# Applications of Pseudofinite Model Theory

Lecture 11 (18 May 2020)

**Recall** (10.1) $G$ pseudofinite saturated. $\Gamma \leq G$ ctbly-type-def., normal, bounded index.

Then $\Gamma = \bigcap_{i=0}^{\infty} X_i$ with $\tau_i : H_i \to \mathbb{T}^{n_i}$ st $\Gamma \leq \ker \tau_i \subseteq X_i \leq H_i$.

**Lemma 11.1** Fix $k \geq 1$, $\varepsilon > 0$, and $\gamma : \mathbb{Z}^+ \times \mathbb{Z}^+ \times (0,1) \to (0,1)$. Then $\exists$ $n = n(k, \varepsilon, \gamma)$ st the following holds. Suppose $G$ is a finite group and $A \subseteq G$ is $k$-NIP. Then there are:

$$\left.\begin{array}{l} * \text{ a normal subgroup } H \leq G \text{ of index } \ell \leq n, \\ * \text{ a } (\delta, r)\text{-Bohr set } B \text{ in } H, \text{ with } \delta^{-1}, r \leq n, \text{ and} \\ * \text{ a set } Z \subseteq G, \text{ with } |Z| < \varepsilon |G|, \end{array}\right\} (*)$$

st $\forall g \in G \setminus Z$, either $|gB \cap A| < \gamma(\ell, r, \delta)|G|$ or $|gB \setminus A| < \gamma(\ell, r, \delta)|G|$.

**Proof** Suppose not. Then $\forall n \geq 1$ $\exists$ a finite group $G_n$ + $k$-NIP $A_n \subseteq G_n$ st $(*)$ fails. View $G_n$ as a finite structure in the group language with a unary relation symbol naming $A_n$. Let $\mathcal{U}$ be n.p.u.f. on $\mathbb{Z}^+$, and set $\mathcal{M} = \prod_{\mathcal{U}} G_n$. Let $G \succ \mathcal{M}$ be suff. saturated. Let $A \subseteq G$ be defined by the unary relation symbol (so $A(\mathcal{M}) = \prod_{\mathcal{U}} A_n$). Then $A$ is $k$-NIP by Łoś. Let $\alpha = \delta(\overset{\mathbb{T}}{S^1})$ be as in Thm 3.5. Let $G_A^{00} = \bigcap_{j=0}^{\infty} X_j$ be as in Prop 10.1. By Corollary 9.3, $\exists$ def. set $Z \subseteq G$, with $\mu(Z) < \varepsilon$, and $\exists$ $j$ st if $X = X_j$ then $\forall g \in G \setminus Z$, either $\mu(gX \cap A) = 0$ or $\mu(gX \setminus A) = 0$. We have a def. finite-index normal subgroup $H \leq G$, and a def. hom. $\tau : H \to \mathbb{T}^r$ st $\cancel{G_A^{00}} \leq \underbrace{\ker \tau}_{\Gamma} \subseteq X \leq H$. By Prop 10.3, $\exists$ a def. $\tau$-approx Bohr chain $(W_m)_{m=0}^{\infty}$. So $\bigcap_{m=0}^{\infty} W_m = \ker \tau \subseteq X$. Thus $W_m \subseteq X$ for $m$ suff. large (by Exc Sa). Choose $m > 0$ st $W_m \subseteq X$

and $\frac{1}{m} < \alpha$. Set $W = W_m$. Since $W \in X$, we still have

$\mu(gW \cap A) = 0$ or $\mu(gW \setminus A) = 0$ $\forall$ $g \notin Z$. Let $\ell = [G:H]$, and

$\delta = \frac{1}{m}$. There is a def. $\delta$-approx. hom. $f : H \to \mathbb{T}^r$ st $f(H)$ is finite

and $W = \{x \in H : d(f(x), 0) < 3\delta\}$. Set $\Lambda = f(H)$. If $\lambda \in \Lambda$

then $f^{-1}(\lambda)$ is a definable subset of $H$. Choose formulas (over $\emptyset$) $\varphi(x;\bar{y})$,

$\psi(x;\bar{z})$, $\theta(x;\bar{u})$, $\zeta_\lambda(x;\bar{v}_\lambda)$ for $\lambda \in \Lambda$ st $H$, $W$, $Z$, $f^{-1}(\lambda)$ are defined

by instances of $\varphi$, $\psi$, $\theta$, $\zeta_\lambda$, respectively. Let $I$ be the set of $n \geq 1$ st

$\exists$ tuples $\bar{a}_n$, $\bar{b}_n$, $\bar{c}_n$, $\bar{d}_{n,\lambda}$ from $G_n$ satisfying:

i) $\varphi(x, \bar{a}_n)$ defines a normal subgroup $H_n$ of $G_n$ of index $\ell$.

ii) $\theta(x, \bar{c}_n)$ defines $Z_n \subseteq G_n$ st $|Z_n| < \varepsilon |G_n|$,

iii) $\forall$ $\lambda$, $\zeta_\lambda(x, \bar{d}_{n,\lambda})$ defines $F_{n,\lambda} \subseteq H_n$ + $(F_{n,\lambda})_{\lambda \in \Lambda}$ forms a partition of $H_n$,

iv) if $f_n : H_n \to \Lambda$ is the function determined by the partition in (iii), then

$f_n$ is a $\delta$-approx. hom to $\mathbb{T}^r$ (see the proof of Thm 3.1)

v) $\psi(x, \bar{b}_n)$ defines $W_n = \{x \in H_n : d(f_n(x), 0) < 3\delta\}$, and

vi) $\forall$ $g \in G_n \setminus Z_n$, $|gW_n \cap A_n| < \gamma(\ell, r, \delta)|G_n|$ or $|gW_n \setminus A_n| < \gamma(\ell, r, \delta)|G_n|$.

Then $I \in \mathcal{U}$ by Łoś. Since $\mathcal{U}$ is nonprincipal, $\exists$ $n \in I$ st $n \geq \ell, r, \delta^{-1}$

By Thm 3.5 (and Exc. 9), $\exists$ a hom. $\tau_n : H_n \to \mathbb{T}^r$ st $d(\tau_n(x), f_n(x)) < 2\delta$

$\forall$ $x \in H_n$. Let $B_n = \{x \in H_n : d(\tau_n(x), 0) < \delta\}$. Then $B_n$ is a $(\delta, r)$-Bohr

set in $H_n$ and $B_n \subseteq W_n$ $[x \in B_n,\ d(f_n(x), 0) \leq d(f_n(x), \tau_n(x)) + d(\tau_n(x), 0) < 3\delta]$.

We have vi) with $B_n$. This contradicts the choice of $n$.