The Higher Harmonic Signature for Foliations

Joint work with Moulay-Tahar Benameur

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Main Theorems

Theorem

Suppose that M is a compact Riemannian manifold with oriented Riemannian foliation F of dimension 4ℓ. Then the leafwise signature $\sigma(F)$ of F_s is a leafwise homotopy invariant, and

$$\sigma(F) = \int_F \mathrm{L}(TF).$$

Can twist by a leafwise flat $\mathbb{C}$ bundle $E = E^+ \oplus E^- \to M$ with an (indefinite) non-degenerate Hermitian metric, preserved by the leafwise flat structure, i.e. a leafwise $U(p,q)$ flat bundle.

Theorem

M, F, $E \to M$, as above. $\dim F = 2\ell$.
Assume that $\rho^E : \Omega^*_{(2)}(F_s \otimes r^*(E)) \to \mathrm{Ker}(\Delta^E_\ell)$, is transversely smooth.
Then $\sigma(F, E)$, the leafwise (for F_s) signature with coefficients in $r^*(E)$, is a leafwise homotopy invariant.

ρ^E is transversely smooth

- if $E = M \times \mathbb{C}$, i.e., the untwisted case;
- if leafwise parallel translation on E defined by the flat structure is a bounded map;
- if the preserved metric on E is positive definite;
- if E is a bundle associated to the normal bundle of F;
- for important examples, e.g., the examples of Lusztig which proved the Novikov conjecture for $\mathbb{Z}^n$.

Conjecture (B-H)

$$\sigma(F, E) = \int_F \mathrm{L}(TF)\Big(\mathrm{ch}(E^+) - \mathrm{ch}(E^-)\Big).$$

H-Lazarov (improved by Benameur-H) proved this for foliations with nice spectra. Azzali, Goette & Schick prove it for globally flat E.

Applications of B-H Thm, assuming Conj if necessary.

Conjecture (Novikov)

Suppose $f : N \to B\pi_1 N$ *classifies the* $\pi_1 N$ *bundle* $\widetilde{N} \to N$*, and* $x \in \mathrm{H}^*(B\pi_1 N; \mathbb{Q})$*. Then* $\int_N \mathrm{L}(TN) f^*(x)$ *is a homotopy invariant.*

B-H implies this immediately for $\mathbb{Z}^n$ and for all surface groups.
Originally proved by Lusztig.
B-H implies this for $\mathrm{ch}(E^+) - \mathrm{ch}(E^-)$, where $E^+ \oplus E^-$ is a $U(p,q)$ flat bundle over $B\pi_1 N$.

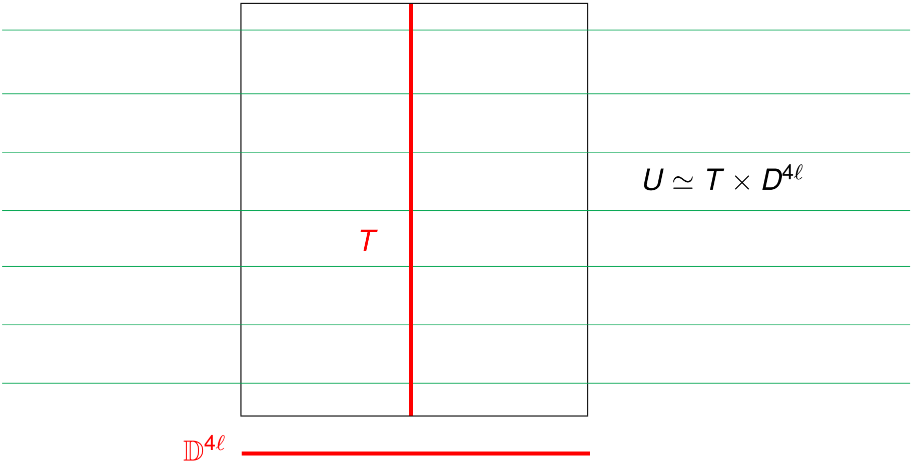
Conjecture (Baum-Connes Novikov conjecture)

$f : M \to B\mathcal{G}$ *a classifying map for* F*. Then, for any* $x \in H^*(B\mathcal{G}; \mathbb{Q})$*,* $\int_F \mathrm{L}(TF) f^* x$ *is a leafwise homotopy invariant.*

