552 HOMEWORK 4

This assignment is due September 26, via upload to Gradescope. Unless otherwise specified, we work over an algebraically closed field k.

The *general point* of an irreducible variety X is said to have some property P if there is an open nonempty subset U of X such that every point in U satisfies P.

Exercise 1. Show that the locus of colinear triples of points in $(\mathbb{P}^2)^3$ is closed of dimension 5, and hence that the general triple of points in $(\mathbb{P}^2)^3$ is not colinear. In this problem, if you use an incidence correspondence, you *do not* need to explicitly verify that it is closed in whatever product you use.

Exercise 2. Let S be a general quadric surface in $\mathbb{P}^3$. Show that S contains a one-dimensional family of lines. This has two steps:

(1) Set up an incidence correspondence describing lines on all quadric surfaces in $\mathbb{P}^3$. Show that it is an irreducible variety and compute its dimension.
(2) Find a quadric surface in $\mathbb{P}^3$ with a one-dimensional family of lines. Use the theorem on fiber dimension to prove the result.

Exercise 3. (1) Let C be a rational normal curve in $\mathbb{P}^n$, i.e., the image of $\mathbb{P}^1$ by the inclusion $i(s,t) = (s^n, s^{n-1}t, \dots, t^n)$. Show that if we regard $\mathbb{P}^n$ as the projective space of all matrices of the form

$$\begin{bmatrix} x_0 & x_1 & \dots & x_{n-1} \\ x_1 & x_2 & \dots & x_n, \end{bmatrix}$$

then C is the locus of such matrices having rank at most 1.
(2) In the case $n = 3$, use the previous part to determine a set of polynomials that cut out the twisted cubic.

Exercise 4. Given two regular maps $f : X \to Z$, $g : Y \to Z$, show that the set $X \times_Z Y \subseteq X \times Y$ given by

$$X \times_Z Y := \{(x,y) | f(x) = g(y)\}$$

is a closed subvariety of $X \times Y$. This variety is called the *fiber product* of X and Y over Z.

Exercise 5. Show the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ is deficient (i.e., has secant variety of smaller than expected dimension). Bonus: can you generalize this fact to other Segre varieties $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{(n+1)(m+1)-1}$?