Index Theory and Non-Commutative Geometry.
Moulay-Tahar Benameur & James L. Heitsch

Families Index Theorem follows from commutative diagram:

$$\begin{array}{ccc} \mathrm{K}^0_c(N) & \xrightarrow{\quad f_! \quad} & \mathrm{K}^0(M)) \\ \mathrm{ch}(\cdot)\wedge \mathrm{Td}(f) \ \downarrow & & \downarrow \ \mathrm{ch} \\ \mathrm{H}^*_c(N;\boldsymbol{R}) & \xrightarrow{\quad f_{**} \quad} & \mathrm{H}^*(M:\boldsymbol{R}). \end{array}$$

$f : N \to M$ a K-oriented map, i.e. $TN \oplus f^*TM$ has Spin^c structure. $\mathrm{Td}(f) = \mathrm{Td}(TN)/\mathrm{Td}(f^*(TM))$. $f_{**} = PD \circ f_* \circ PD$, $f_* : H_*(N;\boldsymbol{R}) \to H_*(M;\boldsymbol{R})$. If f a submersion, $f_{**} = \int$ over the fibers of f.
We extend this to foliations.
M compact manifold, F oriented foliation. M/F = "space of leaves" of F. $f : N \to M/F$ a K-oriented map. $\mathcal{G}$ the holonomy groupoid of F.

Theorem: For k large, the diagram commutes

$$\begin{array}{ccc} \mathrm{K}^0_c(N) & \xrightarrow{\quad f_! \quad} & \mathrm{K}_0(C^\infty_c(\mathcal{G}\times \boldsymbol{R}^{2k})) \\ \mathrm{ch}(\cdot)\wedge \mathrm{Td}(f) \ \downarrow & & \downarrow \ \mathrm{ch}_a \\ \mathrm{H}^*_c(N;\boldsymbol{R}) & \xrightarrow{\quad f_{**} \quad} & \mathrm{H}^*_c(M/F). \end{array}$$

$f_!$ the Connes-Skandalis push forward map. $H^*_c(M/F)$ Haefliger cohomology of F. ch_a and f_{**} to be defined.

Haefliger Cohomology

$\mathcal{U} = \{U_i\}$ cover by foliation charts for F. $T_i \subset U_i$ a transversal, $T = \cup T_i$, disjoint union. $\Omega^k_c(T) = C^\infty$ k forms with compact support. $\mathcal{H}_k$ = vector space generated by $\alpha - h^*\alpha$, $h \in$ holonomy pseudogroup and $\alpha \in \Omega^k_c(T)$. $\Omega^k_c(F) = \Omega^k_c\big(T\big)/\mathcal{H}_k$. $d : \Omega^k_c(T) \to \Omega^{k+1}_c(T)$ induces $d : \Omega^k_c(F) \to \Omega^{k+1}_c(F)$. Construction independent of all choices. $H^*_c(M/F)$ = cohomology of this complex. If F given by a fibration $M \to B$, then $H^*_c(M/F) = H^*_c(B;\boldsymbol{R})$.

Integration over the fiber of F

$\displaystyle\int_F : \Omega^{p+k}(M) \to \Omega^k_c(F)$, $\quad p = \dim F$. $\omega \in \Omega^{p+k}(M)$. Write $\omega = \displaystyle\sum_i \omega_i$, where $\omega_i \in \Omega^*_c(U_i)$. Integrate ω_i along the fibers of $\pi_i : U_i \to T_i$. Get $\displaystyle\int_{\pi_i} \omega_i \in \Omega^k_c(T_i)$. $\displaystyle\int_F \omega \equiv$ class of $\displaystyle\sum_i \int_{\pi_i} \omega_i$. $\displaystyle\int_F \omega \in \Omega^k_c(F)$ well defined; $\displaystyle\int_F$ commutes with d. Get $\displaystyle\int_F : H^{p+k}(M) \to H^k_c(M/F)$.