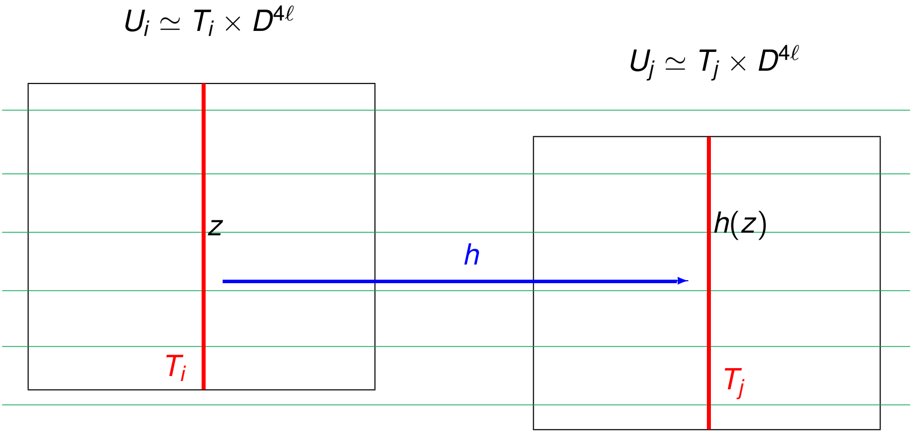
B-H implies this for the Chern characters of leafwise $U(p,q)$ flat bundles over $B\mathcal{G}$.

A Foliation Chart on M for F

F is a partition of M into disjoint dim 4ℓ submanifolds.
Locally F is a product $T \times \mathbb{D}^{4\ell}$. F Riemannian if there is a metric on M so that the distance between leaves is constant.



Holonomy and Local Integration



- $h^* : \Omega_c^k(T_j \cap h(T_i)) \to \Omega_c^k(T_i)$.
- If $\omega_i \in \Omega_c^{4\ell+k}(U_i)$, get $\int \omega_i \in \Omega_c^k(T_i)$.

Haefliger Cohomology

- Write $\quad M = \bigcup U_i \qquad U_i$ foliation charts for F.
- Choose transversals $T_i \subset U_i$ so that $T = \bigcup T_i$ is disjoint union.
- In C^∞ k forms with compact support $= \Omega_c^k(T)$, consider $L^k = \overline{\text{span}\{\alpha - h^*\alpha\}}, \quad h \in$ holonomy pseudogroup.
- Set $\qquad \Omega_c^k(M/F) = \Omega_c^k(T)/L^k$.
- $d : \Omega_c^k(T) \to \Omega_c^{k+1}(T)$ induces $d_H : \Omega_c^k(M/F) \to \Omega_c^{k+1}(M/F)$.
- $\mathrm{H}_c^*(M/F) =$ cohomology of this complex.
- If F given by a fibration $M \to B$, then $H_c^*(M/F) = H^*(B;\mathbb{R})$.
- Independent of all choices.

Integration over the fiber of F

- $\displaystyle\int_F : \Omega^{4\ell+k}(M) \to \Omega_c^k(M/F).$

- Given $\omega \in \Omega^{4\ell+k}(M)$, write $\quad \omega = \displaystyle\sum_i \omega_i, \quad$ where $\omega_i \in \Omega_c^{4\ell+k}(U_i)$.

- Integrate ω_i along the fibers of $U_i \to T_i$. Get $\displaystyle\int \omega_i \in \Omega_c^k(T_i)$.

- $\displaystyle\int_F \omega \equiv$ class of $\displaystyle\sum_i \int \omega_i$. $\qquad \displaystyle\int_F \omega \in \Omega_c^k(M/F)$ well defined.

