Applications of Pseudofinite Model Theory

Lecture 4 (1 May 2020)

G pseudofinite group. $\tau : G \to C$ def. comp., C compact Lie group

Goal: C has an abelian subgroup of finite index.

$\forall n \in I$, we built a homomorphism $\tau_n : G_n \to C$ st G_n is finite and $H_n = \tau_n(G_n)$ is a $1/n$-net in C.

By Thms 3.2(b) + 3.3, $\exists\, k = k(C)$ st H_n has an abelian subgroup K_n of index $\leq k$. Let $\mathcal{U}$ be a npuf on I and $\boxed{H = \prod_{\mathcal{U}} H_n.}$

Define $\sigma : H \to C$ st $\sigma([(x_n)_{n \in I}]) = \lim_{\mathcal{U}} x_n \in C$

Exercise 10 σ is a surjective homomorphism.

$\sigma(\prod_{\mathcal{U}} K_n)$ is an abelian subgroup of C of index $\leq k$.

K index $\leq k$ in H

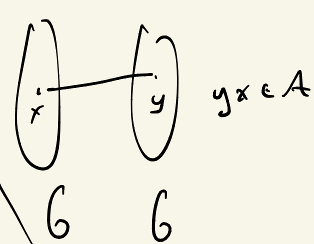
$\square$

NIP and VC-dimension

Notation $[n] = \{1, \ldots, n\}$

bipartite graphs $\Gamma = (V, W; E)$ st $E \subseteq V \times W$

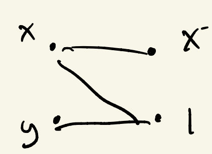
Def 4.1 Given a group G and $A \subseteq G$, define $\Gamma_G(A) = (G, G; E_A)$ where $E_A = \{(x,y) \in G \times G : yx \in A\}$.



Proposition 4.2 A nonempty subset A of a group G is a coset of a subgroup of G iff $\Gamma_G(A)$ omits $([2], [2]; \leq)$. (as an induced subgraph)

Proof: WLOG $1 \in A$.

If $x, y \in A$ st $x^{-1}y \notin A$ then



$\hookrightarrow \Gamma_G(A)$

<u>Thesis</u> Omitted subgraphs in $\Gamma_G(A) \rightsquigarrow$ structure for A.

<u>Def 4.3</u>: $A \subseteq G$ is <u>k-NIP</u> if $\Gamma_G(A)$ omits $([k], \mathcal{P}([k]); \in)$.

<u>Exercise 11</u> (special case) If $\Gamma_G(A)$ omits some finite $\Gamma = (V, W; E)$ then A is k-NIP for some $k \leq |V| + \lceil \log_2 |W| \rceil$.

<u>Note</u>: Cosets of subgroups are 2-NIP since $([2],[2]; \leq) \hookrightarrow ([2], \mathcal{P}([2]); \in)$.

<u>Theorem 4.4</u> (Terry-Wolf (abelian); C.-Terry-Pillay)

"k-stable"

Suppose G is a finite group and $A \subseteq G$ is st $\Gamma_G(A)$ omits $([k],[k]; \leq)$. Then $\forall\, \varepsilon > 0$ $\exists$ a normal subgroup $H \leq G$ of index $O_{k,\varepsilon}(1)$ st $\forall\, g \in G$, either $|gH \cap A| < \varepsilon |H|$ or $|gH \setminus A| < \varepsilon |H|$.

<u>Goal</u>: Something like this for k-NIP sets.

<u>Example 4.5</u> Let $G = \mathbb{Z}/p\mathbb{Z}$ (p is odd prime) and $A = \{0, 1, \ldots, \frac{p-1}{2}\}$. Then A is 3-NIP. (more later)

<u>Notation</u> $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ with metric d given by the product of the normalized arclength metric on $S^1 = \mathbb{R}/\mathbb{Z}$.

<u>Def 4.6</u> Let G be a group and let $\tau : G \to \mathbb{T}^n$ be a homomorphism. Given $\delta > 0$, set $B_\tau(\delta) = \{x \in G : d(\tau(x), 0) < \delta\}$

A <u>(δ, n)-Bohr set in G</u> is a set of the form $B_\tau(\delta)$ for some τ.

<u>Remark 4.7</u> $A = \{0, 1, \ldots, \frac{p-1}{2}\}$ is a translate of a $(\frac{1}{4}, 1)$-Bohr set in $\mathbb{Z}/p\mathbb{Z}$.

<u>Exercise 12</u> Properties of Bohr sets.

Def 4.8 Let X be a set and fix $\mathcal{S} \subseteq \mathcal{P}(X)$.

1) $\mathcal{S}$ shatters $A \subseteq X$ if $\mathcal{P}(A) = \{S \cap A : S \in \mathcal{S}\}$.

2) The VC-dimension of $\mathcal{S}$ is

$$VC(\mathcal{S}) = \sup\{n \in \mathbb{N} : \mathcal{S} \text{ shatters an } n\text{-element subset of } X\}.$$

Exercise 13 Basic properties of VC-dimension.

Exercise 14

a) Let $X = \mathbb{R}$ and $\mathcal{S} = \{(r,s) : r,s \in \mathbb{R}\}$ (interval). Then $VC(\mathcal{S}) = 2$.

b) Let $X = \mathbb{R}^2$ and $\mathcal{S} = \{\text{axis-parallel rectangles}\}$. Then $VC(\mathcal{S}) = 4$.

c) Let $X = \mathbb{R}^2$ and $\mathcal{S} = \{\text{convex sets}\}$. Then $VC(\mathcal{S}) = \infty$.

d) (special case) A subset A of a group G is k-NIP iff

$$VC(\{gA : g \in G\}) \leq k-1.$$

Exercise 15 Any (δ, n)-Bohr set in a group is $(2+o(1))n\log_2 n$-NIP.

* can improve to $\lfloor \frac{3n+3}{2} \rfloor$-NIP if $\delta \leq 1/4$.

Def 4.10: Given a set X and $\bar{a} \in X^n$, define the prob. measure $Av_{\bar{a}}$ on $\mathcal{P}(X)$ st $Av_{\bar{a}}(S) = \frac{1}{n}|\{1 \leq i \leq n : a_i \in S\}|$

Recall: Weak Law of Large Numbers

If X is a finite set and $S \subseteq X$ then $\forall\, \varepsilon > 0$,

$$\left|\left\{\bar{a} \in X^n : \left|Av_{\bar{a}}(S) - |S|/|X|\right| \geq \varepsilon\right\}\right| \leq \frac{1}{4\varepsilon^2 n}|X|^n$$