VANISHING OF SECONDARY CLASSES FOR COMPACT FOLIATIONS

STEVEN HURDER

Let $\mathscr{F}$ be a C^2-foliation of codimension q on a compact manifold without boundary. If each leaf of $\mathscr{F}$ is a compact submanifold of M, then $\mathscr{F}$ is a *compact* foliation. The purpose of this paper is to show that the Godbillon–Vey class of a compact foliation vanishes, as must all of the 'residuable' secondary classes

$$H_q^*(\mathrm{WO}_q) = \{y_I c_J \in H^*(\mathrm{WO}_q) : \deg c_J = 2q\}\,.$$

THEOREM. *Let $\mathscr{F}$ be a compact, C^2-foliation on a closed manifold M. For each $y_I c_J \in H_q^*(\mathrm{WO}_q)$ the secondary class $\Delta_*(y_I c_J) \in H^*(M)$ of $\mathscr{F}$ is zero. In particular, the Godbillon–Vey class $\mathrm{gv}(\mathscr{F}) = \Delta_*(y_1 c_1^q)$ is zero.*

In codimension one, remarkable progress has been made on the problem of relating the Godbillon–Vey class of a general foliation $\mathscr{F}$ with the geometry of $\mathscr{F}$, especially the rates of growth of the leaves [**3**, **4**, **11**]. The solution of the Sullivan conjecture by Duminy [**4**] implies that the leaves of $\mathscr{F}$ must have exponential growth if $\mathrm{gv}(\mathscr{F}) \neq 0$. For higher codimensions, the secondary classes in $H_q^*(\mathrm{WO}_q)$ appear to be the best candidates for admitting an extension of the codimension one results. The above theorem asserts that the classes from $H_q^*(\mathrm{WO}_q)$ vanish if all leaves of $\mathscr{F}$ have growth of degree zero. The terminology 'residuable' is used for an element $y_I c_J$ in $H_q^*(\mathrm{WO}_q)$ because the evaluation of the class $\Delta_*(y_I c_J) \in H^*(M)$ can be obtained using an integral formula which is often expressible as a generalized residue along the leaves of $\mathscr{F}$ (for example, see [**8**]). The reduction to an integral formula is the key to the proof of the theorem.

It is well known that a compact foliation of codimension one with orientable normal bundle is defined by a submersion $M \to S^1$ [**7**]. For a codimension two compact foliation, the beautiful study of compact foliations by Edwards, Millet and Sullivan [**5**] showed that M has a covering by saturated open sets $U_1, \ldots, U_n$ for which $\mathscr{F}|_{U_i}$ has the property that there is a finite cover $\tilde{U}_i$ of U_i such that the lifted foliation $\tilde{\mathscr{F}}|_{\tilde{U}_i}$ on $\tilde{U}_i$ is defined by a submersion onto the disc D^2. A metric on D^2 pulls back to a metric on the normal bundle of $\mathscr{F}$ which is invariant under the holonomy of $\mathscr{F}$. Thus, $\mathscr{F}$ is a riemannian foliation and *all* of its secondary classes vanish. For $q > 2$, a compact foliation need not be riemannian, as evidenced by the examples of Sullivan and others, even if $\mathscr{F}$ is analytic [**13**]. These examples show that the geometry of a compact foliation of higher codimension can be very complicated. In spite of this, the restrictions on the holonomy of $\mathscr{F}$ imposed by the condition that all leaves are compact leads to the conjecture that all of the secondary classes of $\mathscr{F}$ must vanish [**10**, Question 1.22].

We give now the idea of the proof of the theorem. Passing to a double cover of M if necessary, we can assume that M is orientable. Multiplying the form on M

Received 23 February, 1982.

J. London Math. Soc. (2), 28 (1983), 175–183

representing our secondary class by a closed form, we see that it is enough to show that an integral over M vanishes. We use the Epstein filtration [5, 6] of the bad set of $\mathscr{F}$ to decompose this integral into a sum of integrals over saturated sets $Y_\alpha = T_\alpha \times L_\alpha$ on which $\mathscr{F}$ is the product foliation. We then show that the secondary class decomposes into a corresponding product, and the integral of one factor over a leaf L_α is proportional to a leaf class of $\mathscr{F}$ for L_α. The leaf classes of a compact foliation are identically zero, and so the integrals over the sets Y_α all vanish.