Holonomy Graph $\mathcal{G}$ of F

$\mathcal{G}$ = quivalence classes of leafwise paths in M. Paths equivalent if start at same point, end at same point, and have same holonomy. s,$r : \mathcal{G} \to M$: $s\big([\gamma]\big) = \gamma(0)$, $r\big([\gamma]\big) = \gamma(1)$. F_s foliation of $\mathcal{G}$, leaves are $\widetilde{L}_x = s^{-1}(x)$. $r : \widetilde{L}_x \to L_x$ is the holonomy cover of L_x. $\mathcal{G}_0$ = units of $\mathcal{G}$ = classes of constant paths.
$i : M \to \mathcal{G}$, $\quad i(x)$ = class of constant path at x. $\mathcal{G}_0 = i(M)$. $C^\infty_c(\mathcal{G})$ is a non-commutative algebra with product $(f\cdot g)(\gamma) \equiv \displaystyle\int_{\widetilde{L}_{s(\gamma)}} f(\gamma\gamma_1^{-1})g(\gamma_1)\, d\gamma_1$. $C^\infty_c(\mathcal{G})$ plays role of $C^\infty_c(M/F)$.

Definition of f_{**}. Case 1: f comes from $f : N \to M$ transverse to F. f induces an oriented foliation F_N of N. F_N and F are locally transversely diffeomorphic, so get $f_* : H_c^*(N/F_N) \to H_c^*(M/F)$.

$$f_{**} : H_c^*(N;\boldsymbol{R}) \xrightarrow{\int_{F_N}} H_c^*(N/F_N) \xrightarrow{f_*} H_c^*(M/F)$$

Case 2: f locally in Case 1. $f : N \to M/F$ is a $\mathcal{G}$ valued cocycle $(V_\alpha, f_{\alpha\beta})$. $\{V_\alpha\}$ locally finite open cover of N. $f_{\alpha\beta} : V_\alpha \cap V_\beta \to \mathcal{G}$. $f_{\alpha\beta}(x) f_{\beta\gamma}(x) = f_{\alpha\gamma}(x) \implies f_{\alpha\alpha} : V_\alpha \to \mathcal{G}_0 = M$. f a submersion if each $f_{\alpha\alpha}$ transverse to F, i.e. if a submersion to "space of leaves" of F. If f a submersion, it induces an oriented foliation F_N of N. F_N and F are locally transversely diffeomorphic, so get $f_* : H_c^*(N/F_N) \to H_c^*(M/F)$.

$$f_{**} : H_c^*(N;\boldsymbol{R}) \xrightarrow{\int_{F_N}} H_c^*(N/F_N) \xrightarrow{f_*} H_c^*(M/F)$$

Case 3: Arbitrary f. Construct a manifold W, and K-oriented maps $i : N \to W$ and $g : W \to M/F$. g is a submersion, and $f = g \circ i$.

$$f_{**} : H_c^*(N;\boldsymbol{R}) \xrightarrow{i_{**}} H_c^*(W;\boldsymbol{R}) \xrightarrow{g_{**}} H_c^*(M/F)$$

Definition of $\mathrm{ch}_a : K_0(C_c^\infty(\mathcal{G})) \to H_c^*(M/F)$.

Choose connection $\nabla : C^\infty(\mathcal{G}) \to C^\infty(T^*\mathcal{G})$. ν_s^* = normal bundle of $F_s \simeq s^*(T^*M)$. Projection $T^*\mathcal{G} \to \nu_s^*$ gives partial connection $\nabla^\nu : C^\infty(\mathcal{G}) \to C^\infty(\nu_s^*) \simeq C^\infty(s^*(T^*M)) \simeq C^\infty(\mathcal{G}) \otimes_{C^\infty(M)} \Omega^1(M)$, so, $\nabla^\nu : C^\infty(\mathcal{G}) \to C^\infty(\mathcal{G}) \otimes_{C^\infty(M)} \Omega^1(M)$. $\phi \in C_c^\infty(\mathcal{G})$ acts on $C^\infty(\mathcal{G})$ by: $\phi(g)([\gamma]) = \displaystyle\int_{\widetilde{L}_{s(\gamma)}} \phi(\gamma\gamma_1^{-1}) g(\gamma_1) d\gamma_1$. ϕ is a $C^\infty(M)$ equivariant smoothing operator. Extend ϕ to act on $C^\infty(\mathcal{G}) \otimes_{C^\infty(M)} \Omega^*(M)$. Consider $\partial_\nu(\phi) \equiv [\nabla^\nu, \phi]$. Essentially a transverse deRham operator, i.e. $\partial_\nu(\phi) \in C_c^\infty(\mathcal{G}) \otimes_{C^\infty(M)} \Omega^1(M)$. Since $\partial_\nu^2 \neq 0$, use Connes' X-construction to extend to δ with $\delta^2 = 0$. Want $\mathrm{Tr} : C_c^\infty(\mathcal{G}) \otimes_{C^\infty(M)} \Omega^*(M) \to \Omega_c^*(F)$. Any $K \in C_c^\infty(\mathcal{G}) \otimes_{C^\infty(M)} \Omega^*(M)$ is a smoothing operator on $C^\infty(\mathcal{G}) \otimes_{C^\infty(M)} \Omega^*(M)$. Schwartz kernel denoted $K(\alpha, \beta)$.

