Applications of Pseudofinite Model Theory

Lecture 6 (6 May 2020)

Proof of Thm 5.9 $\mathcal{M}$ pseudofinite, $\varphi(x;\bar{y})$ NIP. Fix $\varepsilon \geq 0$.

Set $D_\varepsilon = \{\bar{b} \in \mathcal{M}^{\bar{y}} : \mu(\varphi(x;\bar{b})) \leq \varepsilon\}$. WTS D_ε is an intersection of ctbly many φ^*-definable sets. Given $m \geq 1$, apply Prop 5.4 to obtain $\bar{a}^m \in \mathcal{M}^{n_m}$ st $\bar{b} \in \mathcal{M}^{\bar{y}}$

$$\left| Av_{\bar{a}^m}(\varphi(x;\bar{b})) - \mu(\varphi(x;\bar{b})) \right| < \frac{1}{m}.$$

Set $X_m = \{\bar{b} \in \mathcal{M}^{\bar{y}} : Av_{\bar{a}^m}(\varphi(x;\bar{b})) < \varepsilon + \frac{1}{m}\}$. So $D_\varepsilon = \bigcap_{m=1}^{\infty} X_m$.

Fix $m \geq 1$. For $s \in [n_m]$ set $\theta_s(\bar{y}) := \bigwedge_{i \in s} \varphi(a_i^m, \bar{y}) \wedge \bigwedge_{i \notin s} \neg\varphi(a_i^m, \bar{y})$.

Let $\Sigma = \{ s : |s|/n_m < \varepsilon + \frac{1}{m}\}$. Then X_m is defined by the φ^*-formula

$$\bigvee_{s \in \Sigma} \theta_s(\bar{y}). \qquad \square$$

NIP Sets + Stabilizers

Setting: G is a sufficiently saturated pseudofinite group.

$A \subseteq G$ is definable + NIP (ie., $x \in yA$ is NIP)

$\mathcal{B}$ is the Boolean algebra generated by $\{gAh : g, h \in G\}$.

Note: $\mathcal{B} = \{\varphi\text{-definable sets}\}$ where $\varphi(x; y, z) := x \in yAz$

This formula may not be NIP! (Exercise 19)

But any set in $\mathcal{B}$ is NIP (Exercise 20)

(i.e., if $\theta(x)$ is a φ-formula then $\psi(x;y) := \theta(y \cdot x)$ is NIP).

Def 6.1: $X \subseteq G$ is $\mathcal{B}$-type-definable if $X = \bigcap_{i \in I} X_i$ where I is small and each X_i is in $\mathcal{B}$. If I is countable then X is countably $\mathcal{B}$-type-definable.

Def 6.2: Given $\varepsilon \geq 0$, set $\mathrm{Stab}_\varepsilon^\mu(A) = \{g \in G : \mu(gA \triangle A) \leq \varepsilon\}$

Prop 6.3 a) $\mathrm{Stab}^M_\varepsilon(A)$ is symmetric and contains 1.

b) $\mathrm{Stab}^M(A) := \mathrm{Stab}^M_0(A)$ is a subgroup of G.

Theorem 6.4 $\mathrm{Stab}^M(A)$ is a ctbly $\mathcal{B}$-type-def. subgroup of G of index $\leq 2^{\aleph_0}$.

Proof Note: $\mathrm{Stab}^M(A) = \bigcap_{\varepsilon > 0} \mathrm{Stab}^M_\varepsilon(A)$. We fix $\varepsilon > 0$ and show that $\mathrm{Stab}^M_\varepsilon(A) =: S$ is countably $\mathcal{B}$-type-def, and is left generic.

Let $\varphi(x;y)$ be $x \in y^{-1}A \triangle A$. Note $\varphi(x;y)$ is NIP. So $S = \{g \in G : \mu(\varphi(x;g)) \leq \varepsilon\}$

By Thm 5.9 S is ctbly φ^*-definable. Given $a \in G$, $\varphi(a;y)$ defines Aa^{-1} if $a \notin A$ and $G \setminus Aa^{-1}$ if $a \in A$. So S is ctbly $\mathcal{B}$-type-definable.

Let $\psi(x;y,z)$ be $x \in yA \triangle zA$. Then ψ is NIP.

By Cor 5.5, $\exists$ finite $F \subseteq G$ st $\forall\, g,h \in G$ if $\mu(gA \triangle hA) > \varepsilon$ then $(gA \triangle hA) \cap F \neq \emptyset$. Define an eq. rel. $\sim$ on G by $g \sim h$ iff $gA \cap F = hA \cap F$.

Choose rep's $g_1, \dots, g_n \in G$ for all $\sim$-classes. If $a \in G$ then $\exists\, i$ st $a \sim g_i$, ie., $F \cap (aA \triangle g_iA) = \emptyset$. So $\mu(aA \triangle g_iA) \leq \varepsilon$, ie. $g_i^{-1}a \in S$

So $G = g_1 S \cup \dots \cup g_n S$. We've shown $\mathrm{Stab}^M(A)$ is ctbly $\mathcal{B}$-type-def.

Now write $\mathrm{Stab}^M(A) = \bigcap_{n=1}^{\infty} X_n$ where $X_n = \mathrm{Stab}^M_{1/n}(A)$. Fix a ctble set $E \subseteq G$ st $G = EX_n$ $\forall\, n \geq 1$. For $a \in G$, set $I_a = \{(g,n) \in E \times \mathbb{Z}^+ : a \in gX_n\}$.

Given $a,b \in G$, if $I_a = I_b$ then $\forall\, n$, $a^{-1}b \in X_n^{-1}X_n \subseteq X_{\lceil n/2 \rceil}$, so $a^{-1}b \in \mathrm{Stab}^M(A)$.

So $\mathrm{Stab}^M(A)$ has index $\leq 2^{\aleph_0}$. $\square$

Def 6.5: $G^{00}_A = \bigcap_{g \in G} g\, \mathrm{Stab}^M(A)\, g^{-1}$

Corollary 6.6 G_A^{00} is a ctbly B-type-def. normal subgroup of index $\leq 2^{\aleph_0}$.

Proof: $G_A^{00} = \bigcap_{g \in I} g\,\mathrm{Stab}^{M}(A)\,g^{-1}$ where I is a set of left coset reps for $\mathrm{Stab}^{M}(A)$ so $|I| \leq 2^{\aleph_0}$.

So G_A^{00} is B-type-definable of index $\leq 2^{2^{\aleph_0}}$.

Note: There is a countable set $C \subseteq G$ st G_A^{00} is invariant under automorphisms of G fixing C ptwise.

Using model theoretic tricks (Exc. 5d, 6c), we can show that G_A^{00} is ctbly B-type-def. of index $\leq 2^{\aleph_0}$.