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On the secondary classes of foliations with trivial normal bundles

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The secondary classes of a smooth foliation on a manifold are invariants of the concordance class of the foliation. A central problem in the theory of foliations is to determine which of these invariants are non-zero, and also to determine their geometric significance. In this paper we examine one aspect of how the secondary classes of a foliation depend upon the topology of the foliated manifold. This dependence is used to construct many examples of foliated manifolds with framed normal bundles having non-trivial secondary invariants. This implies the non-triviality of the corresponding universal secondary classes in $H^*(F\Gamma_G)$.

There are two basic ideas which we exploit to deduce our theorems. The first regards the construction of a universal map $k_*: H^*(W(\mathfrak{G})_q) \to H^*(F\Gamma_G)$ using the theory of foliation invariants as given by Kamber–Tondeur [17]. There is a map of spectral sequences converging to k_* which, when combined with a knowledge of the connectivity of $F\Gamma_G$, leads to the proofs of Theorems 1 and 3 below. The second idea is a type of permanence theorem which is related to a concept first explored by Kamber–Tondeur [18] and Lazarov [19]. Coupled with the results on evaluating the secondary classes via residues of Baum–Bott [1], Lazarov–Pasternack [21] and Heitsch [13], this yields Theorems 2, 4, and 5.

In a previous paper [14], Sullivan's theory of minimal models was used to show that many of the secondary classes of foliated manifolds are non-trivial. This was done by an analysis of the homotopy properties of $F\Gamma_G$; more precisely, the Whitehead product action of $\pi_*(BG)$ on $\pi_*(F\Gamma_G)$ was used to construct spherical classes in $H_*(F\Gamma_G)$ which support universal secondary classes. Then following an idea of Haefliger [11], this yields much information on the higher homotopy groups of $F\Gamma_G$. The present paper significantly extends the results of [14] on the independence of the secondary cohomology invariants. However, it is no longer possible to assert that the classes in $H^*(F\Gamma_G)$ are spherically supported. The contrast between the two approaches is seen as follows. There is a sequence of fibrations $G \to F\Gamma_G \to B\Gamma_G \to BG$, and the first map defines a G-action on the space $F\Gamma_G$. Assume that $F\Gamma_G$ is N-connected. Then for $n \le N$ there is a smash

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product operation

$$\pi_n(G)\times\pi_m(F\Gamma_G)\xrightarrow{\wedge}\pi_{n+m}(F\Gamma_G)$$

which is induced from this G-action. The minimal model techniques of [14] essentially analyze this smash product, and establish that it is non-trivial for many n and m. Our present method is to study the more general G-action, which defines for all n a map

$$H_n(G)\times H_m(F\Gamma_G)\to H_{n+m}(F\Gamma_G).$$

This $H_*(G)$-action is shown to be non-trivial for many n and m by the use of the Permanence Theorems 3.3 and 3.5 of §3.

Our main results are stated below as Theorems 1 to 5. The terminology is explained in §1. The techniques required for the proofs of these theorems are developed in §§2 and 3; the proofs are then given in §4. The author would like to thank James Heitsch for pointing out a gap in the original proof of Theorem 3.3, and whose questions led to a general reformulation of the results of this paper.

The first two theorems deal with the invariants of a smooth codimension q framed foliation on a manifold M. Recall that the secondary invariants in this case form a real vector space $H^*(W_q)$. A class in $H^*(W_q)$ is said to be rigid if its value in $H^*(M)$ must remain constant under deformations of the given foliation on M; it is said to be variable otherwise [12].

THEOREM 1. (a) *Let $q\geq 4$ be even. Then there exists a framed codimension q foliation on a compact manifold M such that the set of rigid secondary classes*

$$\mathcal{R}_q=\{y_2y_{2i_2}\cdots y_{2i_s}c_q \mid 1<i_2<\cdots<i_s\}$$

is independently realized in $H^*(M)$.

(b) *Let $q=4k-2$ with $k\geq 2$. Then there exists a framed codimension q foliation on a compact manifold M such that the set of rigid secondary classes*

$$\mathcal{R}'_q=\{y_{2k}y_{2i_2}\cdots y_{2i_s}c_{2k} \mid k<i_2<\cdots<i_s\}$$

is independently realized in $H^*(M)$. Further, the set $\mathcal{R}_q$ is mapped to zero in $H^*(M)$ for this foliation.

(c) *For q even, the set $k_*(\mathcal{R}_q)\subseteq H^*(F\Gamma_q)$ is linearly independent. For* $q\equiv 2 \bmod (4)$, *the set $k_*(\mathcal{R}_q\cup\mathcal{R}'_q)\subseteq H^*(F\Gamma_q)$ is linearly independent.*

It can be shown that many of these rigid classes in $H^*(F\Gamma_q)$ are supported on spherical cycles in $H_*(F\Gamma_q)$, [16]. It is also interesting to contrast Theorem 1 with the results of Pittie [23].

Our second theorem considers the variable secondary classes. This uses the results of Heitsch [13].

THEOREM 2. *For each $q \geq 3$, there exists a set $\mathcal{V}_q \subseteq H^{2q+1}(W_q)$, having at least 3 elements, and a family of framed codimension q foliations on a compact manifold M such that the set of variable classes*

$$\overline{\mathcal{V}_q} = \{y_I c_J = y_i y_{2i_2} \cdots y_{2i_s} c_J \mid y_i c_J \in \mathcal{V}_q \text{ and } i < 2i_2 < \cdots < 2i_s\}$$

is mapped to a set of independent and variable classes in $H^*(M)$. Consequently, the set $k_*(\bar{\mathcal{V}}_q) \subseteq H^*(F\Gamma_q)$ is independently variable.

Note that Theorems 1 and 2 give an alternate proof of some of the results of Fuks [6] on $H^*(F\Gamma_q)$.

A foliation which has a metric on its normal bundle that is parallel along the leaves is said to be Riemannian [24]. The characteristic classes of these foliations have been studied by Kamber–Tondeur [18] and Lazarov–Pasternack [20], [21].

THEOREM 3. *There exists a Riemannian foliation with SO_q-framed normal bundle on a manifold M such that all of the secondary classes are independently realized in $H^*(M)$. Thus, the universal map $k_*: H^*(W(so(q))_{[q/2]}) \to H^*(F\Gamma_{SO_q})$ is injective.*

It was shown by Lazarov and Pasternack that for $q = 2, 3 \bmod (4)$, there are secondary classes for Riemannian foliations of codimension q which can vary under deformations of the foliation. These classes are called variable, and using the results of [21] we show:

THEOREM 4. *If $q = 2, 3 \bmod 4$, then there exists a family of Riemannian foliations of codimension q with SO_q-framed normal bundles on a compact manifold M such that all of the variable secondary classes are independent and variable in $H^*(M)$. Hence, they are independently variable in $H^*(F\Gamma_{SO_q})$.*

A foliation which is locally defined by submersions into $\mathbf{C}^n$, with biholomorphic transition functions, is called a complex foliation. The secondary invariants of a complex foliation with trivial normal bundle form a complex vector space

$H^*(W(Gl(n, \mathbf{C}))_n)$. Examples of complex foliations on S^{2n+1} with variable secondary classes have been constructed by Baum–Bott [1]; their results are used to show:

THEOREM 5. *For $n \geq 2$, there exists a family of complex foliations with $Gl_n\mathbf{C}$-framed normal bundles on a compact manifold M such that all of the variable secondary classes in $H^*(W(Gl(n, \mathbf{C}))_n)$ are independent and variable in $H^*(M; \mathbf{C})$. Thus, they are independently variable in $H^*(F\Gamma_n^{\mathbf{C}}; \mathbf{C})$.*

§1. Definitions and basic constructions

All manifolds are assumed to be smooth (C^∞) and paracompact; all foliations and maps between manifolds are assumed to be smooth. For a topological space X we use $H^*(X)$ to denote the singular cohomology of X with real coefficients. For a real number x, let $[x]$ denote the greatest integer $\leq x$.

