D. ADDENDUM TO LECTURE 6 (G_A^{00} AND EXERCISES 5 & 6)

Let G be a sufficiently saturated pseudofinite group, and suppose $A \subseteq G$ is definable and NIP. Let $\mathcal{B}$ be the Boolean algebra generated by the collection $\{gAh : g, h \in G\}$ of bi-translates of A. In Lecture 6, we defined the stabilizer subgroup

$$\mathrm{Stab}^{\mu}(A) = \{g \in G : \mu(gA \triangle A) = 0\}$$

(where, as usual, μ denotes the normalized pseudofinite counting measure on definable subsets of G). We then proved the following result:

Theorem 6.4. $\mathrm{Stab}^{\mu}(A)$ *is a countably* $\mathcal{B}$*-type-definable subgroup of index at most* $2^{\aleph_0}$*.*

As it may not be the case that $\mathrm{Stab}^{\mu}(A)$ is *normal* in G, we defined:

$$G_A^{00} := \bigcap_{g \in G} g\,\mathrm{Stab}^{\mu}(A)g^{-1}.$$

Then I sketched the proof of the following corollary (and here I give more details).

Corollary 6.6. G_A^{00} *is a countably* $\mathcal{B}$*-type-definable normal subgroup of index at most* $2^{\aleph_0}$*.*

Proof. By construction, G_A^{00} is a normal subgroup. We first observe that it is $\mathcal{B}$-type-definable of bounded index. Indeed, if $I \subseteq G$ is a set of left coset representatives for $\mathrm{Stab}^{\mu}(A)$, then $|I| \leq 2^{\aleph_0}$ and $G_A^{00} = \bigcap_{g \in I} g\,\mathrm{Stab}^{\mu}(A)g^{-1}$. Since $\mathrm{Stab}^{\mu}(A)$ is an intersection of countably many sets in $\mathcal{B}$, and $\mathcal{B}$ is bi-invariant, it follows that G_A^{00} is an intersection of at most $2^{\aleph_0}$ sets in $\mathcal{B}$. Also, since any conjugate of $\mathrm{Stab}^{\mu}(A)$ still has index at most $2^{\aleph_0}$, it follows that G_A^{00} has index at most $2^{2^{\aleph_0}}$.

Next, we prove that G_A^{00} is C-invariant for some countable set $C \subseteq G$, i.e., $\sigma(G_A^{00}) = G_A^{00}$ for any (model-theoretic) automorphism σ of G that fixes C pointwise. To see this, recall that $\mathrm{Stab}^{\mu}(A)$ is an intersection of countably many definable sets, and so we may let C be the collection of all parameters used in the formulas defining these sets. So C is countable. Now suppose σ is an automorphism of G that fixes C pointwise. Then σ fixes any C-definable set setwise, and thus fixes $\mathrm{Stab}^{\mu}(A)$ setwise. Now,

$$\sigma(G_A^{00}) = \bigcap_{g \in G} \sigma(g\,\mathrm{Stab}^{\mu}(A)g^{-1}) = \bigcap_{g \in G} \sigma(g)\,\mathrm{Stab}^{\mu}(A)\sigma(g)^{-1} = G_A^{00}$$

(note that σ is, in particular, a group-theoretic automorphism of G).

We now know that G_A^{00} is type-definable and C-invariant. It follows from Exercise 5(d) that G_A^{00} is type-definable over C. Without loss of generality, we can assume we are working in a finite language (since G_A^{00} is $\mathcal{B}$-type-definable, we only need the group language and enough symbols to define A). So there are only countably many formulas with parameters from C, and thus only countably many C-definable subsets of G. So G_A^{00} is *countably* type-definable. Since G_A^{00} has bounded index, it follows from Exercise 6(c) that $[G : G_A^{00}] \leq 2^{\aleph_0}$.

The final issue is that we don't yet have countable type-definability using sets in $\mathcal{B}$ (in general, Exercise 5(d) introduces quantifiers). But this can be fixed with a very useful saturation trick. In particular, we have two representations of G_A^{00}, namely, as a countably type-definable set and a $\mathcal{B}$-type-definable set. So let $G_A^{00} = \bigcap_{n=0}^{\infty} X_n$, where each X_n is definable; and let $G_A^{00} = \bigcap_{i \in I} Y_i$ where I is small and each Y_i is in $\mathcal{B}$. For any $n \geq 0$, we have

$$\bigcap_{i \in I} Y_i = G_A^{00} \subseteq X_n,$$

and so by saturation (specifically, Exercise 5(a)), there is some finite $I_n \subseteq I$ such that $Z_n := \bigcap_{i \in I_n} Y_i \subseteq X_n$. Note that $Z_n \in \mathcal{B}$ for any $n \geq 0$. By construction,

$$G_A^{00} = \bigcap_{i \in I} Y_i \subseteq \bigcap_{n=0}^{\infty} Z_n \subseteq \bigcap_{n=0}^{\infty} X_n = G_A^{00}.$$

Therefore $G_A^{00} = \bigcap_{n=0}^{\infty} Z_n$ is countably $\mathcal{B}$-type-definable. □

Further Reading

Since Exercises 5 and 6 are a little more involved than some of the other basic saturation exercises, I have included proofs (of the relevant parts) below.

