Topological Hochschild Homology and p-adic Hodge Theory

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Abstract

This is the final report from a summer project guided by the wonderful Niles Johnson. The goal was to try to develop the necessary background to understand the anticipated paper of Bhargav Bhatt, Matthew Morrow, and Peter Scholze in which the results of their paper 'Integral p-adic Hodge Theory' will be explained using Topological Hochschild homology. The focus of our project was to try to understand topological Hochschild homology, and begin to understand the connection to p-adic cohomology theories in anticipation of this paper. In this paper an outline of the new construction of THH from [10] is given, and a very rough sketch of the connection of THH to crystalline cohomology is given.

Contents

1 Introduction

Topological Hochschild Homology (THH) is a generalization of Hochschild homology, which is a classical homology theory on associative rings.

For now suppose that A is a commutative, flat k-algebra, for some field k of characteristic 0; although most of the following results work more generally.

Definition 1. For an associative k-algebra A, the n'th Hochschild homology group of A, $\mathrm{HH}_n(A/k)$, is defined as the n'th homology of the complex

$$A \otimes_k \cdots \otimes_k A \to \cdots \to A \otimes_k A \otimes_k A \to A \otimes_k A \to A,$$

where the differential is given by the alternating sum of replacing a tensor with a product in the ring, i.e.

$$d(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} + \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n.$$

For example, $a \otimes b \otimes c \mapsto ab \otimes c - a \otimes bc + ca \otimes b$.

A slightly more sophisticated way to look at this, is $\mathrm{HH}_*(A/k) = \mathrm{Tor}_*^{A \otimes_k A}(A, A)$. This is easy to generalize to spectra, by replacing k with some commutative ring spectrum and replacing the tensor product with the smash product of spectra.

Definition 2. The Topological Hochschild homology of a ring spectrum A is the spectrum defined as $A \wedge_{A \wedge_{\mathbb{S}} A} A$, and $\mathrm{THH}_n(A) := \pi_n \mathrm{THH}(A)$, where $\mathbb{S}$ is the sphere spectrum; the initial commutative ring spectrum.

In this definition we use $\wedge$ to remind ourselves that these are spectra; but from here on we will use $\otimes$ to denote the smash product of spectra.

It has long been known that the Hochschild homology of a commutative algebra is closely related to the deRham cohomology of that algebra.

Theorem 3 (Hochschild-Kostant-Rosenberg)**.** *If A is a smooth commutative algebra of finite type over k then* $\mathrm{HH}_*(A) \simeq \Omega^*_{A/k}$.

This isn't quite the same as recovering the deRham complex though, as *a priori* there are no differentials on the $\mathrm{HH}(A/k)$ modules - however $\mathrm{HH}(A/k)$ has some additional structure which will let us recover the deRham differential! This extra structure comes from an S^1-action on HH; this action can be seen in the above chain complex as the cyclic actions, where the action of C_n permutes the copies of A in the term $\underbrace{A \otimes ... \otimes A}_{n}$ from the above complex. We can make this precise by realizing the above complex as a *cyclic object* in $\mathrm{Vect_k}$, which we will do later.

For now we will arrive at this S^1 action in a different way, which is unfortunately quite complicated and will require us to think about HH in a slightly different way. We want to think about $\mathrm{HH}(A/k)$ now as a simplicial commutative k-algebra, which we can do by replacing the complex in definition 1 with its Dold-Kan simplicial object, and letting $\mathrm{HH}(A/k)$ be the geometric realization of this simplicial k-algebra, or by thinking about the derived tensor product over k as the coproduct in sCAlg_k and then letting

$$\mathrm{HH}(A/k) = A \coprod_{A \sqcup A} A \in \mathrm{sCAlg}_k,$$

where A is considered as an object of sCAlg_k.

So now $\mathrm{HH}(A/k) \in \mathrm{sCAlg}_k$ - the category of simplicial commutative algebras over k, which is a category tensored over spaces. It turns out that $\mathrm{HH}(A/k) = S^1 \otimes A \in \mathrm{sCAlg}_k$, where we consider A now as a simplicial commutative ring. This makes it clear that $\mathrm{HH}(A/k)$ has an S^1-action. Then we have a forgetful functor $\mathrm{sCAlg}_k \to \mathcal{D}(k)$, where this is the derived category of k-algebras. This S^1-action on the derived category now gives an action of $C^*(S^1, k)$ on $\mathrm{HH}_*(A/k)$ which makes $\mathrm{HH}_*(A/k)$ a module over $C^*(S^1, k)$. Well, we know that $H_*(C^*(S^1, k)) = k[\varepsilon]/(\varepsilon^2)$, where $|\varepsilon| = 1$, so we can define a map $\mathrm{HH}_m(A/k) \to \mathrm{HH}_{m+1}(A/k)$ by $\eta \mapsto \varepsilon \cdot \eta$, and since $\varepsilon^2 = 0$ this map is a differential. If A is smooth over k then this is the deRham differential!

The idea is that we might be able to recover more by replacing $\mathrm{HH}(A/k)$ with $\mathrm{THH}(A/k)$ and repeating the above. There is an upcoming paper of Bhatt-Morow-Scholze which will attempt to reprove the results in [3] using THH, and the goal of this summer project was to learn the relevant background in anticipation for this paper.