We work with the general notion of a G-foliation as this encompasses the usual idea of a foliation, when $G = Gl_q$, and the special cases of Riemannian and complex foliations when $G = O_q$ and $G = Gl_n\mathbf{C}$ respectively. Recall that a foliation of codimension q on a manifold M is an integrable subbundle $\mathcal{F}$ of the tangent bundle TM, where the normal bundle $Q = TM/\mathcal{F}$ has rank q. Let G be a closed subgroup of Gl_q. We say that $\mathcal{F}$ is a G-foliation if Q has a G-structure which is invariant under the natural parallelism along the leaves of $\mathcal{F}$. This is equivalent to the existence of the following data [10], [17]:

(i) a model manifold B of dimension q with a G-structure on TB;

(ii) an open covering $\{U_\alpha\}$ of M and submersions $\phi_\alpha : U_\alpha \to B$ with $\mathcal{F}|_{U_\alpha} = \ker d\phi_\alpha$ such that the local transition functions $\gamma_{\alpha\beta}$ preserve the G-structure on B.

An O_q-foliation is one with a bundle-like metric in the sense of Reinhart [24] on the normal bundle Q, and is also called a Riemannian foliation.

If the model manifold B can be taken to be $\mathbf{R}^q$ with the natural flat G-structure, then the G-foliation is said to be G-integrable. For example, a complex foliation is a $Gl_n\mathbf{C}$-integrable $Gl_n\mathbf{C}$-foliation of real codimension $q = 2n$. In a similar way, by an appropriate choice of group G and model manifold B, one obtains the notion of conformal, flag or symplectic foliations.

Associated to a G-foliation $\mathcal{F}$ on M is a principal G-bundle $\pi : P \to M$. This is constructed using the local defining charts $\{\phi_\alpha : U_\alpha \to B\}$ of $\mathcal{F}$ and G-bundle $P' \to B$ associated to the G-structure on TB. Locally, $P|_{U_\alpha}$ is defined to be the pullback of P' via ϕ_α. These local bundles are then patched together using condition (ii) above. The bundle $P \to M$ is trivial if it admits a section $s: M \to P$; from a section s one can construct a bundle isomorphism $P \cong M \times G$.

A section s of the associated bundle $P \to M$ of $\mathcal{F}$ is called a G-framing of the normal bundle of the foliation. The pair $(\mathcal{F}, s)$ is called a G-framed G-foliation.

Given a group G and manifold B with a G-structure on TB, Haefliger shows in [10] that there exists a classifying space for G-foliations modeled on B. In our examples the model B is dictated by the choice of G, so by abuse of notation this classifying space is denoted $B\Gamma_G$. Let $\nu: B\Gamma_G \to BG$ classify the G-structure on $B\Gamma_G$. If $f: M \to B\Gamma_G$ classifies a G-foliation on M, then $\nu \circ f$ classifies the associated G-bundle $P \to M$. Let $F\Gamma_G$ denote the homotopy theoretic fiber of ν. If a G-foliation on M has trivial associated bundle $P \to M$, then the map $\nu \circ f$ is homotopic to a constant. Hence there is induced a map $\bar{f}$ giving a homotopy commutative diagram

$$\begin{tikzcd} M \arrow[rr, "\bar{f}"] \arrow[dr, "f"'] & & F\Gamma_G \arrow[dl] \\ & B\Gamma_G & \end{tikzcd}$$

In this sense we say that $F\Gamma_G$ classifies G-foliations with trivial associated G-bundle.

For the group Gl_q with model $B = \mathbf{R}^q$, we adopt the standard notation $B\Gamma_q = B\Gamma_{Gl_q}$ and $F\Gamma_q = F\Gamma_{Gl_q}$. For complex foliations, we use $F\Gamma_n^{\mathbf{C}} = F\Gamma_{Gl_n\mathbf{C}}$. The space $F\Gamma_{SO_q}$ is sometimes denoted in the literature by $FR\Gamma_q$.

Next assume that M has a fixed G-foliation $\mathcal{F}$ of codimension q and there is given a section s of $P \to M$. We outline the construction of the secondary classes for $\mathcal{F}$ using truncated Weil algebras. This approach is the most useful for our purposes; it is developed in detail by Kamber and Tondeur in [17]. When G is either Gl_q or $Gl_n\mathbf{C}$, the secondary classes which are obtained coincide with those constructed by Bernstein and Rosenfel'd [2] or Bott and Haefliger [3], [4].

Let $\mathfrak{G}$ denote the Lie algebra of G. Recall that the Weil algebra $W(\mathfrak{G}) = \Lambda\mathfrak{G}^* \otimes S\mathfrak{G}^*$ is a differential graded algebra (dga), where $\Lambda\mathfrak{G}^*$ is the exterior algebra on the dual space $\mathfrak{G}^*$ of $\mathfrak{G}$ (each $x \in \mathfrak{G}^*$ has degree 1) and $S\mathfrak{G}^*$ is the symmetric algebra on $\mathfrak{G}^*$ (each $x \in \mathfrak{G}^*$ has degree 2). This algebra has a decreasing filtration by differential ideals

$$F^{2p}W(\mathfrak{G}) = \bigoplus_{r \geq 2p} \Lambda\mathfrak{G}^* \otimes S^r\mathfrak{G}^*.$$

The quotient dga $W(\mathfrak{G})_q = W(\mathfrak{G})/F^{2q+2}W(\mathfrak{G})$ is called a truncated Weil algebra.

The secondary invariants of the G-foliation $\mathcal{F}$ are defined as follows. Choose a Bott connection ω on the G-bundle $P \to M$, [17]. If $\mathscr{E}(P)$ denotes the deRham complex of P, then the Chern–Weil construction yields a *dga* homomorphism $k(\omega): W(\mathfrak{G})_q \to \mathscr{E}(P)$. The fundamental result is that the induced map in cohomology $k(\omega)_*$ is independent of the choice of the Bott connection ω and depends only on the G-concordance class of $\mathcal{F}$ (see Theorem 4.43 of [17]). Further, the map $k(\omega)_*$ is functorial with respect to pull-backs along maps $f: M' \to M$ which are transverse to $\mathcal{F}$.