Preparatory lemmas are given in §§1 and 2. The proof in a special case is given in §3, while in §4 we consider the general integral over the bad set.

It is a pleasure to thank André Haefliger for helpful conversations and comments on this work. The author is grateful to the referee for several suggestions which have improved and clarified the presentation.

1. *The bad set of a compact foliation*

Let $\mathscr{F}$ be a compact foliation on M, and fix a riemannian metric on M. Each leaf L of $\mathscr{F}$ has an induced metric, and therefore a volume form and a total volume $\mathrm{vol}(L)$. Define a function on M by assigning to a point x the number $\mathrm{vol}(L_x)$, where L_x is the leaf through x. For details on the properties of the function $x \mapsto \mathrm{vol}(L_x)$, the reader is referred to the concise summary given in Proposition 4.1 of [5]. Note that the function $\mathrm{vol}(L_x)$ is continuous at x precisely when L_x has no holonomy.

DEFINITION 1.1. The *bad set* X_1 of $\mathscr{F}$ is the union of the leaves with holonomy:

$$X_1 = \{x_0 \in M \mid x \mapsto \mathrm{vol}(L_x) \text{ is not continuous at } x_0\}.$$

The bad set is a saturated, closed and nowhere dense subset of M. The Lebesgue measure $\mu(X_1)$ need not be zero, and this forces us to introduce a nice decomposition of X_1. The *Epstein filtration* of the bad set is a decreasing collection $\{X_\alpha \mid \alpha \in \mathscr{A}\}$ of closed subsets of X_1 on whose successive differences the function $\mathrm{vol}(L_x)$ is continuous. See [5, §6] or [6, §6] for complete details. The indexing set $\mathscr{A}$ consists of the ordinals, and $X_\beta = \varnothing$ for some countable successor ordinal $\beta \in \mathscr{A}$. For each $\beta > \alpha$, X_β is a saturated, closed and nowhere dense subset of X_α.

For each $\alpha \in \mathscr{A}$, set $Y_\alpha = X_\alpha - X_{\alpha+1}$, and let $Y_0 = M - X_1$. Then $\mathrm{vol}(L_x)$ is a continuous function when restricted to Y_α, whence $\mathscr{F}|_{Y_\alpha}$ has no holonomy [5, Proposition 4.1]. The set Y_α is relatively open in the closed set X_α. We observe that Y_α has an exhaustion by closed, saturated subsets; this will be used in §4.

LEMMA 1.2. *For $\alpha \in \mathscr{A}$ and given $\varepsilon > 0$, there is a closed, saturated subset $K \subseteq X_\alpha$ with $K \cap X_{\alpha+1} = \varnothing$ and $\mu(Y_\alpha - K) < \varepsilon$.*

Proof. Let $W \subseteq M$ be open with $X_{\alpha+1} \subseteq W$ and $\mu(W - X_{\alpha+1}) < \varepsilon$. Let K denote the $\mathscr{F}$-saturation of the closed set $Z = (M - W) \cap X_\alpha$. Since $\mathscr{F}|_{Y_\alpha}$ is without holonomy, by [6, Proposition §8] the quotient map $\pi : Y_\alpha \to Y_\alpha/\mathscr{F}$ is open and proper. The set $\pi(Z)$ is compact and $K = \pi^{-1}(\pi(Z))$, so that K is compact and hence closed in M. Both X_α and $X_{\alpha+1}$ are saturated, and so $K \subseteq Y_\alpha$. Finally, $X_\alpha - K \subseteq W$ implies that

$$\mu(Y_\alpha - K) = \mu(X_\alpha - X_{\alpha+1} - K) \leqslant \mu(W - X_{\alpha+1}) < \varepsilon.$$

