Asymptotic Syzygies of Secant Varieties of Curves

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Setup

- Let $C \subseteq \mathbb{P}H^0(C, L_d)$ be a smooth curve of genus g over $\mathbb{C}$ embedded by a complete linear system of degree d.
- **Problem:** Describe the minimal free resolution of the homogeneous coordinate ring of the secant variety of k-planes to C over $S = \operatorname{Sym} H^0(C, L_d)$ as $d \to \infty$.

Background

- Given a minimal graded free resolution

$$0 \leftarrow \oplus_q S(-q)^{\oplus \kappa_{0,q}} \leftarrow \cdots \leftarrow \oplus_q S(-p-q)^{\oplus \kappa_{p,q}} \leftarrow \cdots$$

we record the numerical data in the *Betti table*

$q \backslash p$	0	1	2	$\cdots$
0	$\kappa_{0,0}$	$\kappa_{1,0}$	$\kappa_{2,0}$	$\cdots$
1	$\kappa_{1,0}$	$\kappa_{1,1}$	$\kappa_{2,1}$	$\cdots$
2	$\kappa_{2,0}$	$\kappa_{2,1}$	$\kappa_{2,2}$	$\cdots$
$\vdots$		$\vdots$		$\ddots$

- By [1], the Betti table of the secant variety of k-planes to C has the form

$q \backslash p$	0	1	$\cdots$	$c_d - g$	$c_d - g + 1$	$\cdots$	c_d
0	1	$-$	$\cdots$	$-$	$-$	$\cdots$	$-$
$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$k+1$	$-$	$*$	$\cdots$	$*$	$*$	$\cdots$	$*$
$k+2$	$-$	$-$	$\cdots$	$-$	$*$	$\cdots$	$*$
$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$2k+2$	$-$	$-$	$\cdots$	$-$	$*$	$\cdots$	$*$

where c_d is the codimension of the secant variety in $\mathbb{P}H^0(C, L_d)$.

- *Boij-Söderberg theory* [2] provides a decomposition of the Betti table as a rational linear combination of *pure diagrams* (diagrams with one nonzero entry in each column).

Slogan

The Betti table of the secant variety of k-planes to C is dominated by a single pure diagram as $d \to \infty$.

The Betti Table of the Secant Variety of k-Planes to C is Asymptotically Pure

Theorem A. Let a_d be the coefficient of the pure diagram

$q \backslash p$	0	1	$\cdots$	$c_d - g$	$c_d - g + 1$	$\cdots$	c_d
0	$*$	$-$	$\cdots$	$-$	$-$	$\cdots$	$-$
$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$k+1$	$-$	$*$	$\cdots$	$*$	$-$	$\cdots$	$-$
$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$2k+2$	$-$	$-$	$\cdots$	$-$	$*$	$\cdots$	$*$

in the Boij-Söderberg decomposition of the (normalized) Betti table of the secant variety of k-planes to C. Then $\lim_{d\to\infty} a_d = 1$.

- Notice that the pure diagram that dominates the Betti table depends only on g and k.
- When $k = 0$, this result is due to Erman [3], who coined the term asymptotic purity.
- The proof of this theorem follows Erman's strategy in [3]. We employ a semicontinuity argument with the *Hilbert numerator*. The crucial piece is the fact that $\kappa_{r-2k-1,2k+2}(\Sigma_k(C, L_d)) = \binom{g+k}{k+1}$ [1].
- There can be more than one supporting diagram for each d. When $k = 0$, the number of such diagrams depends on the *gonality* of C (i.e. the smallest degree of a map $C \to \mathbb{P}^1$) [4].
- Directly from Theorem A, we see that for $d \gg 0$, every entry in the bottom row which is permitted to be non-zero by result of Ein-Niu-Park actually becomes non-zero.
- We use Theorem A below to better understand the numbers appearing in the main row of the Betti table (i.e. when $p = k + 1$).

The Betti Numbers of the Secant Variety of k-Planes to C are Asymptotically Normally Distributed

Theorem B. Let $r_d = \dim_{\mathbb{C}} H^0(C, L_d)$. Fix a sequence $\{p_d\}$ of integers such that $p_d \to \frac{r_d}{2} + \frac{a\sqrt{r_d}}{2}$ for some real number a. Recall that $\kappa_{p,q}$ is the number in the qth row and pth column in the Betti diagram. Then

$$F_{g,k}(d)\kappa_{p_d,k+1} \to e^{-a^2}$$

where

$$F_{g,k}(d) = \frac{(k+1)!}{2^{r_d-2k} r_d^k} \cdot \sqrt{\frac{2\pi}{r_d}}$$

- When $k = 0$, this result is due to Ein, Erman, and Lazarsfeld [5]. They used an explicit formula for the Betti numbers in the linear strand of C.
- There is no analogous closed formula for the Betti numbers of the secant variety. Instead, we use Theorem A and the explicit formula for the Betti numbers of the pure diagrams to show that the pure diagram from Theorem A is the only contributing term as $d \to \infty$.

Further Questions

- Does the number of supporting diagrams stabilize as $d \to \infty$?
- Is there a formula for the coefficients of the Boij-Söderberg decomposition for the general curve?
- Can one describe the syzygies of the secant varieties of higher dimensional varieties?

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