2 A brief introduction to p-adic Hodge theory

The subject of p-adic Hodge theory is the study of cohomology theories on varieties over p-adic fields. We will focus on the easiest possible setting, and find that it is still sufficiently complicated. Consider $\mathbb{Q}_p$ be the field of p-adic numbers, and we will only be thinking about schemes with good reduction over $\mathbb{Q}_p$. To explain this, recall that $\mathbb{Q}_p$ has ring of integers $\mathbb{Z}_p$, which is a discrete valuation ring and has the unique maximal ideal (p) such that $\mathbb{Z}_p/(p) \simeq \mathbb{F}_p$. Since $\mathbb{Z}_p$ is DVR, $\mathrm{Spec}(\mathbb{Z}_p)$ contains two points; (p) and (0), which have residue fields $\mathbb{F}_p$ and $\mathbb{Q}_p$ respectively. A scheme X over $\mathbb{Q}_p$ has *good reduction* if there is a proper, smooth scheme $\mathfrak{X}$ over $\mathbb{Z}_p$ where $X \simeq \mathfrak{X} \times_{\mathbb{Z}_p} \mathbb{Q}_p$, i.e. there is a scheme over $\mathbb{Z}_p$ whose fiber over the point (0) is isomorphic to X. In this case, the other fiber of $\mathfrak{X}$ gives a scheme over $\mathbb{F}_p$ given by $\mathfrak{X} \times_{\mathbb{Z}_p} \mathbb{F}_p$

which one should think of as a characteristic p reduction of X. Heuristically, cohomological invariants of these two fibers should be similiar.

We will also primarily be interested in an algebraicly closed and p-adically completed version of $\mathbb{Q}_p$, we denote this by $\mathbb{C}_p := \hat{\overline{\mathbb{Q}_p}}$, which has ring of integers $\mathcal{O}_{\mathbb{C}_p}$ with residue field $\overline{\mathbb{F}_p}$. Statements below will be for much more general fields, but we are particularly interested in the case $K = \mathbb{C}_p$, and $k = \overline{\mathbb{F}_p}$.

Since we are assuming that our schemes have good reduction, we will usually start with a scheme X over $S := \mathcal{O}_{\mathbb{C}_\mathrm{p}}$. We will break this scheme X into two parts; the *generic fiber*, $X_K := X \times_S \mathbb{C}_p$, and the *special fiber* $X_k := X \times_S \overline{\mathbb{F}_p}$.

There are three cohomology theories that we'll be interested in, deRham cohomology, étale cohomology, and crystalline cohomology.

First let's talk about the deRham cohomology. In the definition k will be an arbitrary field.

Definition 4 (deRham cohomology)**.** For a k-variety X, the deRham cohomology of X, denoted $\mathrm{H}^*_{dR}(X/k)$, is defined to be $\mathbb{H}^*(X, \Omega^\bullet_{X/k})$.

This is the *hypercohomology*, which is defined by taking a term-wise injective resolution of the cochain complex $\mathcal{O}_X \to \Omega^1_{X/k} \to \Omega^2_{X/k} \to ...$, which gives a bicomplex $C^{p,q}$ of sheaves, we then apply global sections to get a bicomplex of k-vector spaces $\Gamma(C^{p,q})$, which we then totalize and take cohomology. One of the great parts about deRham cohomology is that it comes equipped with the Hodge filtration, where $F^i H^*_{dR}(X/k) = \mathbb{H}^*(Xk, \Omega^{\geq i}_{X/k})$ which gives rise to the *Hodge to deRham Spectral Sequence* (HtdRSS) $E_1^{p,q} = \mathrm{H}^q(X, \Omega^p_{X/k}) \Rightarrow H^{p+q}_{dR}(X/k)$.

Intuition can be misleading - the deRham cohomology doesn't behave as expected in characteristic $p > 0$; for example, for $X = \mathbb{A}^1_k$, the affine line over $k = \overline{\mathbb{F}_p}$, we would expect that $H^0_{dR}(X) = k$ since the affine line is connected. We compute this with the complex $0 \to \overline{\mathbb{F}_p}[x] \to \Omega^1_{\overline{\mathbb{F}_p}[x]} \to ....$, so $H^0_{dR}(X) = \ker(d : \overline{\mathbb{F}_p}[x] \to \Omega^1_{\overline{\mathbb{F}_p}[x]})$, but $d(x^p) = px^{p-1}dx = 0x^{p-1}dx = 0$. We therefore have that $H^0_{dR}(X) = \overline{\mathbb{F}_p}[x^p]$, so the deRham cohomology isn't even finite dimensional in degree 0 (or degree 1 for that matter)! For this reason, we will always consider the deRham cohomology of X relative to $\mathcal{O}_{\mathcal{C}_p}$.

One way to fix this, if we start with a variety X_0 over $\mathbb{F}_p$, would be to try to lift X_0 to a variety over a characteristic 0 ring and take deRham cohomology there. There is a universal lift of $\mathbb{F}_p$ to a characteristic 0 ring, given by the quotient $\mathbb{Z}_p \to \mathbb{F}_p$, and if we can find a proper variety X over $\mathbb{Z}_p$ such that $X_{\mathbb{F}_p} \simeq X_0$ then we can compute $H^*_{dR}(X/\mathbb{Z}_p)$.

Theorem 5. *In the situation described above, $H^*_{dR}(X/\mathbb{Z}_p)$ only depends on X_0.*

This gives us an invariant of X_0, but ideally we want an intrinsic definition of this invariant which doesn't require us to choose some X lifting X_0 - this is what the crystalline cohomology does! In fact the crystalline cohomology does more than this - if there is such a lift of X_0 to a characteristic 0 version, the crystalline cohomology will be computed by the deRham cohomology of the lift, but even if such a lift doesn't exist the crystalline cohomology still exists and is a well behaved Weil cohomology theory over $\mathbb{Z}_p$.

We will always refer to crystalline cohomology of the special fiber, and denote it as $\mathrm{H}^*_{\mathrm{crys}}(X_k/W(k))$, where $W(k)$ is the Witt vectors of k. The definition of the crystalline cohomology is extremely involved (see [2]), but there is a more intuitive way to think about the crystalline cohomology, which comes from the *deRham-Witt complex*. The deRham-Witt complex is a projective system of dgas (indexed by $n \in \mathbb{N}$), $W_n\Omega^\bullet_{X/k}$, where W_n denotes the Witt-vectors. From afar the idea is simple - we add new elements to $\Omega^\bullet_{X/k}$ with identities that make it behave as if it were in characteristic 0.