2. *Leaf and secondary classes*

Let $\mathscr{A}^{\cdot}(M)$ denote the deRham algebra of M. The choice of a Bott connection for $\mathscr{F}$ defines a differential algebra map $\Delta : \mathrm{WO}_q \to \mathscr{A}^{\cdot}(M)$. The algebra WO_q is the product of an exterior algebra with a truncated polynomial algebra

$$\mathrm{WO}_q \cong \Lambda(y_1, y_3, \ldots, y_{2[(q+1)/2]-1}) \otimes \mathbb{R}[c_1, \ldots, c_q]_q$$

where $\deg y_i = 2i-i$, $\deg c_i = 2i$ and the differential is determined by $dy_i = c_i$ and $dc_i = 0$. The construction of Δ and its properties are described in detail in the foundational paper of Bott [**1**]; see also [**2**]. The image of the map in cohomology, $\Delta_* : H^*(\mathrm{WO}_q) \to H^*(M)$, consists of the secondary classes of $\mathscr{F}$.

Next recall the construction of the leaf classes for a leaf $L \subseteq M$. Let $Q = TM/\mathscr{F}$ be the normal bundle to $\mathscr{F}$ and let $\Gamma(M, Q^*) \subseteq \mathscr{A}^1(M)$ be the space of 1-forms which annihilate $\mathscr{F}$. The crucial property of the map Δ is that, for all i,

$$\Delta(c_i) \in \Gamma(M, \Lambda^i Q^*) \wedge \mathscr{A}^i(M),$$

a consequence of constructing Δ using a Bott connection. For the leaf L, this implies that the form $\Delta(c_i)$ vanishes when restricted to L, and therefore each $\Delta(y_i)|_L$ is a closed form. We remark that this vanishing corresponds to the observation that the restricted bundle $Q|_L \to L$ has a natural flat structure obtained by restricting the Bott connection on Q. The curvature of a flat connection is zero, and hence $\Delta(c_i)|_L = 0$.

For an index $I = (i_1, \ldots, i_s)$ with $1 \leqslant i_1 < \ldots < i_s \leqslant q$ and all i_l odd, set $y_I = y_{i_1} \ldots y_{i_s} \in \mathrm{WO}_q$. The form $\Delta(y_I) \in \mathscr{A}^{\cdot}(M)$ is not closed in general, but the above remarks imply that $\Delta(y_I)|_L$ is closed and determines a class in $H^*(L)$. The restriction of Δ thus defines an algebra map

$$\chi_L : (\mathbf{gl}_q, \mathrm{O}_q) \to H^*(L)$$

where we identify the relative Lie algebra cohomology $H^*(\mathbf{gl}_q, \mathrm{O}_q)$ with the exterior algebra in WO_q. The image of χ_L consists of the leaf classes of $\mathscr{F}$ for L. The map χ_L is an invariant of the germ of $\mathscr{F}$ about L, and in fact depends only on the flat bundle structure of $Q|_L \to L$. To be precise, recall that the foliation in a neighborhood of L determines the linear holonomy $dh : \pi_1(L, x) \to \mathrm{Gl}_q\mathbb{R}$ where $x \in L$. The flat structure on $Q|_L$ is classified by the induced map $B(dh) : L \to B\mathrm{Gl}_q^{\delta}\mathbb{R}$ where $\mathrm{Gl}_q^{\delta}\mathbb{R}$ has the discrete topology.

In the study of the leaf classes by Shulman and Tischler [**12**] (see also [**9**, Chapter 6]) the following relationship is proven.

PROPOSITION 2.1. *There is a commutative diagram*

$$\begin{tikzcd}
H^*(\mathbf{gl}_q, \mathrm{O}_q) \arrow[rr, "\chi_L"] \arrow[dr, "\mathrm{VE}"'] & & H^*(L) \\
& H^*(B\mathrm{Gl}_q^{\delta}\mathbb{R}) \arrow[ur, "B(dh)^*"'] &
\end{tikzcd}$$

where VE *is the Van Est map defining the continuous cohomology of* $\mathrm{Gl}_q^{\delta}\mathbb{R}$.

