MATH 180-CALCULUS- Solutions of Exam 1 10-4-02-Friedland

1. The average time to lose half of the money at the roulette table in the casino of Paradise Island is 143 minutes. Assume exponential decay.
a. (10pts) What is the rate of decay?
b. (10pts) A person came to the roulette table with the amount of P_0 dollars at 3PM and left at 7PM with \$100. How much did he lose?

Answer:
(a) $P(t) = P_0 e^{-kt}$. k is the rate of (exponential) decay. If T is the half time of life then

$$\frac{1}{2} = e^{-kT} \Rightarrow k = \frac{\ln 2}{T} = 0.00484718 \, \frac{1}{\text{minute}} = 0.29087 \, \frac{1}{\text{hour}}.$$

(b) 4 hours of play are 240 minutes.

$$P(240) = 100 = P_0 e^{-0.00484718*240} \Rightarrow P_0 = 100e^{0.00484718*240} = \$320.06.$$

The person lost $220.06 = 320.06 - 100$ dollars.

2. Let $f(x)$ be defined as below

$$f(x) = \begin{cases} -kx - 3, & \text{if } x \geq 2; \\ \frac{k}{6-x}, & \text{if } x < 2. \end{cases}$$

a. (15pts) Find the value of k for which $f(x)$ is continuous for all values of x.
b. (5pts) For this value of k is $f(x)$ differentiable for all values of x?

Answer: Note that $-kx - 3$ is continuous and and differentiable everywhere. $\frac{k}{6-x}$ is continuous and differentiable except for $x = 6$ (when $k \neq 0$). When you "glue" two functions together the potential problematic point is the glue point $x = 2$.

(a) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{k}{6-x} = \dfrac{k}{6-2} = \dfrac{k}{4}$, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -kx - 3 = -k2 - 3.$

For the continuity the left and the right limit must be the same:

$$\frac{k}{4} = -2k - 3 \Rightarrow k = -8k - 12 \Rightarrow 9k = -12 \Rightarrow k = -\frac{4}{3}.$$

(b) Let $k = -4/3$. Then the function $f(x)$ has a corner at $x = 2$. ($\frac{k}{6-x} = \frac{4}{3(x-6)}$ is decreasing at $x = 2$, while $-kx - 3 = \frac{4x}{3} - 3$ is increasing at $x = 2$.) Hence it is not differentiable at $x = 2$, i.e. not differentaible for all values of x.

3. Let

$$f(x) = \frac{2}{4x - 3} + 5.$$

a. (15pts) Use the definition of the derivative as the limit of the difference quotient to find the first derivative of f.

b. (5pts) Find the vertical and horizontal asymptotes of f.

Answer:
(a) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Then

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\Big(\frac{2}{4(x+h)-3} + 5 - \big(\frac{2}{4x-3}+5\big)\Big) = \frac{1}{h}\Big(\frac{2}{4(x+h)-3} - \frac{2}{4x-3}\Big) =$$

$$\frac{2}{h}\,\frac{4x-3-(4(x+h)-3)}{(4(x+h)-3)(4x-3)} = \frac{2}{h}\,\frac{-4h}{(4(x+h)-3)(4x-3)} = \frac{-8}{(4(x+h)-3)(4x-3)}$$

Let $h \to 0$, that is when taking the limit we substitute $h = 0$ in the above last expression to obtain $f'(x) = \frac{-8}{(4x-3)^2}$.
(b) Vertical asymptote happens when you divide by 0. That is $4x - 3 = 0 \Rightarrow x = \frac{3}{4}$. Horizotal asymptote happens when $f(x)$ tends to a limit either for $x \to \infty$ or $x \to -\infty$. As

$$\lim_{x\to\infty} \frac{2}{4x-3} + 5 = 0 + 5 = 5, \quad \lim_{x\to-\infty} \frac{2}{4x-3} + 5 = 0 + 5 = 5,$$

the horizontal asymptote is $y = 5$.

4. (20pts) Determine which of the four functions $f(x)$, $g(x)$, $h(x)$, $i(x)$ is linear, concave up, concave down or none of the above:

x	-2	-1	0	1	2
$f(x)$	0.14	0.4	1	2.7	7.4
$g(x)$	0	2	0	-2	0
$h(x)$	-7.2	-4.1	-1	2.1	5.2
$i(x)$	-3	0	1	0	-3

Answer: For a function $F(x)$ we substitue the derivative $F'(x)$ by $\frac{F(x+h)-F(x)}{h}$. In this particular example we let $h = 1$.
$F' \nearrow \Rightarrow$ CU, $F' \searrow \Rightarrow$ CD, $F' \equiv Constant \Rightarrow$ LINEAR, F' sometimes increasing sometimes decreasing NOTA.

x	-2	-1	0	1	*answer*
$f(x+1)-f(x)$	0.26	0.6	1.7	4.7	*CU*
$g(x+1)-g(x)$	2	-2	-2	2	*NOTA*
$h(x+1)-h(x)$	3.1	3.1	3.1	3.1	*LINEAR*
$i(x+1)-i(x)$	3	1	-1	-3	*CD*

5. Let

$$f(x) = e^{\sin x}.$$

a. (10 pts) Calculate $f'(1)$ to two decimal places. (Don't forget to set your calculator to radian mode.)

b. (10 pts) Assume that $f'(\pi) = -1$. Find the *tangent approximation* to f at π. Find *without the calculator* the approximate value of $f(\pi + 0.01)$ using the tangent approximation.

Answer: (a) recall that $f'(x)$ is approximated by $\frac{f(x+h)-f(x)}{h}$

h	0.1	0.01	-0.01	0.001	-0.001	0.0001	-0.0001
$\frac{f(2+h)-f(2)}{h}$	1.18295	1.24694	1.25969	1.25274	1.254018	1.253317	1.25344

So $f'(2) = 1.253$ (1.25).
b. Tangent approximation at $x = a$ is the tangent line to the graph of f at a: $y = f(a) + f'(a)(x - a)$. Note that $f(\pi) = e^{\sin \pi} = e^0 = 1$. So the tangent line approximation

$$e^{\sin x} \sim 1 - (x - \pi) = -x - \pi + 1.$$

To find the approximate value of $e^{\sin(\pi+0.01)}$ you plug the value $\pi + 0.01$ into the tangent approximation to get 0.99. The correct value is 0.99005.

SCORES: TOP - 92, BOTTOM - 5; MEDIAN - 44;

A: 76-100 (6); B: 59 - 75 (15); C: 40 - 58 (23); D: 33-39 (13); F: 0 -32 (13).