Applications of Pseudofinite Model Theory

Lecture 12 (20 May 2020)

Proposition 12.1 Suppose G is a finite group and B is a (δ,r)-Bohr set in a subgroup H of index ℓ. Then $\forall\, X \subseteq G$ $\exists\, F \subseteq X$ st $|F| \leq \ell(2/\delta)^r$ and $X \subseteq FB$.

Proof: Let $B = \{x \in H : d(\tau(x),0) < \delta\}$ for some $\tau : H \to \mathbb{T}^r$. Let $B_0 = \{x \in H : d(\tau(x),0) < \delta/2\}$. By Exc 12b, $|B_0| \geq (\delta/2)^r |H| = \ell^{-1}(\delta/2)^r |G|$. Fix $X \subseteq G$. Let $F \subseteq X$ be maximal st $\forall$ distinct $g,h \in F$, $gB_0 \cap hB_0 = \emptyset$. So $|F| \leq \ell(2/\delta)^r$. If $g \in X$ then $gB_0 \cap hB_0 \neq \emptyset$ for some $h \in F$ and so $h^{-1}g = B_0^2 \subseteq B$. So $X \subseteq FB$.

Theorem 12.2 $\forall\, k \geq 1$, $\varepsilon > 0$ $\exists\, n = n(k,\varepsilon)$ st the following holds.
Suppose G is a finite group and $A \subseteq G$ is k-NIP. Then there are:

* a normal subgroup $H \trianglelefteq G$ of index $\ell \leq n$,
* a (δ,r)-Bohr set B in H, with $\delta^{-1}, r \leq n$, and
* a set $Z \subseteq G$, $|Z| < \varepsilon|G|$

st i) $\forall\, g \in G \setminus Z$, either $|gB \cap A| < \varepsilon|B|$ or $|gB \setminus A| < \varepsilon|B|$.

ii) There is $D \subseteq G$, which is a union of at most $\ell(2/\delta)^r$ translates of B, st

$$|(A \triangle D) \setminus Z| < \varepsilon|B|. \qquad (|A \triangle D| < \varepsilon|B| + \varepsilon|G|).$$

Proof: Fix k,ε. Define $\gamma(x,y,z) = \varepsilon\left(x^{-1}\left(\overset{z/2}{\cancel{2/z}}\right)^y\right)\left(x^{-1}\left(\overset{z}{\cancel{1/z}}\right)^y\right)$. Let $n = n(k,\varepsilon,\gamma)$ be as in Lemma 11.1. Fix a finite group G and k-NIP $A \subseteq G$. By Lemma 11.1, $\exists\, H \trianglelefteq G$ index $\ell \leq n$, a (δ,r)-Bohr set B in H ($\delta^{-1}, r \leq n$), and $Z \subseteq G$ with $|Z| < \varepsilon|G|$ st $\forall\, g \in G \setminus Z$, $|gB \cap A| < \underbrace{\gamma(\ell,r,\delta)}_{\eta}|G|$ or $|gB \setminus A| < \gamma(\ell,r,\delta)|G|$.

By Exc. 12b, $|B| \geq \underbrace{\ell^{-1}\left(\overset{\delta}{\cancel{1/\delta}}\right)^r}_{\alpha}|G|$. So $\eta|G| = \varepsilon\underbrace{\left(\ell^{-1}\left(\overset{\delta/2}{\cancel{2/\delta}}\right)^r\right)}_{\beta}\alpha|G| \leq \varepsilon\alpha|G| \leq \varepsilon|B|$

So we have (i). By Prop 12.1, $\exists\, F \subseteq G\setminus Z$ st $|F| \leq \beta^{-1}$ + $G\setminus Z \subseteq FB$.

Let $I = \{g \in F : |gB \cap A| \geq \eta|G|\}$. Let $J = F \setminus I$. If $g \in I$ then $|gB \setminus A| < \eta|G|$. If $g \in J$ then $|gB \cap A| < \eta|G|$. So

$$\big|(IB\setminus A) \cup (JB \cap A)\big| < |I|\eta|G| + |J|\eta|G| = |F|\eta|G|$$
$$\leq \beta^{-1}\varepsilon\beta\alpha|G| = \varepsilon\alpha|G| \leq \varepsilon|B|.$$

Set $D = IB$. We show $A \triangle D \subseteq Z \cup (IB\setminus A) \cup (JB \cap A)$ (yields (ii))

$$\underbrace{A \triangle D}_{(A\setminus IB)\cup(IB\setminus A)}$$

ETS: $A \setminus IB \subseteq Z \cup (JB \cap A)$. Fix $x \in A \setminus IB$. Suppose $x \notin Z$.

Since $G\setminus Z \subseteq FB$ we have $x \in gB$ for some $g \in F$. Note $g \notin I$ since $x \notin IB$.

So $g \in J$. So $x \in JB \cap A$. $\square$

Corollary 12.3 $\forall\, \varepsilon > 0, k \geq 1\ \exists\, m = m(k,\varepsilon)$ st if G is a nonabelian finite simple group with $|G| > m$, and $A \subseteq G$ is k-NIP then $|A| < \varepsilon|G|$ or $|A| > (1-\varepsilon)|G|$.

Proof Let $m = n(k, \varepsilon/2)$ from Thm 12.2. Given G, A, we let H, B, Z, D be as in Thm 12.2. So $|A \triangle D| < |Z| + \varepsilon/2|B| \leq \varepsilon|G|$. H is normal of index $\leq m < |G|$. So $H = G$. B contains $\ker\tau$ for some $\tau : G \to \mathbb{T}^r$. So $\ker\tau = G$ since G is nonabelian. So $B = G$. So $D = \emptyset$ or $D = G$. $\square$.

Remark: One can show $m(k,\varepsilon) \leq \exp\left(25^{\log 25}\left(90/\varepsilon\right)^{6k-6}\right)$.