- $d_H \circ \displaystyle\int_F = \int_F \circ d \qquad$ so get $\qquad \displaystyle\int_F : H^{4\ell+k}(M) \to H_c^k(M/F).$

Homotopy groupoid $\mathcal{G}$ of F

- $\mathcal{G} =$ equivalence classes of leafwise paths in M.
- Paths equivalent if homotopic in their leaf rel their end points.
- $s : \mathcal{G} \to M$: $\quad s\big([\gamma]\big) = \gamma(0)$, $\quad$ is a fiber bundle.
- F_s foliation of $\mathcal{G}$, leaves are $\widetilde{L}_x = s^{-1}(x)$, the fibers.
- $r\big([\gamma]\big) = \gamma(1)$. $\quad r : \widetilde{L}_x \to L_x$ is the simply connected cover of L_x.
- $x \in M$ gives $\overline{x} \in \mathcal{G}$, $\overline{x} =$ class of constant path at x.
- So, $x \to \overline{x}$ $\quad$ gives $\quad M \simeq \mathcal{G}_0 \subset \mathcal{G}$, and $M \subset \mathcal{G}$.

Connections on Transversely Smooth Idempotents

- M a compact Riemannian manifold with oriented Riemannian foliation F of dimension 4ℓ.

- $\Omega^*_{(2)}(F_s) \to M$ is the bundle of L^2 differential forms on leaves of F_s. For $x \in M$, $\quad (\Omega^*_{(2)}(F_s))_x = L^2(\widetilde{L}_x, \wedge T^*\widetilde{L}_x)$.

- Any connection on $\wedge T^*F$ induces one on $\wedge T^*F_s$,
$$\nabla^{F_s} : C^\infty(T^*F_s \otimes \wedge T^*\mathcal{G}) \to C^\infty(T^*F_s \otimes \wedge T^*\mathcal{G}).$$

- $\nu_s^* \equiv s^*(T^*M) \subset T^*\mathcal{G}$ is dual normal bundle of F_s.

- $p_\nu : \wedge T^*\mathcal{G} \to \wedge\nu_s^*$, the projection.

- An idempotent $\rho : \Omega^*_{(2)}(F_s) \to \Omega^*_{(2)}(F_s)$ assigns to each $x \in M$, an idempotent $\rho_x : L^2(\widetilde{L}_x, \wedge T^*\widetilde{L}_x) \to L^2(\widetilde{L}_x, \wedge T^*\widetilde{L}_x)$.

Definition

A connection ∇ on ρ is a $\mathcal{G}$ invariant operator on $\Omega^*_{(2)}(F_s) \otimes \Omega^*(M)$, which can be written as

$$\nabla \;=\; \rho\Big(\rho_\nu \nabla^{F_s} \;+\; A\Big)\rho.$$

A a transversely smooth $\mathcal{G}$ invariant leafwise operator on $\Omega^*_{(2)}(F_s)$.

- ∇ is $\mathcal{G}$ invariant means $\nabla \,|\, \widetilde{L}_{x_1} = \nabla \,|\, \widetilde{L}_{x_2}$, where $x_1, x_2 \in L$. Same for A.
- A is transversely smooth if all the transverse derivatives of its Schwartz kernel define leafwise operators on $\Omega^*_{(2)}(F_s)$ which are smoothing and are globally bounded.

Recall: Schwartz kernel of A is $K_x^A(y,z) \in \mathrm{Hom}((\wedge T^*\widetilde{L}_x)_z, (\wedge T^*\widetilde{L}_x)_y)$.
$\xi \in L^2(\widetilde{L}_x, \wedge T^*\widetilde{L}_x)$ and $y \in \widetilde{L}_x$, then $A(\xi)(y) \;=\; \int_{\widetilde{L}_x} K_x^A(y,z)\xi(z)\,dz$.

All powers ∇^{2k} of the curvature ∇^2 are transversely smooth $\mathcal{G}$ invariant leafwise operators. $K_x^{\nabla^{2k}}(y,z)$ the Schwartz kernel of ∇^{2k}.

For each $x \in M$, denote by $\overline{x}$ the class of the constant path at x.

Definition

Set $$\mathrm{Tr}(\nabla^{2k}) = \int_F \mathrm{tr}(K_x^{\nabla^{2k}}(\overline{x},\overline{x}))dx \quad \in \quad \Omega_c^k(M/F).$$

Proposition

$\mathrm{Tr}(\exp(-\nabla^2/2i\pi))$ *is a closed Haefliger form. Its class is indep. of* ∇.