Defn: $\mathrm{Tr}(K) \equiv \displaystyle\int_F K(\bar{x}, \bar{x}) dx \in \Omega_c^*(F)$. $\bar{x}$ is class of constant path at $x \in M$. Tr is a graded trace and $\mathrm{Tr} \circ \delta = d \circ \mathrm{Tr}$.

Theorem: $B = [(e_1, \lambda_1)] - [(e_2, \lambda_2)] \in K_0(C_c^\infty(\mathcal{G}))$, where $(e_i, \lambda_i) \in M_N(C_c^\infty(\mathcal{G}) \oplus \boldsymbol{C})$ are idempotents. $\mathrm{tr} : M_N(C_c^\infty(\mathcal{G}) \otimes_{C^\infty(M)} \Omega^*(M)) \to C_c^\infty(\mathcal{G}) \otimes_{C^\infty(M)} \Omega^*(M)$ the usual trace. The Haefliger form

$$\mathrm{Tr} \circ \mathrm{tr}\Big(e_1 \exp\left[-(\delta e_1)^2/2i\pi\right] - e_2 \exp\left[-(\delta e_2)^2/2i\pi\right]\Big)$$

is closed, and its Haefliger cohomology class depends only on B.

$\mathrm{ch}_a(B)$ is the Haefliger class of this form.

The Connes-Skandalis push forward map

To define must replace M by $M \times \boldsymbol{R}^{2k}$, $\mathcal{G}$ by $\mathcal{G} \times \boldsymbol{R}^{2k}$, F by $\widehat{F}$ on $M \times \boldsymbol{R}^{2k}$. Leaves of $\widehat{F}$ are $L \times \{x\}, x \in \boldsymbol{R}^{2k}$. Details later. Get $f_! : K_c^0(N) \to K_0(C_c^\infty(\mathcal{G} \times \boldsymbol{R}^{2k}))$.

Theorem: Following diagram commutes

$$\begin{array}{ccc} \mathrm{K}_c^0(N) & \xrightarrow{\quad f_! \quad} & \mathrm{K}_0(C_c^\infty(\mathcal{G} \times \boldsymbol{R}^{2k})) \\ \mathrm{ch}(\cdot) \wedge \mathrm{Td}(f) \ \downarrow & & \downarrow \ \mathrm{ch}_a \\ \mathrm{H}_c^*(N;\boldsymbol{R}) & \xrightarrow{\quad f_{**} \quad} & \mathrm{H}_c^*(M \times \boldsymbol{R}^{2k}/\widehat{F}). \end{array}$$

Proof uses naturalness of ch and Td to reduce to a complicated direct computation. As $H_c^*(M \times \boldsymbol{R}^{2k}/\widehat{F}) \simeq H_c^*(M/F \times \boldsymbol{R}^{2k}) \simeq H_c^*(M/F) \otimes H_c^*(\boldsymbol{R}^{2k};\boldsymbol{R}) \simeq H_c^*(M/F)$, we get the main theorem.

Foliation Index Theorem: $\mathrm{Ind}_t : K^0_c(TF) \longrightarrow K_0(C^\infty_c(\mathcal{G} \times \boldsymbol{R}^{2k}))$ the Connes-Skandalis topological index map. $\pi_{F!} : H^*_c(TF) \to H^*(M)$ integration over fibers. Then, for all $u \in K^0_c(TF)$,

$$\mathrm{ch}_a(\mathrm{Ind}_t(u)) = (-1)^p \int_F \pi_{F!}(\mathrm{ch}(u))\, \mathrm{Td}\,(TF \otimes \boldsymbol{C})\,, \text{ in } H^*_c(M/F).$$

Direct corollary of theorem above using classical results of Atiyah-Singer and Connes-Skandalis.