Finally we have étale cohomology. The definition of étale cohomology is also involved (see [9]). The idea is that we define a new topology on our scheme X, called the étale topology and take sheaf cohomology with respect to this topology. This is an oversimplification, as the 'étale topology' isn't actually a topology, instead it is a collection of schemes together with étale maps into X which one should think about as generalized open subsets of X. Étale cohomology was introduced to be a replacement for singular cohomology, and when we consider a variety X over $\mathbb{C}$ with suitable coefficients these two agree, for example $\mathrm{H}^*(X_{\text{ét}}, \mathbb{Q}_p) \simeq \mathrm{H}^*_{\text{sing}}(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$, where $\mathrm{H}^*_{\text{sing}}$ denotes singular cohomology, and $X(\mathcal{C})$ is X considered as a subset of $\mathbb{C}^n$ with the euclidean topology.

There are two standard ways to denote étale cohomology, $\mathrm{H}^*(X_{\text{ét}}, \mathcal{F})$ and $\mathrm{H}^*_{\text{ét}}(X, \mathcal{F})$, the prior is preferable as it reminds us that we are changing the topology on $X_{\bar{K}}$, whereas the notation $\mathrm{H}^*_{\text{ét}}$ makes it seem as if we are using something other than sheaf cohomology. We will often use the latter in cases where the variety involved already has a subscript, since X_k looks nicer than $X_{k,\text{ét}}$.

There is a special case of étale cohomology called ℓ-adic cohomology for some prime ℓ, this is the case $\mathrm{H}^*(X_{\text{ét}}, \mathbb{Q}_\ell)$ (secretely this is defined as $\mathrm{H}^*(X_{\text{ét}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell := \varprojlim H^*(X_{\text{ét}}, \mathbb{Z}/\ell^m\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, because étale cohomology really only behaves well with finite coefficients). One of the motivations for considering ℓ-adic cohomology is that $G_K := \mathrm{Gal}(\bar{K}/K)$ naturally acts on $X_{\bar{K}}$ so for each $\sigma \in G_K$, and each $m \in \mathbb{N}$ we get an automorphism $\sigma^* : \mathrm{H}^m(X_{\bar{K},\text{ét}}; \mathbb{Q}_\ell) \to \mathrm{H}^m(X_{\bar{K},\text{ét}}; \mathbb{Q}_\ell)$, i.e. a $\mathbb{Q}_\ell$-representation of G_K, which is referred to as the ℓ-adic Galois representation.

When X is smooth and proper over $\mathbb{Z}_p$, then we have $\mathrm{H}^*_{\text{ét}}(X_{\mathbb{F}_p}, \mathbb{Z}_\ell) \simeq \mathrm{H}^*_{\text{ét}}(X_{\mathbb{Q}_p}, \mathbb{Z}_\ell)$ as long as $\ell \neq p$, and in particular $\mathrm{H}^*_{\text{ét}}(X_{\mathbb{F}_p}, \mathbb{Q}_\ell) \simeq \mathrm{H}^*_{\text{ét}}(X_{\mathbb{Q}_p}, \mathbb{Q}_\ell)$. In other words the étale cohomology stays the same on each fiber. This fails miserably when we consider $\ell = p$, and in fact this is easy to see:

Example 1. Consider an elliptic curve E over $\mathbb{Z}_p$, and let $E_k = E_{\overline{\mathbb{F}_p}}$, and $E_K = E_{\overline{\mathbb{Q}_p}}$ be the special and generic fibers, which are themselves elliptic curves over $\mathbb{F}_p$ and $\mathbb{Q}_p$ respectively. Recall that elliptic curves are Abelian varieties; we will compute the étale cohomology of an elliptic curve in terms of its group structure. We can compute $\mathrm{H}^1(X_{\text{ét}}, \mathbb{Z}_p)$ as $\mathrm{Hom}_{\text{cont}}(\pi_1(X), \mathbb{Z}_p)$ where $\pi_1(X)$ is the étale fundamental group, but the étale fundamental group of an Abelian variety is easy to compute, as any connected finite étale cover of an Abelian variety is itself an Abelian variety. This lets us identify $\pi_1(X)$ with its Tate modules - i.e. $\pi_1(X) = \prod_\ell T_\ell(E)$ where T_ℓ is $\varprojlim_n E[\ell^n]$, where $E[\ell^n]$ denotes the ℓ^n torsion part of E.

These two ideas let us identify $\mathrm{H}^1(E_{\text{ét}}, \mathbb{Z}_p)$ with $T_p(E)$. First applying this to E_K which is an elliptic curve over a characteristic 0 field, and recalling that $E[p^n] = (\mathbb{Z}/(p)^n)^2$ for elliptic curves over a ch 0 field [11], we see that $\mathrm{H}^1_{\text{ét}}(E_K, \mathbb{Z}_p) = \mathbb{Z}_p \times \mathbb{Z}_p$.

Now applying to E_k which is an elliptic curve over a characteristic p field, which means $E_k[p^n] = 0$ or $\mathbb{Z}/(p)^n$, we have that $\mathrm{H}^1_{\text{ét}}(E_k, \mathbb{Z}_p) = \mathbb{Z}_p$ or 0.

We will only be thinking about the p-adic étale cohomology of the generic fiber, $H^*_{\text{ét}}(X_K, \mathbb{Z}_p)$.