Definition

$$\mathrm{ch}_a(\rho) \;=\; [\mathrm{Tr}(\exp(-\nabla^2/2i\pi))].$$

The Higher Harmonic Signature for Foliations

- $\Omega^*_{(2)}(F_s) \to M$ is the bundle of L^2 differential forms on leaves of F_s. For $x \in M$, $\quad (\Omega^*_{(2)}(F_s))_x = L^2(\widetilde{L}_x, \wedge T^*\widetilde{L}_x)$.

- τ the usual leafwise involution (a $\mathbb{C}$ multiple of deRham $*$) gives

$$\Omega^*_{(2)}(F_s) = \Omega^*_+(F_s) \oplus \Omega^*_-(F_s).$$

- The leafwise Laplacian Δ preserves this splitting.

- The leafwise operator $D = d + d^*$ reverses splitting, $D^2 = \Delta$.

- D defines $D^+ : \Omega^*_+(F_s) \to \Omega^*_-(F_s)$, the leafwise signature operator.

- $\mathrm{Ind}_c^\infty(D^+) \in K_0(C_c^\infty(\mathcal{G}; \mathrm{Hom}(\wedge T^* F_s)))$.

Theorem

The projections $\rho_\pm : \Omega^*_{(2)}(F_s) \to \mathrm{Ker}(\Delta^\pm_{2\ell}) = \mathrm{Ker}(\Delta) \cap \Omega^{2\ell}_\pm(F_s)$ *are transversely smooth.*

$$(\rho_\pm)_x : L^2(\widetilde{L}_x, \Lambda T^*\widetilde{L}_x) \to (\mathrm{Ker}(\Delta^\pm_{2\ell}))_x = \mathrm{Ker}(\Delta_x) \cap L^2_\pm(\widetilde{L}_x, \Lambda^{2\ell} T^*\widetilde{L}_x).$$

Definition

The Higher Harmonic Signature $\sigma(F)$ of (M, F) is the Haefliger class

$$\sigma(F) = \mathrm{ch}_a(\rho_+) - \mathrm{ch}_a(\rho_-).$$

Recall First Main Theorem

Theorem

Suppose that M *is a compact Riemannian manifold with oriented Riemannian foliation* F *of dimension* 4ℓ. *Then the leafwise signature* $\sigma(F)$ *of* F_s *is a leafwise homotopy invariant, and*

$$\sigma(F) = \int_F \mathrm{L}(TF).$$

Recent results of Azzali, Goette, and Schick improving results of H-Lazarov and Benameur-H immediately give:

Theorem

Suppose that M is a compact Riemannian manifold with oriented Riemannian foliation F of dimension 4ℓ. Then

$$\mathrm{ch}_a(\mathrm{Ind}_c^\infty(D^+)) = \sigma(F).$$

Theorem

$$\mathrm{ch}_a(\mathrm{Ind}_c^\infty(D^+)) = \int_F \mathrm{L}(TF).$$

Outline of the Proof of the First Main Theorem

1. $f : M, F \to M', F'$ a LHE induces a leafwise map $\widetilde{f} : \mathcal{G}, F_s \to \mathcal{G}', F'_s$, and an isomorphism $\quad f^* : H^*_c(M'/F') \to H^*_c(M/F)$.

2. $$\rho^f_\pm = \widetilde{f}^*(\rho'_\pm)$$ is a transversely smooth idempotent.

3. $$f^* \operatorname{ch}_a(\rho'_\pm) = \operatorname{ch}_a(\rho^f_\pm).$$

4. $$\operatorname{ch}_a(\rho^f_\pm) = \operatorname{ch}_a(P_{2\ell}\rho^f_\pm).$$ $P_{2\ell}$ proj. to $\operatorname{Ker}(\Delta_{2\ell})$.

5. $$\operatorname{ch}_a(P_{2\ell}\rho^f_\pm) = \operatorname{ch}_a(\rho_\pm).$$

Use $\widetilde{f}^*$ to pull back the idempotents $\rho'_\pm$ to idempotents $\rho^f_\pm$.

Problem. In general, $\widetilde{f}^*$ does NOT induce a map on L^2 leafwise forms.

Solution. Adapt Hilsum-Skandalis and H-Lazarov (à la Dodziuk) to re-define $\widetilde{f}$, get a leafwise submersion. Prove $\widetilde{f}$ induces bounded maps on all leafwise Sobolev spaces.