Recall that a section $s: M \to P$ is given. Set

$$\Delta_*^s = s^* \circ k(\omega)_*: H^*(W(\mathfrak{G})_q) \to H_{DR}(M).$$

The elements in the image of Δ_*^s are defined to be the secondary classes for the G-framed G-foliation $\mathcal{F}$ on M.

For certain groups G, the *dga* homomorphism $k(\omega)$ annihilates the ideal $F^{2q'+2}W(\mathfrak{G})$ where $q' = [q/2]$. The corresponding map $k(\omega): W(\mathfrak{G})_{q'} \to \mathscr{E}(P)$ is then used to construct the characteristic homomorphism Δ_*^s. This is the case for both Riemannian and complex foliations [17].

An additional useful property of the Chern–Weil homomorphism $k(\omega): W(\mathfrak{G})_q \to \mathscr{E}(P)$ is that it preserves the basic filtration on both algebras and hence induces a map of spectral sequences. The basic filtration on $\mathscr{E}(P)$ is that given by

$$F^p\mathscr{E}(P) = \mathscr{E}(P) \wedge \pi^*\mathscr{E}^p(M),$$

where $\pi: P \to M$. The curvature of any connection is a horizontal form so we have $k(\omega)F^{2p}W(\mathfrak{G})_q \subseteq F^{2p}\mathscr{E}(P)$. The induced map of the spectral sequences associated to these filtrations is denoted by

$$E_r k(\omega): E_r^{2p,t}W(\mathfrak{G})_q \to E_r^{2p,t}(P).$$

We conclude this section by exhibiting a set of cocycles in $W(\mathfrak{G})_q$ which span $H^*(W(\mathfrak{G})_q)$. Let $A(\mathfrak{G})_q$ be the standard model for $W(\mathfrak{G})_q$. More precisely, define $A(\mathfrak{G})_q = \Lambda(y_1, \ldots, y_r) \otimes I(\mathfrak{G})_q$ where $I(\mathfrak{G})$ denotes the *dga* of $ad(\mathfrak{G})$-invariant polynomials on $\mathfrak{G}$. This is identified with a symmetric algebra $I(\mathfrak{G}) \cong \mathbf{R}[c_1, \ldots, c_r]$; each c_j is homogeneous and of even degree. We assume that $\deg c_i \leq \deg c_j$ if $i \leq j$. The element $y_j \in W(\mathfrak{G})$ is defined to be the suspension of c_j and satisfies $dy_j = c_j$. There is an inclusion $A(\mathfrak{G})_q \subseteq W(\mathfrak{G})_q$ which induces an isomorphism in cohomology [8], [17]. In fact, if we give $A(\mathfrak{G})_q$ the basic filtration from $W(\mathfrak{G})_q$, then there are natural isomorphisms

$$A(\mathfrak{G})_q \cong E_2 A(\mathfrak{G})_q \cong E_2 W(\mathfrak{G})_q.$$

For the group Gl_q, the algebra $A(gl(q,\mathbf{R}))_q$ is commonly denoted by W_q.

We adopt the standard notation for the elements of $A(\mathfrak{G})_q$: Given index sets $I=(i_1,\ldots,i_s)$ and $J=(j_1,\ldots,j_r)$ with each $j_k\geq 0$, let

$$y_Ic_J = y_{i_1}\cdots y_{i_s}\otimes c_1^{j_1}\cdots c_r^{j_r}.$$

According to Vey [7], the following set of cocycles (which are called admissible) gives a basis for $H^*(W(\mathfrak{G})_q)$:

$$\{y_Ic_J \mid 1\leq i_1<\cdots<i_s\leq r;\ \deg y_{i_1}c_J>2q;\ l<i_1\Rightarrow j_l=0\}.$$

For a complex $H^*(W(\mathfrak{G})_{[q/2]})$ one modifies this by requiring that $\deg y_{i_1}c_J>2[q/2]$.

The admissible cocycles are divided into two types. The classes in the image of the natural map $H^*(W(\mathfrak{G})_{q+1})\to H^*(W(\mathfrak{G})_q)$ are said to be *rigid*. This terminology is used because the rigid secondary classes give invariants for a foliation $\mathcal{F}$ which are constant under deformations of $\mathcal{F}$ [12]. With respect to the Vey basis, the rigid classes correspond to the set

$$\{y_Ic_J \mid y_Ic_J \text{ admissible and } \deg y_{i_1}c_J>2q+2\}.$$

The remaining admissible cocycles, for which $\deg y_{i_1}c_J=2q+1$, are said to be *variable*.

§2. The universal characteristic map

In this section we review the construction of the universal map $k_*: H^*(W(\mathfrak{G})_q)\to H^*(F\Gamma_G)$, and also analyze the geometric significance of the lifts $\tilde{f}: M\to F\Gamma_G$ of a given map $f: M\to B\Gamma_G$.

First recall that there is a fibration sequence $G\to F\Gamma_G\to B\Gamma_G\to BG$. We therefore can consider $F\Gamma_G$ as the total space of the G-bundle over $B\Gamma_G$ constructed as follows. Let $EG\to BG$ denote the universal G-bundle. Define $F\Gamma_G$ to be the pullback in the diagram

$$\begin{tikzcd} G \arrow[d] & G \arrow[d] \\ F\Gamma_G \arrow[r, "\tilde{\nu}"] \arrow[d] & EG \arrow[d] \\ B\Gamma_G \arrow[r, "\nu"] & BG \end{tikzcd}$$

It is then standard that there exists a weak homotopy equivalence from $F\Gamma_G$ to the fiber of the fibration induced from $\nu: B\Gamma_G \to BG$.

Given a map $f: M \to B\Gamma_G$, the set of lifts $\tilde{f}$ of f

$$\begin{tikzcd} & F\Gamma_G \arrow[d] \\ M \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B\Gamma_G \end{tikzcd}$$

corresponds to the set of maps $M \to G$: Given one lift $\tilde{f}_0$, every other lift $\tilde{f}$ is obtained by choosing some function $\varphi: M \to G$ and setting $\tilde{f} = \tilde{f}_0 \cdot \varphi$, where for $m \in M$, the element $\varphi(m) \in G$ acts on the fiber over $f(m) \in B\Gamma_G$. Note that the composition $\nu \circ f: M \to BG$ fixes a representation of the bundle $P \to M$, and the lift $\tilde{\nu} \circ \tilde{f}_0$ defines a G-framing of P. Hence a different lift $\tilde{f} = \tilde{f}_0 \cdot \varphi$ corresponds to a change in the framing $\tilde{\nu} \circ \tilde{f}_0$.

Conversely, given a map $f: M \to B\Gamma_G$ we construct the associated G-bundle as the pullback via $\nu \circ f$ of $EG \to BG$. This gives a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\tilde{f}"] \arrow[d, "f_s"] & F\Gamma_G \arrow[r, "\tilde{\nu}"] \arrow[d] & EG \arrow[d] \\ M \arrow[u, bend left, "s"] \arrow[ur, "s"] \arrow[r, "f"] & B\Gamma_G \arrow[r, "\nu"] & BG \end{tikzcd} \tag{2.1}$$

A section s of $P \to M$ then gives rise to a lift $f_s = \tilde{f} \circ s$ of the map f.