There is also a new cohomology theory, due to Bhatt-Morrow-Scholze [3] which is a generalization of all 3 of these. We need some new terminology to make this precise. Define $\mathrm{A}_{\text{inf}} := W(\mathcal{O}^\flat_{\mathbb{C}_p})$, where $-^\flat$ denotes the tilting operation - i.e. the inverse limit $\varprojlim_{n\in\mathbb{N}} \mathcal{O}_{\mathbb{C}_p}/p = \varprojlim(\ldots \to \mathcal{O}_{\mathbb{C}_p}/p \to \mathcal{O}_{\mathbb{C}_p}/p \to \mathcal{O}_{\mathbb{C}_p}/p)$ where the map is always given by the Frobenius $x \mapsto x^p$. The idea is that $\mathcal{O}_{\mathbb{C}_p}$ is a characteristic p field, and this tilting procedure is adjoining arbitrary p-power roots to every element in $\mathcal{O}_{\mathbb{C}_p}/p$ which makes it into a perfect field of characteristic p. In the paper [3], a new cohomology theory is introduced, which for a variety X over $\mathcal{O}_{\mathbb{C}_p}$ is denoted as $\mathrm{R\Gamma}_{\mathrm{A}_{\text{inf}}}(X)$, and is a perfect complex of A_{inf}-modules with the properties that:

Theorem 6 (Bhatt-Morrow-Scholze)**.** *The cohomology theory $\mathrm{R\Gamma}_{\mathrm{A}_{\text{inf}}}(X)$ recovers all three of the other cohomology theories, in the sense that for X/S as above:*

1. $\mathrm{R\Gamma}_{\mathrm{A}_{\text{inf}}}(X) \otimes_{\mathrm{A}_{\text{inf}}} W(k) \simeq \mathrm{H}^*_{crys}(X_k, W(k))$
2. $\mathrm{R\Gamma}_{\mathrm{A}_{\text{inf}}}(X) \otimes_{\mathrm{A}_{\text{inf}}} \mathcal{O}_{\mathbb{C}_p} \simeq \mathrm{H}^*_{dR}(X/S)$

3. $\mathrm{R}\Gamma_{\mathrm{A_{inf}}}(X)[1/\mu] \simeq \mathrm{H}^*_{\acute{e}t}(X_K, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathrm{A_{inf}}[1/\mu]$.

It might seem as if we haven't fully recovered the étale cohomology, since we are tensoring with $\mathrm{A_{inf}}[1/\mu]$, but $\mathrm{A_{inf}}$ comes equipped with a Frobenius action and we can recover the integral étale cohomology by taking Frobenius fixed points, i.e.

$$H^*_{\text{ét}}(X_K, \mathbb{Z}_p) = \left(\mathrm{H}^*_{\text{ét}}(X_K, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathrm{A_{inf}}[1/\mu]\right)^{\varphi=1}$$

There is a new paper coming out soon, also due to Bhatt-Morrow-Scholze, where they will prove this same result again, but this time using THH; it seems as if the starting point for this idea was the following calculation:

Theorem 7 (Hesselholt)**.** *With the notation above,*

$$\pi_0 \left(\mathrm{THH}(\mathcal{O}_{\mathbb{C}_p})\hat{}_p\right)^{hS^1} = \mathrm{A_{inf}}.$$

This is the homotopy fixed points of the S^1-action on the p-completion of $\mathrm{THH}(\mathcal{O}_{\mathbb{C}_p})$.

3 Construction of THH

The definition of topological Hochschild homology given in the introduction is classical, but there is a more modern approach to THH. This new constructions of THH comes with technical advantages, at the cost of more sophistication. We first give a brief overview of the ∞-category of spectra, for more details see [5].

A spectrum is the data of a collection of based spaces $\{X_n\}_{n\in\mathbb{Z}}$, as well as homeomorphisms $\phi_n : X_n \to \Omega X_{n+1}$. An object of the ∞-category of spectra will quite literally be the same thing, we just encode this data in a different way. To be precise, let S denote the ∞-category of spaces, and $S_* := S_{*/}$ the ∞-category of pointed spaces. Let $\mathbb{Z}$ be a category by its ordering.

Definition 8. A *prespectrum* is a functor

$$X : \mathrm{N}(\mathbb{Z} \times \mathbb{Z}) \to S_*,$$

such that for all $i \neq j$ the value $X(i,j)$ is a zero object in S_*.

Here $\mathrm{N}(\mathbb{Z} \times \mathbb{Z})$ is the nerve of the category $\mathbb{Z} \times \mathbb{Z}$. Writing X_n for $X(n,n)$, visually this data is:

$$\begin{tikzcd}
 & & & ... \\
 & & 0 \arrow[r] & X_{n+1} \arrow[r] \arrow[u] & ... \\
 & 0'' \arrow[r] & X_n \arrow[r] \arrow[u] & 0' \arrow[u] \\
... \arrow[r] & x_{n-1} \arrow[r] \arrow[u] & 0''' \arrow[u] \\
 & ... \arrow[u]
\end{tikzcd}$$

Since the suspension and loop functors are homotopy pushouts and pullbacks, respectively, of diagrams of this type, this data induces maps $X_n \to \Omega X_{n+1}$, and we say that X is a *spectrum* if the induced map $X_n \to \Omega X_{n+1}$ is an equivalence for all n. We define the ∞-category of spectra Sp to be the full subcategory of $Fun^\infty(N(\mathbb{Z} \times \mathbb{Z}), S_*)$ spanned by the spectrum objects. The ∞-category Sp is an example of a *stable* ∞-category.

Earlier this year Thomas Nikolaus and Peter Scholze introduced a new construction of THH, which is involved, but simplifies many things. For example, if A is an $\mathbb{E}_\infty$-algebra in the ∞-category Sp of spectra, then

1. $TC(A) = \mathrm{Map}_{\mathrm{CycSp}}(\mathbb{S}, \mathrm{THH}(A))$,
2. $\mathrm{THC}^-(A) = \mathrm{THH}(A)^{hS^1}$,
3. $\mathrm{TP}(A) = \mathrm{THH}(A)^{tS^1}$.

These are, in order, the *topological cyclic homology*, the *topological negative cyclic homology*, and the *topological periodic homology*, and CycSp is the ∞-category of cyclotomic spectra which we will be able to define soon. We will try to outline the construction below.

