if V contains only finitely many rational points over any finite field extension of K.*

It is natural to ask which, if any, of the geometric properties we described so far imply that a variety is of Mordell type. Clearly a variety of Mordell type does not contain any rational or elliptic curves. Similarly any map from an abelian variety to a variety of Mordell type has to be constant. In view of these observations it is not unreasonable to formulate the following conjecture due to Lang.

CONJECTURE 3.3. *A complex projective variety V defined over a number field is of Mordell type if and only if it is hyperbolic.*

In fact Lang has conjectured much more precise statements.

CONJECTURE 3.4 (weak form). *If X is a variety of general type defined over a number field K, then the set of K-rational points of X is not Zariski dense.*

The conjecture that a hyperbolic manifold is of Mordell type follows from Conjecture 3.4 and the geometric Conjecture 2.2 by the following argument. If a hyperbolic manifold X has infinitely many rational points, then the Zariski closure of these points is a subvariety not of general type by Conjecture 3.4. By Conjecture 2.2 every subvariety of X is of general type. Lang has conjectured an even stronger statement.

CONJECTURE 3.5 (strong form). *If X is of general type, then there exists a proper algebraic subset Z of X, such that over any finite extension L of K, the number of L-rational points of $X - Z$ is finite.*

The Lang conjectures are currently open except for the case of curves and more generally for subvarieties of abelian varieties. If true, they would provide a fundamental understanding of the arithmetic of varieties of general type.

The Lang Conjectures 3.4 and 3.5 have some surprising consequences for rational points on curves of genus at least two. These consequences were investigated by Caporaso, Harris and Mazur in the papers cited above.

THEOREM 3.6. *The weak form of Lang's conjecture implies that for every number field K and genus $g \geq 2$, there exists a constant $B(K, g)$ such that any curve of genus g defined over K has at most $B(K, g)$ K-rational points.*

The strong form of Lang's conjecture implies an even more surprising bound.

THEOREM 3.7. *The strong Lang conjecture implies that for any $g \geq 2$ there exists an integer $N(g)$ such that there are only finitely many curves defined over a number field K that have more than $N(g)$ K-rational points.*

Ideas behind the proof: These theorems depend on the following geometric theorem Caporaso, Harris and Mazur prove.

THEOREM 3.8. *(Correlation) Let $f : X \to B$ be a proper morphism of irreducible and reduced schemes whose general fiber is a smooth curve of genus at least two. Then for n sufficiently large the fiber product $h : X_B^n \dashrightarrow W$ admits a dominant rational map to a variety of general type. Moreover, if $f : X \to B$ is defined over a number field K, h and W can also be defined over K.*

D. Abramovich proved a generalization of the Correlation Theorem for morphisms with higher dimensional fibers of general type under suitable hypotheses (see [**Abr**]). However, in order to deduce the bounds on the rational points on curves, we only need the case of curves.

Caporaso, Harris and Mazur deduce the bounds on rational points on curves by applying the Lang conjectures to a global family. More precisely, they start with a family of curves $f : X \to B$ such that for every curve C defined over K, there exists a K-rational point on B such that the fiber over it is isomorphic to C over K.

Using Theorem 3.8, X_B^n, the $n-$th fiber product of X over B, admits a rational map to a variety W of general type. By Lang's conjecture there exists a smallest closed, proper subvariety V of X_B^n that contains all the K-rational points.

By studying the successive images of the complement of V under the projections to various factors of X_B^n, they produce a non-empty open subset U of B and an integer N such that for every rational point $b \in U$ the fiber over b has at most N points. Then by Noetherian induction on the complement of U, they conclude the uniform bound. $\square$

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