Remark 2.2. The above does not show that $F\Gamma_G$ is the classifying space of G-framed G-foliations; it only shows that given any pair $(\mathcal{F}, s)$ on M there exists a natural map $f_s: M \to F\Gamma_G$. The problem arises with the following type of example. Consider the foliation by points on $M = (\mathbf{R}^{n+1} - \{0\}) \times \mathbf{R}^{q-n-1}$ with the canonical flat G-structure. This is classified by a unique homotopy class of maps $M \to F\Gamma_G$. However, the set of homotopy classes of G-framings corresponds to $\pi_n(G)$ which may not be trivial.

In spite of the above remark, we still have the following basic result.

THEOREM 2.3. *There exists a universal map*

$$k_*: H^*(W(\mathfrak{G})_q) \to H^*(F\Gamma_G).$$

That is, given a G-framed G-foliation $(\mathscr{F}, s)$ on M, there exists a classifying map $f_s : M \to F\Gamma_G$ such that $\Delta_*^s = f_s^* \circ k_*$.

If $\mathscr{F}$ is either a Riemannian or complex foliation on M, then the truncation index is taken to be $[q/2]$ instead of q.

Proof. Given a G-foliation $\mathscr{F}$ on M with associated G-bundle $P \to M$, the Chern–Weil homomorphism $k(\omega)_* : H^*(W(\mathfrak{G})_q) \to H^*(P)$ depends only on the concordance class of $\mathscr{F}$ and is functorial with respect to pullbacks along maps $f : N \to M$ transverse to $\mathscr{F}$. Thus, there exists a universal Chern–Weil homomorphism for the universal bundle $F\Gamma_G \to B\Gamma_G$. We denote this map by $k_* : H^*(W(\mathfrak{G})_q) \to H^*(F\Gamma_G)$, and show that it satisfies the conditions of the theorem.

Consider a G-framed G-foliation $(\mathscr{F}, s)$ on M. Let $\mathscr{F}$ be classified by a map $f : M \to B\Gamma_G$. This defines a pullback square as in (2.1), where the section s yields a map $f_s = \bar{f} \circ s$. For a Bott connection ω on $P \to M$ we then have

$$\Delta_*^s = s^* \circ k(\omega)_* = s^* \circ \bar{f}^* \circ k_* = (f_s)^* \circ k_*.$$

Thus k_* is universal as claimed. ∎

Remark 2.4. If one chooses a section s of the G-bundle $E\Gamma_G \to F\Gamma_G$ induced by the composition $F\Gamma_G \to B\Gamma_G \to BG$, then there is a corresponding map $\Delta_*^s : H^*(W(\mathfrak{G})_q) \to H^*(F\Gamma_G)$. However, this map depends upon the choice of the section s. There exists a canonical G-framing $\bar{s}$ of $E\Gamma_G \to F\Gamma_G$, which is defined by the diagram

$$\begin{tikzcd} E\Gamma_G \arrow[r, "\pi"] \arrow[d, "\pi"] & F\Gamma_G \arrow[d] \\ F\Gamma_G \arrow[u, bend left, "\bar{s}"] \arrow[ur, "\mathrm{id}"] \arrow[r] & B\Gamma_G. \end{tikzcd}$$

The map k_* then corresponds to $\Delta_*^{\bar{s}}$, for $\Delta_*^{\bar{s}} = \bar{s}^* \circ k(\bar{\omega})_* = \bar{s}^* \circ \pi^* \circ k_* = \mathrm{id}^* \circ k_* = k_*$.

The map $\Delta_*^{\bar{s}}$ is the universal secondary map with respect to the canonical G-framing on $F\Gamma_G$ that is constructed in [3], [4], [7], [20]. The advantage of using k_* is that it is a bundle map and has a map of spectral sequences which converge to it. This property is not so obvious for $\Delta_*^{\bar{s}}$.

The next concept is needed in order to formulate the permanence theorem for variable classes in the next section.

DEFINITION 2.5. Let $\{(\mathcal{F}_\alpha, s_\alpha) \mid \alpha \in \mathcal{A}\}$ be a family of framed foliations on M. A set of secondary classes $\{z_1, \ldots, z_d\} \subseteq H^n(W(\mathfrak{G})_q)$ is said to be independent and variable (with respect to this family) if the union over $\alpha \in \mathcal{A}$ of the images of the evaluation homomorphisms

$$\bigoplus_{j=1}^{d} \Delta^{s_\alpha}(z_j) : H_n(M; Z) \to \mathbf{R}^d \tag{2.6}$$

generates $\mathbf{R}^d$ additively.

The point of this definition is that if $\{z_1, \ldots, z_d\}$ is an independent and variable set with respect to some family of framed G-foliations, then the corresponding universal classes $k_*\{z_1, \ldots, z_d\}$ in $H^n(F\Gamma_G)$ are independently variable. That is, the evaluation homomorphism

$$\bigoplus_{j=1}^{d} k_*(z_j) : H_n(F\Gamma_G; Z) \to \mathbf{R}^d \tag{2.7}$$

is an epimorphism. This follows because k_* is universal, so the image of (2.7) contains an additive basis for $\mathbf{R}^d$ and thus is all of $\mathbf{R}^d$.

A useful criterion for when a set is independent and variable is given by the next result, which from the Appendix to Baum–Bott [1].

PROPOSITION 2.8. *Suppose that $V \subseteq \mathbf{R}^d$ is a connected subvariety of dimension ≥ 1 which is not contained in any codimension one hyperplane of $\mathbf{R}^d$. Then V generates $\mathbf{R}^d$ additively.*

For example, if $F = (f_1, \ldots, f_d) : \mathbf{R}^\lambda \to \mathbf{R}^d$ is a polynomial mapping such that $\{1, f_1, \ldots, f_d\}$ is a linearly independent set of functions, then the image of F generates $\mathbf{R}^d$ additively. For each of the foliations constructed in [1], [13], [21], a parameter space $\mathcal{A} = \mathbf{R}^\lambda$ and a set $\{z_1, \ldots, z_d\}$ is given such that the corresponding evaluation map (2.6) contains in its image a variety V as in (2.8). So it follows from Proposition 2.8 that $\{z_1, \ldots, z_d\}$ is an independent and variable set.

§3. Permanence theorems

It was seen in section 2 that given a map $f : M \to B\Gamma_G$, the various framings s of the normal bundle give rise to maps $f_s : M \to F\Gamma_G$ which lift f. It is natural to ask how the image of the map

$$\Delta_*^s = f_s^* \circ k_* : H^*(W(\mathfrak{G})_q) \to H^*(M)$$

depends upon the choice of G-framing s. The answer to this question is implicit in the literature [5], [12]; the formulation we use appears in [20]. Our Theorems 3.3 and 3.5 then follow from an application of this result.