Now we need to understand the Tate construction. Suppose we have a finite group G, and a vector space V with a G action, then we can take the invariants and coinvariants of this action, V^G and V_G, and there is a natural map $V_G \to V^G$ called the norm map given by $v \mapsto \sum\limits_{g\in G} gv$, i.e. we sum across every element in the orbit of v, which is a well defined element of V^G since $h\sum gv = \sum hgv = \sum gv$.

We will now generalize this to groups acting on spectra.

Definition 9. For a group G, a G-equivariant spectrum is an element in the ∞-category $\mathrm{Sp}^{BG} := \mathrm{Fun}(\mathrm{BG}, \mathrm{Sp})$, where BG is a classifying space for G.

Explicitly, this is the data of a spectrum X, a map $f_g : X \to X$ for every element $g \in G$, homotopies $H_{g,h} : f_{gh} \to f_g \circ f_h$, and all of the higher homotopies to fill in this homotopy coherent diagram.

Now we can define the *homotopy orbits functor* (homotopy coinvariants) as

$$-_{hG} : \mathrm{Sp}^{BG} \to \mathrm{Sp} : (F : \mathrm{BG} \to \mathrm{Sp}) \mapsto \mathrm{colim}_{\mathrm{BG}} F,$$

and the *homotopy fixed points functor* (homotopy invariants) as

$$-^{hG} : \mathrm{Sp}^{\mathrm{BG}} \to \mathrm{Sp} : (F : \mathrm{BG} \to \mathrm{Sp}) \mapsto \lim_{\mathrm{BG}} F.$$

This definition can be made to feel intuitive; to see this, consider a G space X, which we think of as a functor $F : \mathrm{BG} \to \mathrm{Top}$, where BG is G realized as a groupoid. By definition the limit of this functor fits in this diagram for every element $g \in G$, where $F(g)$ is the action of g i.e. $F(g)(x) = gx$:

$$\begin{tikzcd}
 & Y \arrow[d, dashed, "\exists ! q"] \arrow[ddl, bend right, "q_k"'] \arrow[ddr, bend left, "q_j"] & \\
 & \lim_{\mathrm{BG}} F \arrow[dl, "i"'] \arrow[dr, "i"] & \\
X \arrow[rr, "F(g)"'] & & X
\end{tikzcd}$$

If one stares at this diagram long enough hopefully it will be clear that the fixed points X^G and inclusion map i deserve to be in this limit diagram - the point is that if this diagram commutes for every $g \in G$ then every point the image of Y has to be fixed by the action, thus $q : Y \to X$ has to factor through X^G.

For $X \in Sp$ we can construct a norm map $Nm_G : X_{hG} \to X^{hG}$, and we define the *Tate construction* of X, denoted X^{tG} to be the cofiber of this map - $X^{tG} := \mathrm{cofib}(Nm_G : X_{hG} \to X^{hG})$, in fact this gives a functor $-^{tG} : \mathrm{Sp}^{\mathrm{BG}} \to \mathrm{Sp}$. If H is a normal subgroup of a group G we get a similar functor, also called the Tate construction and denoted $-^{tG}$, but this time the target has a residual action of H/G: i.e. $-^{tG} : \mathrm{Sp}^{\mathrm{BH}} \to \mathrm{Sp}^{\mathrm{B(H/G)}}$.

To define the ∞-category of Cyclotomic spectra we will also need:

Definition 10. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Define the lax equalizer to be the ∞-category $\mathrm{LEq}(F, G)$ which is the pullback in the ∞-category of ∞-categories:

$$\begin{tikzcd} \mathrm{LEq}(F,G) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{D}) \arrow[d, "\mathrm{ev}"] \\ \mathcal{C} \arrow[r] & \mathcal{D} \times \mathcal{D}, \end{tikzcd}$$

where the functor $\mathcal{C} \to \mathcal{D} \times \mathcal{D}$ is given by (F, G), and $\mathrm{ev} = (\mathrm{ev}_0, \mathrm{ev}_1)$.

Objects of $\mathrm{LEq}(F, G)$ are given by pairs (c, f) where $c \in \mathcal{C}$ and $f : F(c) \to G(c)$ is an arrow in $\mathcal{D}$, and for two objects $X = (c_X, f_X)$ and $Y = (c_Y, f_Y)$:

$$\mathrm{Map}_{\mathrm{LEq}(F,G)}(X, Y) = \mathrm{Eq}\left(\mathrm{Map}_{\mathcal{C}}(c_X, c_Y) \rightrightarrows \mathrm{Map}_{\mathcal{D}}(F(c_x), G(c_y))\right),$$

where the arrows in the equalizer are given by $(f_X)^*G$, and $(f_X)_*F$.

We are now ready to define the ∞-category of cyclotomic spectra! Let $\mathbb{P}$ denote the set of all prime numbers.

Definition 11. The ∞-category CycSp is defined as the lax equalizer of

$$\mathrm{CycSp} := \mathrm{LEq}\left(\mathrm{Sp}^{BS^1} \rightrightarrows \prod_{\mathbb{P}} \mathrm{Sp}^{BS^1}\right),$$

where the two maps are given by the product over all primes of the identity map $\mathrm{id} : \mathrm{Sp} \to \mathrm{Sp}$, and the product over all primes of the Tate construction $-^{tC_p} : \mathrm{Sp}^{BS^1} \to \mathrm{Sp}^{B(S^1/C_p)} \simeq \mathrm{Sp}^{BS^1}$.

This is a complete, cocomplete, presentable, stable ∞-category, with a forgetful functor $\mathrm{CycSp} \to \mathrm{Sp}$ which is conservative and preserves colimits. An object in CycSp is an S^1-equivariant spectrum X, together with the data of S^1-equivariant maps $\varphi_p : X \to X^{tC_p}$ for all p; these maps are analogous to the Frobenius map.

