552 HOMEWORK 3

This assignment is due September 19, via upload to Gradescope. Unless otherwise specified, we work over an algebraically closed field k.

Exercise 1. Show that if $n \geq 2$, then the two varieties $\mathbb{A}^n \setminus \{(0,0,\ldots)\}$ and $\mathbb{P}^n \setminus \{(1,0,0,\ldots)\}$ are not isomorphic to a projective variety or an affine variety.

Exercise 2.
(1) Show that the dimension of the space $V_{n,d}$ of homogeneous polynomials of degree d in $n+1$ variables is $N := \binom{n+d}{d}$.
(2) The space of degree d hypersurfaces in $\mathbb{P}^n$ is given by $\mathbb{P}V_{n,d} \cong \mathbb{P}^{N-1}$. For $p \in \mathbb{P}^n$, show that the subvariety $Z_p \subset \mathbb{P}V_{n,d}$ consisting of all hypersurfaces X such that $p \in X$ is a hyperplane in $\mathbb{P}V_{n,d}$.
(3) A collection of k points Γ is said to impose *independent conditions on hypersurfaces of degree* d if the dimension of the set of degree d hypersurfaces that contain Γ has dimension $N-k-1$. Show that a set of points $\Gamma \subset \mathbb{P}^n$ imposes independent conditions on hyperplanes if and only if Γ is in general position.[1]

Exercise 3. Let $\Gamma \subset \mathbb{P}^2$ be a set of 5 points in linearly general position. Show that there is a unique conic passing through the points. (Hint: one possible approach uses the fact that within the space of conics $\mathbb{P}^5$, there is a 4 dimensional projective variety of reducible conics, i.e. pairs of lines).

Exercise 4 (Shafaraevich 1.6.1)**.** Let $\Lambda \subset \mathbb{P}^n$ be a hyperplane, $X \subset \Lambda$ an irreducible projective variety, and $p \in \mathbb{P}^n \setminus \Lambda$ a point. Show that the union of all lines through p and a point $x \in X$ is a variety, and that this variety has dimension $\dim(X)+1$.

Exercise 5. Let $X \subset \mathbb{P}^n$ be a projective variety, and define $\tilde{X} \subset \mathbb{A}^{n+1}$ by

$$\tilde{X} = \{(0,0,\ldots,0)\} \cup \{(x_1,\ldots,x_{n+1}) \in \mathbb{A}^{n+1} | (x_1,\ldots,x_{n+1}) \in X \subset \mathbb{P}^n.$$

(1) Show that $\tilde{X} \subset \mathbb{A}^{n+1}$ is closed. (It's called the *affine cone over* X).
(2) If $X, Y \subset \mathbb{P}^N$ are irreducible projective varieties of dimension n, m. Applying the theorem we proved in class to $\tilde{X}$ and $\tilde{Y}$ (Shafarevich 1.24, first paragraph), show that $X \cap Y$ is nonempty if $n+m-N \geq 0$.

[1] In the sense of Harris, Lecture 1 or class. This condition is referred to as *linearly general position* most of the time.