Definition

$g : M', F' \to M, F$ a homotopy inverse for f.
$P_{2\ell}$ proj. to $\mathrm{Ker}(\Delta_{2\ell})$. Set

$$\rho^f_\pm = \widetilde{f}^* \rho'_\pm \widetilde{g}^* P_{2\ell}.$$

Proposition

The $\rho^f_\pm = \widetilde{f}^* \rho'_\pm \widetilde{g}^* P_{2\ell}$ *are transversely smooth idempotents.*

Proof.

$P_{2\ell}$ and $\rho'_\pm$ are TS, so take any Sobolev space to any Sobolev space.
$\widetilde{f}^*$ and $\widetilde{g}^*$ are bounded maps on all leafwise Sobolev k spaces.

Lemma

$$d_\nu \widetilde{f}^* - \widetilde{f}^* d'_\nu = \widetilde{f}^* d'_s - d_s \widetilde{f}^* \quad \text{and} \quad d'_\nu \widetilde{g}^* - \widetilde{g}^* d_\nu = \widetilde{g}^* d_s - d'_s \widetilde{g}^*.$$

d_ν and d'_ν are the transverse de Rham operators.
d_s and d'_s are the leafwise de Rham operators, so take leafwise Sobolev k spaces to leafwise Sobolev $k-1$ spaces.
Lemma relates transverse derivatives for $(\mathcal{G}, F_s)$ and $(\mathcal{G}', F'_s)$.
A good deal of functional analysis finishes the proof. $\square$

Proposition

$$f^* \, \mathrm{ch}_a(\rho'_\pm) = \mathrm{ch}_a(\rho^f_\pm).$$

Proof.

If ∇' is a connection on ρ'_+, it defines the pull-back connection $\nabla = \widetilde{f}^*(\nabla')$ on ρ^f_+. Then $\nabla^2 = \widetilde{f}^*(\nabla'^2)$ and $\mathrm{Tr}(\nabla^{2k}) = f^* \, \mathrm{Tr}(\nabla'^{2k})$ for all k, which gives the result. $\square$

Proposition

If e_t, $0 \leq t \leq 1$, is a smooth family of $\mathcal{G}$ invariant transversely smooth idempotents, then $\mathrm{ch}_a(e_0) = \mathrm{ch}_a(e_1)$.

Proposition

$$\mathrm{ch}_a(\rho^f_\pm) = \mathrm{ch}_a(P_{2\ell}\rho^f_\pm).$$

Proof.

$(1-t)P_{2\ell}\rho^f_\pm + t\rho^f_\pm$ is a smooth family of TS idempotents. □

Finally,

Proposition

$$\mathrm{ch}_a(P_{2\ell}\rho^f_\pm) = \mathrm{ch}_a(\rho_\pm).$$

Proof.

Restriction of $\rho_\pm$ to $\mathrm{Im}(P_{2\ell}\rho^f_\pm)$ is an isomorphism onto $\mathrm{Im}(\rho_\pm)$ with uniformly bounded inverse.

Main Step: $\varphi_\pm = \rho_\pm^{-1} \circ \rho_\pm : \Omega^{2\ell}_{(2)}(F_s) \to \mathrm{Im}(P_{2\ell}\rho^f_\pm)$ is a TS idempotent.

Proof involves a good deal of heavy functional analysis.

To finish we need two easy results.

1. The TS idempotents $\varphi_\pm$ and $P_{2\ell}\rho^f_\pm$ have the same image, so $t\varphi_\pm + (1-t)P_{2\ell}\rho^f_\pm$ is a smooth family of TS idempotents, and

$$\mathrm{ch}_a(P_{2\ell}\rho^f_\pm) = \mathrm{ch}_a(\varphi_\pm).$$

2. Since $\varphi_\pm$ is projection onto $\mathrm{Im}(P_{2\ell}\rho^f_\pm)$ along $\mathrm{Ker}(\rho_\pm)$, we have $\varphi_\pm\rho_\pm = \varphi_\pm$ and $\rho_\pm\varphi_\pm = \rho_\pm$. Thus, $t\varphi_\pm + (1-t)\rho_\pm$ is a smooth family of TS idempotents, and

$$\mathrm{ch}_a(\varphi_\pm) = \mathrm{ch}_a(\rho_\pm).$$

□