Example 2.
1. There is a trivial cyclotomic structure that can be put on $X \in \mathrm{Sp}$, which we will only consider in the case $X = \mathbb{S}$. This comes from considering $\mathbb{S}$ as an S^1-spectrum via the trivial action, and the maps φ_p come from the composition of the canonical maps $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$.
2. The key example of a cyclotomic spectrum will be given by $\mathrm{THH}(A)$ where A is an $\mathbb{E}_1$-algebra in Sp. We will see where this cyclotomic structure comes from later.

Since CycSp is a stable ∞-category, it is enriched over spectra - i.e. for any $X, Y \in \mathrm{CycSp}$, $\mathrm{Map}_{\mathrm{Cycsp}}(X, Y) \in \mathrm{Sp}$. So for any $X \in \mathrm{CycSp}$ we can define $\mathrm{TC}(X) := \mathrm{Map}_{\mathrm{CycSp}}(\mathbb{S}, X)$. Typically when we refer to TC it will be in the case when we started with an $\mathbb{E}_1$-algebra $A \in \mathrm{Sp}$, and apply TC to $\mathrm{THH}(A)$ - $\mathrm{TC}(A) := \mathrm{TC}(\mathrm{THH}(A))$; we will abusively call this $\mathrm{TC}(A)$.

We already know what THH "should be", and as usual with a higher categorical construction we just need to find combinatorial ∞-categories which encode all of the data in the diagram:

$$
\begin{tikzcd}
\dots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A\otimes A\otimes A \arrow[loop above, "C_3"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A\otimes A \arrow[loop above, "C_2"] \arrow[r, shift left=1] \arrow[r, shift right=1] & A.
\end{tikzcd}
$$

We will reveal the punchline first, and then explain what each of these objects are, and what structure they encode piece by piece afterwards. As opposed to following [10] closely to make everything precise, we settle for giving the main ideas and intuition; letting the reader refer to appendix T of [10] if they wish to make the discussion precise.

Definition 12. For an $\mathbb{E}_1$-algebra $A \in \mathrm{Sp}$, define $\mathrm{THH}(A) \in \mathrm{Sp}^{BS^1}$ to be the geometric realization of the cyclic spectrum

$$\mathrm{N}(\Lambda^{op}) \xrightarrow{V^\circ} \mathrm{N}(\mathrm{Ass}^{\otimes}_{\mathrm{act}}) \xrightarrow{A^{\otimes}} \mathrm{Sp}^{\otimes}_{\mathrm{act}} \xrightarrow{\otimes} \mathrm{Sp}.$$

Here, Λ is Connes' *cyclic category*, and $\mathrm{N} : \mathrm{Cat} \to \infty\mathrm{Cat}$ is the nerve functor. This category is constructed with the property $|\Lambda| \simeq S^1$ in mind; Λ has objects finite ordered sets $|n| = \{0 \to 1 \to \dots \to n\}$, and for morphisms it has the face maps $\delta^n_i : |n-1| \to |n|$ and degeneracy maps $\sigma^n_i : |n+1| \to |n|$ satisfying the same relations as in the simplex category, as well as the new maps $\tau_n : |n| \to |n|$ satisfying a few relations which ensure that these new maps τ_n play nicely with the structure maps. The point is that $\mathrm{Aut}_{\Delta}(|n|) = \{1\}$, but $\mathrm{Aut}_{\Lambda}(|n|) = C_n$. This makes it clear that there is a functor $j : \Delta^{op} \to \Lambda^{op}$.

We call a functor $X^c_\bullet : \Lambda^{op} \to \mathcal{C}$ a *cyclic object* in $\mathcal{C}$. Given a cyclic object $X^c_\bullet \in \mathcal{C}$, there is an underlying simplicial set $X_\bullet$ given by forgetting the maps τ_n; i.e. precomposing the functor $X^c_\bullet : \Lambda^{op} \to \mathcal{C}$ with the functor $j : \Delta^{op} \to \Lambda^{op}$. In this way a cyclic object is just a simplicial object which after passing to geometric realization will have a continuous S^1 action. To make this precise, there is a natural functor $\mathrm{Fun}(\mathrm{N}(\Lambda^{op}), \mathrm{Sp}) \to \mathrm{Sp}^{BS^1}$, which takes a cyclic spectrum $X_\bullet \in \mathrm{Fun}(\mathrm{N}(\Lambda^{op}), \mathrm{Sp})$ to an S^1-equivariant spectrum whose underlying spectrum is the geometric realization $|j^* X_\bullet| \in \mathrm{Sp}$.

Moving on to the next object in the diagram. The category $\mathrm{Ass}^{\otimes}$ is the associative operad, which has a natural functor to the commutative operad Fin_*, which forgets the linear ordering. Define $\mathrm{Ass}^{\otimes}_{\mathrm{act}} := \mathrm{Ass}^{\otimes} \times_{\mathrm{Fin}_*} \mathrm{Fin}$, where the functor $\mathrm{Fin} \to \mathrm{Fin}_*$ just adds a disjoint basepoint. There is a functor $\Lambda \to \mathrm{Fin}$, which lifts to $\Lambda \to \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ due to an equivariance property on morphisms in Λ which is discussed in [10]. We also have that Λ is self dual - i.e. $\Lambda \simeq \Lambda^{op}$, and combining these we get a functor $\tilde{V^\circ} : \Lambda^{op} \to \mathrm{Ass}^{\otimes}_{\mathrm{act}}$. This of course gives rise to $V^\circ : \mathrm{N}(\Lambda^{op}) \to \mathrm{N}(\mathrm{Ass}^{\otimes}_{\mathrm{act}})$.

