552 HOMEWORK 1

This assignment is due September 2, via upload to Gradescope. Unless otherwise specified, we work over an algebraically closed field k.

Exercise 1. The goal of this exercise is to prove the Noether normalization lemma: if R is a finitely generated k-algebra, then there is some nonnegative integer d and elements $y_1, \ldots, y_d \in R$ such that $S = k[y_1, \ldots, y_d]$ is a subring of R and R is a finite S-module. We asserted it in class, but skipped the proof.

The typical argument is by induction on the number n of generators of R over k.

(1) Prove the base case $n = 0$.
(2) Assume we've proven the result through $n - 1$ generators, and let R have n generators $x_1, \ldots, x_n$. If x_n satisfies no relations in R over $k[x_1, \ldots, x_{n-1}]$, prove the result for R.
(3) If x_n satisfies a relation $f(x_1, \ldots, x_n) = 0$ in R for some polynomial f, show that there is some positive integer r such that x_n is integral over $k[z_1, \ldots, z_{n-1}]$, where $z_i = x_i - x_n^{r^i}$. (Hint: what does f look like as a polynomial in the z_i and x_n?) Conclude the theorem for R.

Exercise 2. Show that the following sets are closed in the Zariski topology, and give generators for their ideals:

(1) The union of the coordinate axes in $\mathbb{A}^3$.
(2) The set of points $C = \{(t, t^2, t^3) | t \in k\} \subset \mathbb{A}^3$.
(3) The three points $\{(0,0), (1,0), (0,1)\} \subset \mathbb{A}^2$

Exercise 3. Let C be the *cuspidal affine curve* $V(x^3 - y^2)$ in $\mathbb{A}^2$.

(1) Give a regular map $f : \mathbb{A}^1 \to C$.
(2) Show that this map is not an isomorphism.
(3) Show that there is a rational map $g : C \to \mathbb{A}^1$ such that $fg = \mathrm{id}$ whenever g is regular. Where does this map fail to be regular?

Exercise 4 (Shafarevich 1.3.1)**.** Let k have characteristic distinct from 2. Let $X = V(x^2 + y^2 + z^2, x^2 - y^2 - z^2 - 1)$. Write X as a union of irreducible affine varieties.

Exercise 5. Given sets $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$, define $X \times Y \subset A^{n+m}$ as having all points $(x_1, \ldots, x_{n+m})$ with $(x_1, \ldots, x_n) \in \mathbb{A}^n$ and $(x_{n+1}, \ldots, x_{n+m}) \in \mathbb{A}^m$. Prove that if X and Y are affine varieties, $X \times Y \subset \mathbb{A}^{n+m}$ is also an affine variety.