552 HOMEWORK 2

This assignment is due September 12, via upload to Gradescope. Unless otherwise specified, we work over an algebraically closed field k.

Exercise 1. Verify that the Segre variety $\mathrm{Seg}(n,m) \subseteq \mathbb{P}^{(n+1)(m+1)-1}$, together with its projections to $\mathbb{P}^n$ and $\mathbb{P}^m$, is the categorical product of $\mathbb{P}^n$ and $\mathbb{P}^m$ in the category of varieties Var_k.

Exercise 2. Let $S = k[x_0, \ldots, x_n]$ and let I be a homogeneous ideal in S.

(1) Let $I_{irr} = (x_0, \ldots, x_n)$ denote the *irrelevant ideal* in S. If I contains I_{irr}^m for some m, prove that $V(I) \subset \mathbb{P}^n$ is empty.

(2) Let $\mathbb{A}_i^n$ be the complement of $V(x_i)$ in $\mathbb{P}^n$. The n coordinate functions on $\mathbb{A}_i^n$ are then $\frac{x_j}{x_i}$ for $i \neq j$. If $V(I) \cap \mathbb{A}_i^n = \emptyset$, use the Nullstellensatz to show $x_i^m \in I$ for some m.

(3) Use the previous part to show the converse to (1); if $V(I) = \emptyset$, then I contains the irrelevant ideal. (This question implies that one can multiply in or factor out irrelevant ideals from an ideal I without changing $V(I)$, hence the name).

Exercise 3 (Harris 1.3)**.** Let $\Gamma \subset \mathbb{P}^n$ consist of d points. Show that Γ is the zero set of a collection of polynomials of degree at most d. If not every point of Γ lies on a single line, show that Γ is the zero set of a collection of polynomials of degree at most $d-1$.

Exercise 4 (Harris 1.12)**.** Show that any 4 distinct points on a twisted cubic $C \subset \mathbb{P}^3$ are in *general position*, that is, they do not lie on any hyperplane. (Hint: how many points are in the intersection of C with a hyperplane H?)

Exercise 5. Let $X \subset \mathbb{P}^n$ be a hypersurface. Show that its complement $\mathbb{P}^n \setminus X$ is isomorphic to an affine variety.