The next two maps are more familiar. First, recall that an $\mathbb{E}_1$-algebra A in the ∞-category Sp is given by a functor $A^{\otimes} : \mathrm{Ass}^{\otimes} \to \mathrm{Sp}^{\otimes}$ [7]. We abusively write $A^{\otimes}$ for the map $\mathrm{Ass}^{\otimes}_{\mathrm{act}} \to \mathrm{Sp}^{\otimes}_{\mathrm{act}} := \mathrm{Sp}^{\otimes} \times_{\mathrm{N}(\mathrm{Fin}_*)} \mathrm{N}(\mathrm{Fin})$ which comes from functoriality of $\times$. Then $\otimes : \mathrm{Sp}^{\otimes}_{\mathrm{act}} \to \mathrm{Sp}$ is the symmetric monoidal functor which sends $(X_1, \dots, X_n) \mapsto X_1 \otimes \dots \otimes X_n$ (the symmetric monoidal structure on $\mathrm{Sp}^{\otimes}_{\mathrm{act}}$ is defined on page 76 of [10]).

Phew.

Now for the bad news, remember we need to define $\mathrm{THH}(A)$ as an object in CycSp; so far we have the easy part, $\mathrm{THH}(A) \in \mathrm{Sp}$. To give the cyclotomic structure, we need to use the *Tate diagonal*. This will be a replacement for the diagonal map in Sp.

Proposition 13. *For a prime p, the functor $T_p : \mathrm{Sp} \to \mathrm{Sp}$, given by $X \mapsto (X \otimes \dots \otimes X)^{tC_p}$ is exact, where $(X \otimes \dots \otimes X)$ is p self tensors of X.*

Now, given an exact functor $\mathrm{Sp} \to \mathrm{Sp}$, we can compose with $\Omega^\infty : \mathrm{Sp} \to S$ to get a left exact functor $\mathrm{Sp} \to S$. This is in fact an equivalence $\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Sp}, \mathrm{Sp}) \to \mathrm{Fun}^{\mathrm{LEx}}(\mathrm{Sp}, S)$. Then combining with the equivalence coming from the Yoneda lemma, for any $F \in \mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Sp}, \mathrm{Sp})$ we get an equivalence

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{Sp},\mathrm{Sp})}(\mathrm{id}_{\mathrm{Sp}}, F) \to \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, F(\mathbb{S})) \to \Omega^\infty F(\mathbb{S})$$

In particular, a natural transformation $\mathrm{id}_{\mathrm{Sp}} \to T_p$ is specified by a map $\mathbb{S} \to T_p(\mathbb{S}) = (\mathbb{S} \otimes ... \otimes \mathbb{S})^{tC_p} = \mathbb{S}^{tC_p}$.

Definition 14. The Tate diagonal is the natural transformation $\Delta_p : \mathrm{id}_{\mathrm{Sp}} \to T_p$ which under the above equivalence corresponds to the canonical map $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$. Thus for every $X \in \mathrm{Sp}$, $\Delta_p : X \to (X \otimes ... \otimes X)^{tC_p}$.

For an $\mathbb{E}_\infty$ ring spectrum R, the Tate diagonal gives way to a generalization of the Frobenius map, which is called the *Tate valued Frobenius.*

Definition 15 (Tate valued Frobenius)**.** The Tate valued Frobenius is the map

$$R \xrightarrow{\Delta_p} (R \otimes ... \otimes R)^{tC_p} \xrightarrow{m^{tC_p}} R^{tC_p},$$

where m^{tC_p} comes from the multiplication map $m : (R \otimes ... \otimes R) \to R$.

This map deserves to be called the Frobenius:

Example 3. Given $A \in \mathrm{CRing}$, let $HA \in \mathrm{Sp}$ be its Eilenberg-MacLane spectrum, then the Tate valued Frobenius gives a map $HA \to HA^{tC_p}$, and on π_0 this is the Frobenius map $A \mapsto A/p$, $x \mapsto x^p$.

In the higher homotopy groups the Tate valued Frobenius recovers the Steenrod operations [10].

The Tate valued Frobenius will also give us an easier route to constructing the cyclotomic structure on $\mathrm{THH}(A)$ when A is an $\mathbb{E}_\infty$-algebra, which will suit our needs. There is a natural map $A \to \mathrm{THH}(A)$, given by the inclusion of A into the cyclic object defining $\mathrm{THH}(A)$. This lets us write the following diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\Delta_p"] & \mathrm{THH}(A) \\ (A \otimes \cdots \otimes A)^{tC_p} \arrow[r] & \mathrm{THH}(A)^{tC_p} \end{tikzcd}$$

The cyclotomic structure maps on $\mathrm{THH}(A)$ will be the the maps $\varphi_p : \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$ that make this a commutative diagram of $\mathbb{E}_\infty$-rings. To prove that this makes sense we need to know the universal property of THH - for an $\mathbb{E}_\infty$-algebra A, the map $A \to \mathrm{THH}(A)$ is initial among maps from A to an $\mathbb{E}_\infty$-ring with an S^1 action, which is proven by showing that $\mathrm{THH}(A) = S^1 \otimes A$ in the ∞-category of $\mathbb{E}_\infty$-rings.

Then the map in the bottom of this diagram, $A \to \mathrm{THH}(A)^{tC_p}$ which is an $\mathbb{E}_\infty$-ring map and $\mathrm{THH}(A)^{tC_p}$ has an S^1 action since $S^1/C_p \simeq S^1$, so the universal property gives us a map unique up to canonical isomorphism $\varphi_p : \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$.

This diagram also gives a nice feel for what the cyclotomic structure maps are doing - they come from citing a universal property to factor the Tate valued Frobenius map through $\mathrm{THH}(A)$, so they should be similar to Tate valued Frobenius maps.

4 THH and p-adic Hodge theory

Suppose now that A is an $\mathbb{F}_p$-algebra. In [8] there are outlines of how we can recover things like the crystalline or deRham cohomology from $\mathrm{THH}(A)$. These ideas are currently unpublished, so there are missing details below; we try to get an idea of how these constructions go, in hopes of being prepared to read [4] when it is published.