$\mathrm{Ind}_a : K^0_c(TF) \longrightarrow K_0(C^\infty_c(\mathcal{G}))$, the Connes-Skandalis analytic index map.
$B : K_0(C^\infty_c(\mathcal{G}) \longrightarrow K_0(C^\infty_c(\mathcal{G} \times \boldsymbol{R}^{2k}))$, the Bott map. In general $B \circ \mathrm{Ind}_a \neq \mathrm{Ind}_t$.
Theorem: For all $u \in K^0_c(TF)$,

$$\mathrm{ch}_a(\mathrm{Ind}_t(u)) = \mathrm{ch}_a(\mathrm{Ind}_a(u)).$$

Proof depends on the deep extension theorem of Connes.

Theorem: Let E be an Hermitian bundle over a compact manifold M, and F a codimension q foliation of M with Hausdorff graph. Assume that F is even dimensional, oriented and spin. Assume further that the family of generalized leafwise Dirac operators D_E is regular near zero and that the strong Novikov-Shubin invariants of F are greater than $q/2$. Set $P =$ projection onto $\ker D_E$. Then

$$\mathrm{ch}_a(\mathrm{Ind}_a(D_E)) = \mathrm{ch}_a(P).$$

As $\mathrm{ch}_a(\mathrm{Ind}_a(D_E)) = \mathrm{ch}_a(\mathrm{Ind}_t(D_E)) = \int_F \widehat{A}(TF)\mathrm{ch}(E)$, and $\mathrm{ch}_a(P)$ carries geometric information about F, this relates characteristic classes of F to its geomtery.

Heitsch-Lazarov proved this for NS $> 3q$. Proof requires careful analysis of $e^{-tD_E^2}$ as $t \to \infty$.

Regularity Near Zero

$D_E =$ family of Dirac operators (on $\mathcal{G}$!) along leaves of F_s associated to $r^*(E)$.
$P =$ projection onto $\ker D_E$. $P_\epsilon =$ spectral projection of D_E^2 for $(0, \epsilon)$.

Assumption: both P and P_ϵ (for sufficiently small ϵ) have smooth Schwartz kernels, (i.e. are smooth transversely) and their transverse derivatives define bounded smoothing operators along the leaves of F_s.

