Applications of Pseudofinite Model Theory

Lecture 10 (15 May 2020)

<u>Setting</u>: G is a sufficiently saturated pseudofinite expansion of a group.

<u>Main Task</u> Given definable $A \subseteq G$, NIP, express $G_A^{00} = \bigcap_{n=0}^{\infty} X_n$ where X_n is definable and has nice algebraic structure.

<u>Remark</u>

1) Suppose G/G_A^{00} is profinite. Then $G_A^{00} = \bigcap_{n=0}^{\infty} H_n$ where H_n is a def. finite-index, normal subgroup. (Exc 78).

<u>Thm</u> (Iltis) Any compact Hausdorff torsion group is profinite

<u>Exercise 26</u>: Suppose G is a finite group of exponent r and $A \subseteq G$ is k-NIP. Then $\forall\ \varepsilon > 0$, $\exists$ a normal subgroup $H \leq G$, of index $O_{k,r,\varepsilon}(1)$, and $Z \subseteq G$, $|Z| < \varepsilon |G|$, st $\forall\ g \notin Z$, $|gH \cap A| < \varepsilon |H|$ or $|gH \setminus A| < \varepsilon |H|$.

2) If G_A^{00} is definable. Then it has finite index (Exc. 6c). So G/G_A^{00} is finite.
Also $\mathbb{E}_A = \emptyset$. Then $\forall\ g \in G$, $\mu(gG_A^{00} \cap A) = 0$ or $\mu(gG_A^{00} \setminus A) = 0$.
This happens if A is "k-stable" (i.e. $\Gamma_G(A)$ omits $([k],[k];\leq)$; see Thm 4.4).

<u>Proposition 10.1</u> Suppose $\Gamma \leq G$ is ctbly-def., normal bounded index.
Then $\Gamma = \bigcap_{i=0}^{\infty} X_i$ where $\forall\ i \geq 0$, X_i is def., $X_{i+1} \subseteq X_i$, $\exists$ a def. finite-index normal subgroup $H_i \leq G$ + a def. surj. hom. $\tau_i : H_i \to \mathbb{T}^{n_i}$, for some $n_i \in \mathbb{N}$ st $\Gamma \leq \ker \tau_i \subseteq X_i \subseteq H_i$.

<u>Proof</u>: By Thm 3.2(a) [Peter-Weyl], $G/\Gamma = \varprojlim L_i$ where $(L_i)_{i=0}^{\infty}$ is an inverse system of compact Lie groups with surjective projection maps $f_i : G/\Gamma \to L_i$.
Let $\Gamma_i = \ker(f_i \circ \pi)$ where $\pi : G \to G/\Gamma$. So Γ_i is a ctbly def. normal subgroup of bounded index. Note $\Gamma \leq \Gamma_i$ and $\Gamma = \bigcap_{i=0}^{\infty} \Gamma_i$.

WLOG: $L_i = G/\Gamma_i$ and $f_i : G/\Gamma \to G/\Gamma_i$ is the canonical map.

Fix $i \geq 0$. Then G/Γ_i is a def. compactification of G (Example 2.7).

So by Thm 3.1, $(G/\Gamma_i)^\circ$ is compact connected abelian Lie group.

So $\exists\, n_i \in \mathbb{N}$ st $(G/\Gamma_i)^\circ \cong \mathbb{T}^{n_i}$ (see Notes B.3). Let H_i be the preimage of $(G/\Gamma_i)^\circ$ under $G \to G/\Gamma_i$. So $(G/\Gamma_i)^\circ = H_i/\Gamma_i$.

Now H_i/Γ_i is a clopen finite-index normal subgroup of G/Γ_i (B.3)

By Exc 7b, H_i is a def. finite-index normal subgroup of G.

Let $\tau_i : H_i \to H_i/\Gamma_i \cong \mathbb{T}^{n_i}$. So $\Gamma_i = \ker \tau_i$.

If $i \leq j$ then $\Gamma_j \leq \Gamma_i$ + $H_j \leq H_i$. Let $\Gamma_i = \bigcap_{m=0}^{\infty} X_{m,i}$. Then $\Gamma = \bigcap_{i=0}^{\infty} \bigcap_{m=0}^{\infty} X_{m,i}$. By taking finite intersections, we obtain a decreasing sequence.

<u>Def 10.2</u> Suppose $H \leq G$ is def. and $\tau : H \to \mathbb{T}^n$ is a def. comp. Then a <u>τ-approx. Bohr chain</u> is a decreasing sequence $(W_m)_{m=0}^{\infty}$ of definable subsets of H st $\ker \tau = \bigcap_{m=0}^{\infty} W_m$ + $\exists$ sequence $(f_m)_{m=0}^{\infty}$ st $\forall\, m$, $f_m : H \to \mathbb{T}^n$ is a def. $\frac{1}{m}$-approx. hom. with finite image + $W_m = \{x \in H : d(f_m(x), 0) < \frac{3}{m}\}$

<u>Proposition 10.3</u> Any def. $\tau : H \to \mathbb{T}^n$ as above admits a τ-approx. Bohr chain.

<u>Proof</u>: Fix m. By Lemma 3.6 + Remark 3.7, $\exists$ a def. $\frac{1}{m}$-approx. hom. $f_m : H \to \mathbb{T}^n$ st $f_m(H)$ is finite, $d(\tau(x), f_m(x)) < \frac{1}{3m}$ $\forall\, x \in H$, + if $\tau(x) \in B_{\leq \frac{1}{4m}}(0)$ then $f_m(x) = 0$. Set $W_m = \{x \in H : d(f_m(x), 0) < \frac{3}{m}\}$. So $\ker \tau \subseteq W_m$, and W_m is definable by Exc 8a. If $x \in W_m$ then $d(\tau(x), \overset{f_m(x)}{0}) \leq \frac{1}{3m} + \frac{3}{m} = \frac{10}{3m}$, and so $\bigcap_{m=0}^{\infty} W_m = \ker \tau$. We have $\tau^{-1}(B_{\leq \frac{1}{4m}}(0)) \subseteq W_m \subseteq \tau^{-1}(B_{< \frac{10}{3m}}(0))$

So $W_\ell \subseteq W_m$ if $\frac{10}{3\ell} \leq \frac{1}{4m}$, ie, $\ell \geq \frac{40}{3} m$. So WLOG $(W_m)_{m=0}^{\infty}$ is decreasing.