We work with *types* (i.e. finitely consistent collections of formulas), and follow the convention of listing free variables, e.g., $p(\bar{x})$ denotes a type consisting of formulas with free variables lying in the tuple $\bar{x}$. Note that since types can contain infinitely many formulas, $\bar{x}$ might be infinite. We also say that a type p is *over* a set A if all parameters used in the formulas in p come from A. See *Notes on Model Theory* for more details.

Exercise 5. *Let $\mathcal{M}$ be a sufficiently saturated structure.*

(c) *Let $p(\bar{x}, \bar{y})$ be a type over a small set $A \subseteq \mathcal{M}$, where $\bar{x}$ and $\bar{y}$ are tuples of variables of bounded length. Then the set*

$$X := \{\bar{a} \in \mathcal{M}^{\bar{x}} : p(\bar{a}, \bar{b}) \text{ holds for some } \bar{b} \in \mathcal{M}^{\bar{y}}\}$$

is type-definable over A.

(d) *Suppose $X \subseteq \mathcal{M}^{\bar{x}}$ is type-definable and A-invariant over some small set $A \subseteq \mathcal{M}$, where $\bar{x}$ has bounded length. Then X is type-definable over A.*

Proof. Part (c). Without loss of generality, we may assume that p is closed under finite conjunctions. Let $q(\bar{x})$ be the collection of formulas of the form $\exists \bar{y} \phi(\bar{x}, \bar{y})$ where $\phi(\bar{x}, \bar{y})$ is a formula in $p(\bar{x}, \bar{y})$. (We are abusing notation since any formula in p uses only finitely many variables in $\bar{x}, \bar{y}$.) So q is a type over A. Note also that q contains only boundedly many formulas (in particular, p contains only boundedly many formulas since A, $\bar{x}$, and $\bar{y}$ are bounded). We show that $X = q(\mathcal{M})$, and thus X is type-definable over A.

First, if $\bar{a} \in X$, then there is some $\bar{b} \in \mathcal{M}^{\bar{y}}$ such that $p(\bar{a}, \bar{b})$ holds. So $\bar{b}$ witnesses that $\exists \bar{y} \phi(\bar{a}, \bar{y})$ holds for any $\phi(\bar{x}, \bar{y})$ in p. So $q(\bar{a})$ holds. Conversely, suppose $\bar{a} \notin X$. Consider the type $p(\bar{a}, \bar{y})$ (which is now a type in the free variables $\bar{y}$ and with parameters from $A \cup \bar{a}$). Then $p(\bar{a}, \bar{y})$ is inconsistent. By saturation, there is some finite subset of $p(\bar{a}, \bar{y})$ that is inconsistent. So there is a formula $\phi(\bar{x}, \bar{y}) \in p$ such that $\phi(\bar{a}, \bar{y})$ is inconsistent, i.e., $\bar{a}$ does not realize $\exists \bar{y} \phi(\bar{x}, \bar{y})$. So $\bar{a}$ does not realize q.

Part (d). Since X is type-definable, we may fix a type $r(\bar{x}, \bar{y})$ over $\emptyset$, where $\bar{y}$ has bounded length, such that $X = r(\mathcal{M}, \bar{c})$ for some $\bar{c} \in \mathcal{M}^{\bar{y}}$. Let $q(\bar{y})$ be the complete type of $\bar{c}$ over A. Let $p(\bar{x}, \bar{y}) = r(\bar{x}, \bar{y}) \cup q(\bar{y})$, and note that p is a type over A. We show that

$$X = \{\bar{a} \in \mathcal{M}^{\bar{x}} : p(\bar{a}, \bar{b}) \text{ holds for some } \bar{b} \in \mathcal{M}^{\bar{y}}\},$$

and so X is type-definable over A by part (a).