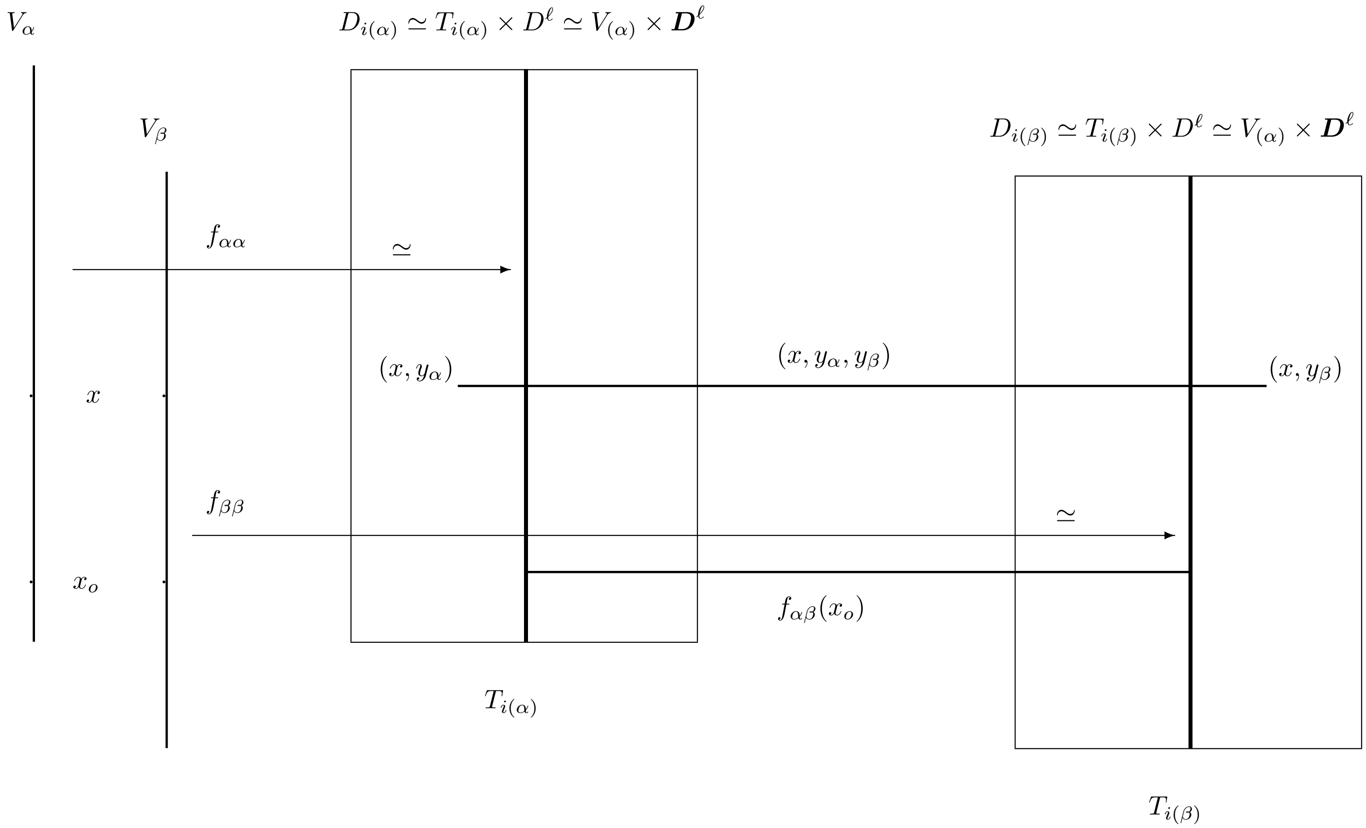
Strong Novikov-Shubin invariants

$K_{P_\epsilon}(\alpha, \beta) =$ Schwartz kernel of P_ϵ.

Assumption: $\mathrm{Tr}(K_{P_\epsilon}) = \mathcal{O}(\epsilon^\beta)$ for $\beta > q/2$ as $\epsilon \to 0$.

The Connes-Skandalis map: details

Reduce to $f : N \to M/F$ is **étale**, i.e. $f_{\alpha\alpha} : V_\alpha \xrightarrow{\simeq} T_{i(\alpha)} \subset U_{i(\alpha)}$. Must replace M by $M \times \boldsymbol{R}^{2k}$, $\mathcal{G}$ by $\mathcal{G} \times \boldsymbol{R}^{2k}$, F by $\widehat{F}$ on $M \times \boldsymbol{R}^{2k}$ with leaves $L \times \{x\}, x \in \boldsymbol{R}^{2k}$. $\varrho : D_i \to T_i$ normal disc bundle in $M \times \boldsymbol{R}^{2k}$. Coordinates x on V_α and y_α on $\boldsymbol{D}^\ell$, give coordinates (x, y_α) on $D_{i(\alpha)} \simeq T_{i(\alpha)} \times \boldsymbol{D}^\ell \simeq V_{(\alpha)} \times \boldsymbol{D}^\ell$.
$U_{\alpha\beta}$ = classes of paths γ where 1. $s(\gamma) \in D_{i(\alpha)}$, and $r(\gamma) \in D_{i(\beta)}$, 2. $\gamma \mid\mid f_{\alpha\beta}(x_o)$, where $x_o \in V_\alpha \cap V_\beta$.
$U_{\alpha\beta}$ charts on $\mathcal{G} \times \boldsymbol{R}^{2k}$, coords (x, y_α, y_β).



Choose $\psi : \boldsymbol{D}^\ell \to \boldsymbol{R}$ with compact support and $\int_{\boldsymbol{D}^\ell} \psi^2 = 1$, and $\{\phi_\alpha\}$ a partition of unity on N subordinate to $\{V_\alpha\}$.

Define $f_! : C_c^\infty(N) \to C_c^\infty(\mathcal{G} \times \boldsymbol{R}^{2k})$as follows. For $g \in C_c^\infty(N)$, $f_!(g) = 0$ except on the $U_{\alpha\beta}$, where

$$f_!(g)(x, y_\alpha, y_\beta) = g(x)\psi(y_\alpha)\psi(y_\beta)\sqrt{\phi_\alpha(x)\phi_\beta(x)}.$$

$f_!$ is an algebra map, so get

$$f_! : K_c^0(N) \to K_0(C_c^\infty(\mathcal{G} \times \boldsymbol{R}^{2k})).$$