For each $1 \leq j \leq r$, the cocycle $c_j \in I(G)$ determines a cohomology class in $H^{m_j+1}(BG)$. Let $\tau_j \in H^{m_j}(G)$ denote the suspension of this class. Next suppose there is given a G-foliation $\mathcal{F}$ on M with two sections s, s' of $P \to M$. Define a mapping $\varphi: M \to G$ by the relation $s = s' \cdot \varphi$, where for $m \in M$, $\varphi(m)$ acts on the fiber over m. Then for each j there is a cohomology class $\varphi^*(\tau_j) \in H^{m_j}(M)$. The next theorem was proved for SO_q-foliations in [Theorem 4.2; 20]. The proof carries over directly to G-foliations to give:

THEOREM 3.1. *Let $\mathcal{F}$ be a G-foliation on M and suppose that two sections s, s' of $P \to M$ are given. Given a cocycle $y_I c_J = y_{i_1} \cdots y_{i_n} c_J \in W(\mathfrak{G})_q$, then*

$$\Delta_*^{s'}(y_I c_J) = \prod_{j=1}^{n} \{\Delta^s(y_{i_j}) - \varphi^*(\tau_{i_j})\}\Delta^s(c_J).$$

The meaning of the right-hand side of this equation is that it is to be expanded multiplicatively and the terms collected to get a well-defined class in $H^*(M)$.

COROLLARY 3.2. *Let $V_G \subseteq H^*(W(\mathfrak{G})_q)$ be the subspace spanned by the following admissible cocycles:*

(i) $y_i c_J \in W(\mathfrak{G})_q$, or

(ii) $y_I c_J \in W(\mathfrak{G})_q$ such that $\tau_i \in H^{m_i}(G)$ is zero for each $i \in I$. Then for any G-foliated manifold M with trivial associated G-bundle, the characteristic map $\Delta_*^s|_{V_G}: V_G \to H^*(M)$ is independent of the choice of G-framing s.

Proof. For a class $y_i c_J$ of type (i), by (3.1) we have $\Delta_*^{s'}(y_i c_J) = \Delta_*^s(y_i c_J) - \varphi^*(\tau_i)\Delta_*^s(c_J)$ in $H^*(M)$. The normal bundle of $\mathcal{F}$ is trivial so $\Delta_*^s(c_J) = 0$, and thus the difference term $\varphi^*(\tau_i) \cdot \Delta_*^s(c_J) = 0$.

For a class $y_I c_J$ of type (ii), use (3.1) to expand $\Delta_*^{s'}(y_I c_J)$, then note that each $\varphi^*(\tau_{i_j}) = 0$ by assumption. ∎

It is useful to know which of the classes $\tau_i \in H^{m_i}(G)$ are non-zero. Let $K \subseteq G$ be a maximal compact subgroup and assume that G is connected. There is a natural restriction map $r: I(G) \to I(K)$ for which there is a commutative diagram, where σ is a map of vector spaces:

$$\begin{array}{ccc} I(K) & \xrightarrow{\sigma} & H^*(K) \\ \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle \cong} \\ I(G) & \xrightarrow{\sigma} & H^*(G). \end{array}$$

The kernel of the map $\sigma: I(K) \to H^*(K)$ is exactly the ideal of products $I^+(K) \cdot I^+(K) \subseteq I(K)$. Thus $\tau_i \neq 0$ precisely when c_i is not mapped to zero under the composition $I(G) \xrightarrow{r} I(K) \to QI(K)$, where $QI(K) = I(K)/\{I^+(K) \cdot I^+(K)\}$. For example, when G is compact each $\tau_i \neq 0$. When $G = Gl_q$, then $K = O_q$ and we have $\{\tau_2, \tau_4, \ldots, \tau_{2[q-1/2]}\}$ is part of an algebra basis for $H^*(Gl_q)$. Also, in this case $\tau_{\text{odd}} = 0$.

The next two theorems are examples of a phenomenon called permanence in [19]. There are a few related remarks to make. Many of the first examples of foliated manifolds with non-trivial secondary classes were constructed using generalized flat bundles. There is a relationship between the secondary classes for these foliations and the characteristic classes of the associated flat bundle. This relationship can be used to show that if one secondary class is non-trivial, then a whole family must be non-trivial. This principle was used by Kamber–Tondeur [18] for locally homogeneous foliations; it was further explored by Lazarov [19], and also used by Heitsch [13] to obtain many new results on the variation of the secondary classes. The theorems we give apply to an arbitrary foliation $\mathcal{F}$ on a manifold M. In one sense, we use the natural flat bundle structure on the leaves of the foliation $\mathcal{F}$. In order to assure the leaves have enough cohomological complexity to carry the classes we construct, we extend them by the group G. This forces us to lift the foliation $\mathcal{F}$ to the product $M \times G$ where we obtain our results.

The following notation is very convenient. Given a subset $\mathcal{S} \subseteq H^*(W(\mathfrak{G})_q)$ of admissible cocycles, define the *extension* of $\mathcal{S}$ to be the set of admissible cocycles

$$\bar{\mathcal{S}} = \{y_I c_J \mid I = I' \,\dot{\cup}\, I'' \text{ with } y_{I'} c_J \in \mathcal{S},\ \tau_{I''} \neq 0, \text{ and if } i \in I' \text{ has } \tau_i \neq 0 \text{ then } \deg y_j > \deg y_i \text{ for all } j \in I''\}.$$

Recall that V_G is the set defined in Corollary 3.2. If $\mathcal{S} \subseteq V_G$, then a typical element of $\bar{\mathcal{S}}$ is either of the form $y_I c_J = y_{i_1} \cdots y_{i_s} c_J$ with $\tau_{i_1} \neq 0$ and $\deg y_{i_j} > \deg y_{i_1}$ for $2 \leq j \leq s$, or of the form $y_I c_J = y_{I'} y_{I''} c_J$ with $\tau_i = 0$ for each $i \in I'$, $y_{I'} c_J \in \mathcal{S}$ and $\tau_{I''} \neq 0$. It follows that if $y_I c_J \in \bar{\mathcal{S}}$ for $\mathcal{S} \subseteq V_G$, then there is a unique decomposition $I = I' \,\dot{\cup}\, I''$ with $y_{I'} c_J \in \mathcal{S}$, $\tau_{I''} \neq 0$ and the degree of $y_{I''}$ maximal. For this decomposition we define $D(y_I c_J) = \deg y_{I''}$. This gives a function $D: \bar{\mathcal{S}} \to \{0, 1, 2, \ldots\}$ which is used in the proofs of the permanence theorems:

THEOREM 3.3. *Let $(\mathcal{F}, s)$ be a G-framed G-foliation on M and $\mathcal{S} \subseteq V_G$ a subset such that $\Delta_*^s(\mathcal{S}) \subseteq H^*(M)$ is a linearly independent set. Then there exists a G-framed G-foliation $(\mathcal{F}', s')$ on $N \stackrel{\text{def}}{=} M \times G$ such that the extended set $\Delta_*^{s'}(\bar{\mathcal{S}}) \subseteq H^*(N)$ is linearly independent.*

By Theorem 2.3 this has the consequence:

COROLLARY 3.4. *If $k_*(\mathcal{S}) \subseteq H^*(F\Gamma_G)$ is a linearly independent set for $\mathcal{S} \subseteq V_G$, then the extended set $k_*(\bar{\mathcal{S}}) \subseteq H^*(F\Gamma_G)$ is also linearly independent.*

In particular, in order to conclude that $k_*: H^*(W(\mathfrak{G})_q) \to H^*(F\Gamma_G)$ is injective, it suffices to show that the set V_G is realized independently for some G-framed G-foliation.