The left-to-right containment is clear, since if $\bar{a} \in X$ then $p(\bar{a}, \bar{c})$ holds. Conversely, fix $\bar{a} \in \mathcal{M}^{\bar{x}}$ such that $p(\bar{a}, \bar{b})$ holds for some $\bar{b} \in \mathcal{M}^{\bar{y}}$. Since $q(\bar{b})$ holds, it follows that $\bar{b}$ and $\bar{c}$ have the same complete type over A. Since $\mathcal{M}$ is strongly homogeneous, there is an automorphism σ of $\mathcal{M}$ fixing A pointwise such that $\sigma(\bar{b}) = \bar{c}$. Since $r(\bar{a}, \bar{b})$ holds, it follows that $r(\sigma(\bar{a}), \bar{c})$ holds, and so $\sigma(\bar{a}) \in X$. So $\bar{a} \in X$ since X is A-invariant. □

Exercise 6. *Let G be a sufficiently saturated structure expanding a group, and suppose Γ is a type-definable subgroup of G.*

(a) *Suppose $\Gamma = \bigcap_{i\in I} X_i$ where I is small, each X_i is definable, and $\{X_i : i \in I\}$ is closed under finite intersections. Then for any $i \in I$ there is $j \in I$ such that $X_j^2 \subseteq X_i$.*

(b) *Suppose Γ has bounded index and X is a definable set containing Γ. Then X is left and right generic.*

(c) *Suppose Γ has bounded index and is an intersection of λ definable sets, where λ is small. Then Γ has index at most $2^{\lambda+\aleph_0}$.*

Proof. Part (a). For $i \in I$, let $\phi_i(x)$ be a formula defining X_i. Fix $i \in I$ and suppose there is no such j. Then the following type (in two singleton variables x and y) is finitely consistent:

$$p(x, y) := \{\phi_j(x) \wedge \phi_j(y) \wedge \neg\phi_i(x \cdot y) : j \in J\}.$$

By saturation, there are $a, b \in G$ such that $p(a, b)$ holds. Then $a, b \in \bigcap_{j\in J} X_j = \Gamma$, while $ab \notin X_i$. This is a contradiction since Γ is a subgroup contained in X_i.

Part (b). Suppose first that X is not left generic. Let λ be a cardinal which is larger than the index of Γ, but still bounded. Let $\bar{x}$ be a tuple of variables of length λ and let $\phi(x)$ be a formula defining X. Consider the type

$$p(\bar{x}) := \{\neg\phi(x_i^{-1} \cdot x_j) : i < j \in \lambda\}.$$

We claim that $p(\bar{x})$ is finitely satisfiable. Specifically, we fix $n \geq 1$ and find $a_1, \ldots, a_n \in G$ such that $a_i^{-1}a_j \notin X$ for all $i < j \leq n$. Choose $a_1 \in G$ arbitrarily and, given $k < n$ and $a_1, \ldots, a_k$, choose $a_{k+1} \notin a_1X \cup \ldots \cup a_kX$ (such an element exists by assumption on X).

Now, by saturation $p(\bar{x})$ is realized in G, and so we have $(a_i)_{i<\lambda}$ such that $a_i^{-1}a_j \notin X$ for all $i < j < \lambda$. In particular, $a_i^{-1}a_j \notin \Gamma$ for all $i < j < \lambda$, which contradicts the choice of λ.

The proof that X is right generic is similar. Or note that X^{-1} is a definable set containing Γ. So X^{-1} is left generic, i.e., X is right generic.

Part (c). (Given parts (a) and (b), the proof is very similar to the proof that $\mathrm{Stab}^{\mu}(A)$ has index at most $2^{\aleph_0}$.) Let $\Gamma = \bigcap_{i\in\lambda} X_i$, where each X_i is definable. After replacing each X_i with $X_i \cap X_i^{-1}$, we can assume that each X_i is symmetric. Without loss of generality, we may also assume λ is infinite and $\{X_i : i \in \lambda\}$ is closed under finite intersections.

By part (b), each X_i is left generic, and so we may fix a set $E \subseteq G$, with $|E| \leq \lambda$, such that $G = EX_i$ for all $i \in \lambda$. Given $a \in G$, set $I_a = \{(g, i) \in E \times \lambda : a \in gX_i\}$. We show that if $a, b \in G$ are such that $I_a = I_b$, then $a^{-1}b \in \Gamma$, and thus G has index at most 2^{λ}.

So suppose we have $a, b \in G$ with $I_a = I_b$. We show $a^{-1}b \in X_i$ for all $i \in I$. So fix $i \in I$. By part (a) there is some $j \in I$ such that $X_j^2 \subseteq X_i$. Since $G = EX_j$ there is some $g \in E$ such that $a \in gX_j$, i.e., $(g, j) \in I_a$. So $(g, j) \in I_b$, i.e., $b \in gX_j$. So $a^{-1}b \in X_j^2 \subseteq X_i$. □