Let's try to understand how the crystalline cohomology of A is related to $\mathrm{THH}(A)$. The punchline is:

Theorem 16 (Hesselholt [6]). *If A is a smooth $\mathbb{F}_p$-algebra, then $\pi_\bullet \mathrm{THH}(A)^{C_{p^{n-1}}} \simeq W_n\Omega^\bullet_A[t]$ where $W_n\Omega^\bullet_A$ is the n'th piece of the deRham Witt complex, and $|t| = 2$.*

Remember, the deRham Witt complex computes the crystalline cohomology!

Let's try to understand why this is reasonable. First, if E is a connective cyclotomic spectrum, then we can realize E as a genuine C_{p^n} spectrum for any n. To see how this is done, first take

$$E^{C_p} = E \times_{E^{tC_p}} E^{hC_p},$$

then inductively:

$$E^{C_{p^n}} = E \times_{E^{tC_p}} (E^{C_{p^{n-1}}})^{hC_p}.$$

In particular, $E = \mathrm{THH}(A)$ is a genuine C_{p^n} spectrum for all n.

The point is that the cyclotomic maps on $\mathrm{THH}(A)$ are the "Frobenius" maps, and taking C_{p^n} fixed points should feel like taking the length n Witt vectors. To try to explain this analogy, lets first review the Witt vectors.

The Witt vectors of a commutative ring A, denoted $W(A)$, are a way to lift A to a characteristic 0 ring. The key example is $W(\mathbb{F}_p) = \mathbb{Z}_p$. We also have the length n Witt vectors, $W_n(A)$ which mediate between A and $W(A)$; for example $W_n(\mathbb{F}_p) = \mathbb{Z}/(p^n)$. There are several important maps between these rings

1. Restriction $R : W_n(A) \to W_{n-1}(A)$,
2. Verschiebung $V : W_{n-1}(A) \to W_n(A)$, and
3. Frobenius $F : W_n(A) \to W_{n-1}(A)$,

with the following properties

1. Everything commutes with the restriction, $RF = FR$, $RV = VR$,
2. $FV = p$, the multiplication by p map,
3. There are "ghost maps" $\gamma_i : W_n(A) \to A$ for $0 \leq i < n$ which are the ring homomorphisms $\gamma_i = R^{n-i-1}F^i$,
4. There are Teichmüller representatives $\tau : A \to W_n(A)$ such that $R\tau = \tau$, and
5. F is a ring homomorphism, and V is semilinear i.e. $V(x)y = V(xFy)$.

The deRham Witt complex satisfies a universal property as an initial dga over A with maps behaving in the way these maps behave, to be precise, the deRham Witt complex is the initial object in the category of *Witt complexes* over A. Thus we can show that $E = \mathrm{THH}(A)$ receives a map from the deRham Witt complex (after passing to homotopy) if we can endow E with a differential, and with these Witt maps.

These maps are difficult to construct, but we try to outline their construction below anyways.
First, the ghost maps. An equivalent way to write our above definition of $E^{C_{p^n}}$ is:

$$E^{C_{p^n}} = E \times_{E^{tC_p}} E^{hC_p} \times_{(E^{tC_p})^{hC_p}} \cdots \times E^{hC_{p^n}}.$$

This lets us define $\gamma_i : E^{C_{p^n}} \to E$ via the factorization $E^{C_{p^n}} \to E^{hC_{p^i}} \to E$, where the first map is the projection, and the latter map is the canonical one. These are the ghost maps.

The restriction maps $R : E^{C_{p^n}} \to E^{C_{p^{n-1}}}$ fit into the pullback square

$$\begin{array}{ccc} E^{C_{p^n}} & \longrightarrow & E^{hC_{p^n}} \\ \downarrow R & & \downarrow \\ E^{C_{p^{n-1}}} & \longrightarrow & E^{tC_{p^n}}, \end{array}$$

where the arrow on the right is the canonical map (remember $E^{tG} := \mathrm{cofib}(Nm_G : E_{hG} \to E^{hG})$).

We get a fiber sequence

$$(E^{C_{p^{n-1}}})_{hC_p} \to E^{C_{p^n}} \to E.$$

The map $V : E^{C_{p^{n-1}}} \to E^{C_{p^n}}$ is given by the transfer, and the map $F : E^{C_{p^n}} \to E^{C_{p^{n-1}}}$ is the inclusion map.

Now we need Teichmüller representatives, which will come from a map $\tau : \Omega^\infty A \to \Omega^\infty \mathrm{THH}(A)^{C_{p^n}}$. This is where the factorization homology picture will help us. An element of $\Omega^\infty \mathrm{THH}(A)$ will be given by choosing finitely many points of S^1 and labeling each with a point of A. To make this a C_{p^n} fixed point we should restrict to labelings of p^n points, where each point has the same label. Now the map τ is clear. It is unclear to the author how to make this map precise.

Since E is an S^1-spectrum, we have a map $S^1_+ \wedge E \to E$, or $E \to \mathrm{Map}(S^1, E) \simeq E \wedge \Sigma^{-1} E$. The map $d : E \to \Sigma^{-1} E$ will be our differential! This is known as *Connes' differential*.

Now by the universal property of the deRham Witt complex $\{W_n \Omega^\bullet_A\}_{n \in \mathbb{N}}$, there is a map of inverse systems of dgas $\{W_n \Omega^\bullet_A\}_{n \in \mathbb{N}} \to \{\pi_\bullet \mathrm{THH}(A)^{C_{p^n}}\}_{n \in \mathbb{N}}$. When A is smooth over $\mathbb{F}_p$, the above computation of Hesselholt tells us that this map exhibits $\{\pi_\bullet \mathrm{THH}(A)^{C_{p^n}}\}_{n \in \mathbb{N}}$ as $\{W_n \Omega^\bullet_A[\sigma]\}_{n \in \mathbb{N}}$, where $|\sigma| = 2$. In particular, the deRham Witt complex is recovered in degree 0!

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