THEOREM 3.5. *Let $\{(\mathcal{F}_\alpha, s_\alpha) \mid \alpha \in \mathcal{A}\}$ be a family of G-framed G-foliations on M and $\mathcal{V} \subseteq V_G$ a subset which is independent and variable with respect to these foliations. Then there exists a family of G-framed G-foliations $\{(\mathcal{F}'_\alpha, s'_\alpha) \mid \alpha \in \mathcal{A}\}$ on $N = M \times G$ such that the extended set $\bar{\mathcal{V}}$ is independent and variable. Consequently, the set $k_*(\bar{\mathcal{V}}) \subseteq H^*(F\Gamma_G)$ is independently variable.*

Given that the set $\mathcal{V}$ is independently variable for the foliations on M, one can conclude, in addition, that for a suitable integer m with $1 \leq m \leq \dim H^*(G)$, the set $\bar{\mathcal{V}}$ is independently variable for an $\mathsf{X}_{j=1}^{m} \mathcal{A}$-parameter family of framed foliations on $\dot{\bigcup}_{j=1}^{m} N$.

Proof of 3.3. Define the G-foliation $\mathcal{F}'$ on $N = M \times G$ to be the pullback of $\mathcal{F}$ via the projection $p: N \to M$. A leaf of $\mathcal{F}'$ is thus a leaf of $\mathcal{F} \times G$. Define $\varphi: N \to G$ to be the projection, so that $\varphi^*: H^*(G) \to H^*(N) \cong H^*(M) \otimes H^*(G)$ is the inclusion. The associated G-bundle to $\mathcal{F}'$ is defined by the pullback

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P \arrow[d] \\ N \arrow[r, "p"] & M \arrow[u, bend right, "s"'] \end{tikzcd}$$

We construct two sections of $P' \to N$. The first, $\bar{s}$, is that induced by $s \circ p$. The second, s', is defined by the relation $s' \cdot \varphi = \bar{s}$. For the pair $(\mathcal{F}', s')$ we show that $\Delta_*^{s'}(\bar{\mathcal{S}}) \subseteq H^*(N)$ is a linearly independent set.

For the section $\bar{s}$, note that $\Delta_*^{\bar{s}} = p^* \circ \Delta_*^{s}$ by functoriality. Then from Theorem 3.1 we have

$$\begin{aligned} \Delta_*^{s'}(y_I c_J) &= \sum_{I' \cup I'' = I} \pm \Delta_*^{\bar{s}}(y_{I'} c_J) \cdot \tau_{I''} \\ &= \sum_{I' \cup I'' = I} \pm p^*(\Delta_*^{s}(y_{I'} c_J)) \cdot \tau_{I''}. \end{aligned} \tag{3.6}$$

Now suppose that $\sum_{(I,J) \in \mathcal{S}} \lambda_{(I,J)} y_I c_J \neq 0$ is an element in the span of $\bar{\mathcal{S}}$ which is in the kernel of $\Delta_*^{s'}$. We show this yields a contradiction to the hypothesis of (3.3). We may assume that the elements $y_I c_J$ for $(I, J) \in \mathcal{I}$ are all distinct and that each $\lambda_{(I,J)} \neq 0$. Then by (3.6) we obtain

$$0 = \sum_{\substack{(I,J)\in\mathcal{I} \\ I'\cup I''=I}} \pm \lambda_{(I,J)} p^* \Delta_*^{s}(y_{I'} c_J) \cdot \tau_{I''}. \tag{3.7}$$

Define a non-negative integer $n_0 = \max\{D(y_I c_J) \mid (\mathcal{I}, J) \in \mathcal{I}\}$ where $D(y_I c_J)$ is the integer defined above for a class $y_I c_J \in \bar{\mathcal{S}}$. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be the subset

$$\mathcal{I}_0 = \{(I, J) \in \mathcal{I} \mid D(y_I c_J) = n_0\}.$$

Note that if $y_I c_J \in \bar{\mathcal{S}}$ and $I = I' \cup I''$ has $\deg y_{I''} = n_0$ and $y_{I'} c_J \in \mathcal{S}$, then by the definition of n_0, either $\tau_{I''} = 0$ or $I' = \varnothing$ and $\Delta_*^{s}(y_{I'} c_J) = 0$.

Thus, in the sum (3.7) we can split off a factor

$$0 = \sum_{(I,J)\in\mathcal{I}_0} \pm \lambda_{(I,J)} p^* \Delta_*^{s}(y_{I_0'} c_J) \cdot \tau_{I_0''}. \tag{3.8}$$

We may assume that $\tau_{I_0''}$ is the same class in $H^{n_0}(G)$ for all $(I, J) \in \mathcal{I}_0$, for otherwise we decompose the sum in (3.8) further. Hence we conclude that

$$0 = \sum_{(I,J)\in\mathcal{I}_0} \pm \lambda_{(I,J)} \Delta_*^{s}(y_{I_0'} c_J)$$

in $H^*(M)$. But the classes $y_{I_0'} c_J$ are all distinct for $(I, J) \in \mathcal{S}_0$, since the classes $y_I c_J = y_{I_0'} c_J \cdot y_{I_0''}$ are assumed to be distinct. This implies that $\Delta_*^{s}(\mathcal{S})$ is not an independent set, contrary to hypothesis. ■

Proof of (3.5). Define a family of G-framed G-foliations $\{(\mathcal{F}'_\alpha, s'_\alpha) \mid \alpha \in \mathcal{A}\}$ on $N = M \times G$ as in the proof of Theorem 3.3. We must show that the union over $\alpha \in \mathcal{A}$ of the images of $\Delta_*^{s'_\alpha}(\bar{\mathcal{V}}): H_*(N: Z) \to \mathbf{R}^{\#\bar{\mathcal{V}}}$ generates $\mathbf{R}^{\#\bar{\mathcal{V}}}$ additvely. The proof will be by induction on the function $D: \bar{\mathcal{V}} \to \{0, 1, 2, \ldots\}$ which defines the "excess" degree of a class $y_I c_J \in \bar{\mathcal{V}}$ from being in $\mathcal{V}$. For each $n \geq 0$, set $\bar{\mathcal{V}}_n = \{y_I c_J \in \bar{\mathcal{V}} \mid D(y_I c_J) \leq n\}$. Note that $\bar{\mathcal{V}}_0 = \mathcal{V}$ and $\bar{\mathcal{V}}_d = \bar{\mathcal{V}}$ where $d = \dim H^*(G)$. In the following, we view $H_*(M; Z) \otimes H_*(G; Z)$ as a subgroup of $H_*(N; Z)$ and will consider $\Delta_*^{s'_\alpha} \bar{\mathcal{V}}$ restricted to it.

It is given that the union over $\alpha \in \mathcal{A}$ of the images of

$$\Delta_*^{s'_\alpha} \bar{\mathcal{V}}_n : \bigoplus_{m \leq n} H_*(M; Z) \otimes H_m(G; Z) \to \mathbf{R}^{\#\bar{\mathcal{V}}_n} \tag{3.9}$$

generates $\mathbf{R}^{\#\bar{\mathcal{V}}_n}$ when $n=0$. Now induct on n: assume this holds for some $n \geq 0$; then we show it holds for $n+1$.

Let $\mathcal{D}_n = \bar{\mathcal{V}}_{n+1} - \bar{\mathcal{V}}_n$ and assume $\mathcal{D}_n$ is not empty. First, note that

$$\Delta_*^{s'_\alpha}\bar{\mathcal{V}}_n : H_*(M; Z) \otimes H_{n+1}(G; Z) \to \mathbf{R}^{\#\bar{\mathcal{V}}_n} \tag{3.10}$$

is the zero map. For any $y_I c_J \in \bar{\mathcal{V}}$ we can use Theorem 3.1 to obtain the expansion

$$\Delta_*^{s'_\alpha}(y_I c_J) = \sum_{I = I' \dot{\cup} I''} \pm \Delta_*^{s_\alpha}(y_{I'} c_J) \otimes \tau_{I''}. \tag{3.11}$$

Given any $y_I c_J \in \bar{\mathcal{V}}_n$ such that in the sum (3.11) there is a term with $\deg y_{I''} = n+1$, then by the definition of $D(y_I c_J)$ either $\tau_{I''} = 0$ or $\Delta_*^{s_\alpha}(y_{I'} c_J) = 0$. Therefore, $\Delta_*^{s_\alpha}(y_I c_J)$ vanishes on the domain of (3.10).

To complete the inductive step, it suffices to show the union over $\alpha \in \mathcal{A}$ of the images of

$$\Delta_*^{s'_\alpha}\mathcal{D}_n : H_*(M; Z) \otimes H_{n+1}(G; Z) \to \mathbf{R}^{\#\mathcal{D}_n} \tag{3.12}$$

generates $\mathbf{R}^{\#\mathcal{D}^n}$. The assumption $\mathcal{V} \subseteq V_G$ implies that for any $y_I c_J \in \mathcal{D}_n$, there is a unique decomposition $I = I'_0 \dot{\cup} I''_0$ such that $\tau_{I''_0} \neq 0$, $D(y_I c_J) = \deg y_{I''_0} = n+1$ and $y_{I'_0} c_J \in \mathcal{V}$. So Formula (3.11) implies

$$\Delta_*^{s'_\alpha}(y_I c_J) = \pm \Delta_*^{s_\alpha}(y_{I'_0} c_J) \otimes \tau_{I''_0}$$

when restricted to the domain of the map (3.12). Since the classes $\tau_{I''_0} \in H_{n+1}(G; Z)$ are as independent as the $y_{I''_0}$ and it is given that $\mathcal{V}$ is independent and variable for the foliations on M, it follows that the images of (3.12) generate the range. ∎

§4. Proofs of the main theorems

In this section we combine some known properties of $F\Gamma_G$ with the machinery and results of sections 1 to 3 to prove Theorems 1 through 5.

Proof of Theorem 1. Our construction of foliated manifold with non-zero rigid classes begins with a theorem of Mather and Thurston [27], [22], [25]:

THEOREM 4.1. *For $q \geq 1$ the space $F\Gamma_q$ is $(q+1)$-connected.*

For $q=2$ this implies the map $\nu: B\Gamma_2 \to BO_2$ is 4-connected. Let $g: S^2 \times S^2 \to BO_2$ be a map such that $g_*: H_4(S^2 \times S^2) \to H_4(BO_2)$ is an isomorphism. Since ν is

4-connected, we can lift g to a map $\bar{g}: S^2 \times S^2 \to B\Gamma_2$. Note that $(\nu \circ \bar{g})^*(p_1) \neq 0$, where $p_1 \in H^4(BO_2)$ is the first pontrjagin class.

Part (a). Given $q = 2q' \geq 4$, set $X = (S^2 \times S^2) \times \cdots \times (S^2 \times S^2)$, q' factors. The map $\bar{g}$ induces a natural map $f: X \to B\Gamma_q$ with $(\nu \circ f)^*(p_1^{q'}) \neq 0$. To obtain a foliated manifold, we let $M' = X \times S^n$ with $\alpha: M' \to X$ the projection, where $n > 2q$ is chosen large enough so that $\nu \circ f \circ \alpha$ determines a splitting of TM' into $F' \oplus Q$. Note that $\alpha^*: H^m(X) \to H^m(M')$ is an isomorphism for $m \leq 2q$. By Theorem 1 of Thurston [28], the data consisting of the splitting $TM' = F' \oplus Q$ and the map $f \circ \alpha$ determines a foliation $\mathcal{F}'$ on M' which is classified by a map homotopic to $f \circ \alpha$.

Let $P' \to M'$ be the Gl_q-bundle associated to $\mathcal{F}'$. Choose a reduction of P' to an O_q-bundle with total space M:

$$\begin{tikzcd} M \arrow[r, phantom, "\subseteq"] \arrow[d, "\pi"'] & P' \arrow[dl] \\ M' & \end{tikzcd}$$

The projection π induces a foliation $\mathcal{F}$ on M. Let $P \to M$ be the Gl_q-bundle associated to $\mathcal{F}$, and define a section s of this bundle by

$$\begin{tikzcd} P \arrow[r, "\bar{\pi}"] \arrow[d, shift left] & P' \arrow[d] \\ M \arrow[u, bend left, "s"] \arrow[ur, "\subseteq"] \arrow[r, "\pi"] & M' \end{tikzcd}$$

Choose a Bott connection ω' on $P' \to M'$. Let ω be the pull-back of ω' to $P \to M$. Then

$$\Delta_*^s = s^* \circ k(\omega)_* = s^* \circ \bar{\pi}^* \circ k(\omega')_* = k(\omega')_*,$$

where we use the isomorphism $H^*(M) \cong H^*(P')$. By Theorem 3.3, it suffices to show that $k(\omega')_*(y_2 c_q) \neq 0$.

Consider the map of spectral sequences of §1 associated to $k(\omega')$:

$$E_2 k(\omega'): W_q^{2p,t} \to E_2^{2p,t}(P') \cong H^{2p}(X) \otimes H^t(O_q),$$

where the last isomorphism holds for $2p \leq 2q$. Denote the elements in the image $E_2 k(\omega')$ by:

$$E_2 k(\omega')(y_i) = \bar{y}_i \in H^{2i-1}(O_q)$$
$$E_2 k(\omega')(c_i) = \bar{c}_i \in H^{2i}(X).$$

We want to show that the cocycle $\bar{y}_2\bar{c}_q$ in $E_2^{2q,3}(P')$ survives to $E_\infty(P')$. To see this, first not that $d_{2r}\bar{y}_{2r}=\bar{c}_{2r}$, and each class $\bar{c}_{2j}$ is a multiple of $\bar{c}_2^j$. Thus $E_5^{2p,t}(P')\cong E_\infty^{2p,t}(P')$ for $2p\leq 2q$, and $E_5^{2p,t}(P')\cong E_2^{2p,t}(P')/\bar{c}_2\cdot E^{2p-4,t}$. The class $\bar{y}_2\bar{c}_q$ survives in this quotient, so we are done.

Part (b). For $q=4k-2$, choose a map $g: S^{4k}\to BO_q$ such that $g^*(p_k)\neq 0$. Then by Theorem 4.1 there exists a lift $\bar{g}: S^{4k}\to B\Gamma_q$ of g. Choose a map $\alpha: M'\to S^{4k}$ as in the proof of (a) with $\alpha^*: H^m(S^{4k})\to H^m(M')$ an isomorphism for $m\leq 2q$. The proof then proceeds exactly as in (a). Note that $\Delta_*^{s'}(y_2y_Ic_q)=0$ in $H^*(M)$ since this class comes from the cocycle

$$\bar{y}_2\bar{y}_I\bar{c}_q\in E_2^{2q,t}(P)\cong H^{2q}(S^{4k})\otimes H^t(O_q)=0.$$

Part (c). This follows directly from Theorem 2.3. ∎

The proof of Theorem 3 is very similar to that of Theorem 1 so we give it next.

Proof of Theorem 3. Let X be a connected, finite CW complex of dimension q with $g: X\to BSO_q$ a map such that $g^*: H^m(BSO_q)\to H^m(X)$ is an isomorphism for $0\leq m\leq q$. Let U be an open manifold which retracts to X; let M' be the total space of the $\mathbf{R}^q$-vector bundle over U induced by $g: U\to BSO_q$. We need the following analog of Theorem 4.1 which is proved in [15]:

THEOREM 4.2. *For* $q\geq 1$ *the space* $F\Gamma_{SO_q}$ *is* $(q-1)$*-connected.*

By this theorem the map g lifts to a map $f: M'\to B\Gamma_{SO_q}$. The bundle TM' has a canonical q-dimensional subbundle tangent to the fibers of $M'\to U$. Thus by the Gromov–Phillips Theorem as applied by Haefliger [10], these data determine a Riemannian foliation $\mathcal{F}'$ on M'. Let M be the total space of the principal SO_q-bundle associated to $\mathcal{F}'$. The bundle map $\pi: M\to M'$ induces a Riemannian foliation $\mathcal{F}$ on M. There is a canonical framing s of the normal bundle defined by the diagram

$$\begin{tikzcd} P \arrow[r, "\bar{\pi}"] \arrow[d, shift left] & M \arrow[d] \\ M \arrow[u, bend left, "s"] \arrow[ur, "\mathrm{id}"] \arrow[r, "\pi"] & M' \end{tikzcd}$$

Let ω' be a Bott connection on $M \to M'$. Then as we have seen before,

$$\Delta_*^s = s^* \circ k(\omega) = s^* \circ \bar{\pi} \circ k(\omega')_* = k(\omega')_*.$$

Now the map of spectral sequences associated to $k(\omega')$ gives

$$E_2 k(\omega'): A(so(q))_{[q/2]} \to E_2^{2p,t}(M) \cong H^{2p}(X) \otimes H^t(SO_q).$$

By the construction of the foliation $\mathcal{F}'$ on $M' \simeq X$, this map is an isomorphism. Thus $k(\omega')_*$ is also an isomorphism. ■

The proofs of the remaining theorems all use the permanence result, Theorem 3.5.

Proof of Theorem 2. For $q \geq 3$ and odd, Heitsch constructs in [Theorem 6.3; 13] a family of framed foliations $\{(\mathcal{F}_\alpha, s_\alpha) \mid \alpha \in \mathcal{A}\}$ of codimension q on a compact manifold M. Also, a set $\mathcal{V}_q \subseteq H^{2q+1}(W_q)$ of admissible cocycles is given such that $\Delta_*^{s_\alpha}(\mathcal{V}_q) \subseteq H^*(M)$ varies independently.

Note that a class of degree $2q+1$ in $H^*(W_q)$ must be of the form $y_i c_J$ and thus lies in the set V_{Gl_q}. So we can apply Theorem 3.5 to get a family of foliations $\{(\mathcal{F}'_\alpha, s'_\alpha) \mid \alpha \in \mathcal{A}\}$ on $N = M \times Gl_q$ with the desired properties. Finally, we note that the inclusion $M \times O_q \subseteq M \times Gl_q$ is transverse to the $\mathcal{F}'_\alpha$, so we can use the compact manifold $M \times O_q = N$.

For $q \geq 2$ and even, let $\mathcal{V}_q \subseteq H^{2q+1}(WO_q) \subseteq H^{2q+1}(W_q)$ be the subset which is realized independently in [Theorem 6.2; 13]. It is shown in [16] that this set can be realized independently on a compact manifold M with a family of framed foliations. Then proceed as in the case q odd. ■

Proof of Theorem 4. For $q = 4k-2$, let $\mathcal{V}_q$ be the basis of $H^{q+1}(W(so_q)_{[q/2]})$ which consists of admissible classes. In Section 3 of [21], Lazarov and Pasternack construct a family of SO_q-framed Riemannian foliations on the compact Lie group U_{2k} for which the set $\mathcal{V}_q$ is independent and variable. By Theorem 3.5, the extended set $\bar{\mathcal{V}}_q$ is independent and variable with respect to a family of SO_q-framed foliations on $M = U_{2k} \times SO_q$. The set $\bar{\mathcal{V}}_q$ is a basis of the variable classes, so this establishes the theorem in this case.

For $q = 4k-1$, start with the family of SO_q-framed Riemannian foliations on $U_{2k} \times S^1$ which are obtained from the family on U_{2k} of [21] by letting S^1, with the standard metric, be normal to the leaves. Then take $M = U_{2k} \times S^1 \times SO_q$ and apply Theorem 3.5. ■

Proof of Theorem 5. Let $\mathcal{V}_n$ be the basis of $H^{2n+1}(W(Gl_n\mathbf{C})_n)$ consisting of admissible cocycles. Let $\{(\mathcal{F}_\alpha, s_\alpha) \mid \alpha \in \mathcal{A}\}$ be the family of $Gl_n\mathbf{C}$-framed complex foliations on the compact Lie group U_{n+1} which is derived from Example 11.1 of [1]. It follows from [1] that $\mathcal{V}_n$ is an independent and variable set with respect to this family. Set $M = U_{n+1} \times U_n$. By Theorem 3.5 the extended set $\bar{\mathcal{V}}_n$ is independent and variable with respect to a family $\{(\mathcal{F}'_\alpha, s'_\alpha) | \alpha \in \mathcal{A}\}$ on M. The set $\bar{\mathcal{V}}_n$ is a basis for the variable classes in $H^*(W(Gl_n\mathbf{C})_n)$, so this finishes the proof. ■

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