COROLLARY 2.2. *If the linear holonomy of a leaf L is trivial, then all leaf classes* $\chi_L(y_I)$ *are zero in* $H^*(L)$.

Actually, a stronger form of 2.2 can be shown which is relevant to compact foliations. Recall that a matrix $A \in \mathrm{Gl}_q\mathbb{R}$ is unipotent if all eigenvalues of A have modulus 1. The linear holonomy of a leaf L is said to be unipotent if for all $\gamma \in \pi_1(L, x)$ the matrix $dh(\gamma)$ is unipotent.

PROPOSITION 2.3. *If the linear holonomy of a leaf L has a solvable subgroup of finite index, then all leaf classes from* $H^m(\mathbf{gl}_q, \mathrm{O}_q)$ *vanish for* $m > 1$. *If the linear holonomy is unipotent, then all leaf classes for L vanish.*

Proof. We can put dh into standard form (for example, see [**14**, Proposition 3.2]) and so we can assume that the image of dh is contained in a closed subgroup $H \subseteq \mathrm{Gl}_q\mathbb{R}$, where H is either unipotent or solvable. For the Lie algebra $\mathbf{h}$ of H, the induced map $H^*(\mathbf{gl}_q, \mathrm{O}_q) \to H^*(\mathbf{h})$ is then zero for H unipotent, or zero when $* > 1$ for H solvable. The composition $B(dh)^* \circ \mathrm{VE}$ can be factored

$$H^*(\mathbf{gl}_q, \mathrm{O}_q) \longrightarrow H^*(\mathbf{h}) \longrightarrow H^*(BH^\delta) \longrightarrow H^*(L)$$

and hence is zero for H unipotent, or vanishes in degrees greater than one for H solvable. The composition is equal to χ_L by Proposition 2.1, and Proposition 2.3 follows.

For a compact foliation the linear holonomy of each leaf L is unipotent (see Lemma 4.5 below). Thus, for every leaf L of a compact foliation, all leaf classes vanish.

3. *Proof of the theorem*

Let $\mathcal{F}$ be a compact foliation on the compact m-manifold M without boundary. By passing to a two or four-fold covering of M, we can assume that both M and the normal bundle Q of $\mathcal{F}$ are orientable. For $y_I c_J \in H^n(\mathrm{WO}_q)$ with $\deg c_J = 2q$, we must show that $\Delta_*(y_I c_J) \in H^n(M)$ is zero. It suffices to show, by Poincaré duality, that for every closed $(m\text{-}n)$-form ϕ the integral $\int_M \phi \cdot \Delta(y_I c_J) = 0$. In §1 a countable decomposition of M associated to the Epstein filtration of X_1 was introduced: $M = \dot{\bigcup_\alpha} Y_\alpha$. It is enough to prove that the integral over each Y_α is zero. We treat the technically simpler case of $Y_0 = M - X_1$ in this section, and the case of a general Y_α in the next. If X_1 has measure zero in M, then the theorem is equivalent to the statement that the integral over Y_0 vanishes. For example, if $\mathcal{F}$ is transversely analytic, or if every leaf in X_1 has non-trivial linear holonomy, then $\mu(X_1) = 0$.

For convenience, set $Y = Y_0$. Recall that Y is an open saturated set with every leaf compact and $\mathcal{F}|_Y$ has no holonomy. The quotient space $T = Y/\mathcal{F}$ is therefore an open, smooth Hausdorff manifold with $\mathcal{F}|_Y$ defined by the submersion $\pi : Y \to T$ [**5**, §8]. Choose a volume form $\tilde{\omega}$ on T. Then $\omega = \pi^*\tilde{\omega}$ is a transverse invariant volume form for $\mathcal{F}$ on Y, satisfying $d\omega = 0$ and $i(v)\omega = 0$ for all vector fields v on Y tangent to $\mathcal{F}$.

Give M a riemannian metric, which determines an embedding $Q \subseteq TM$ orthogonal to $\mathscr{F}$. The exterior power bundle $\Lambda^q Q$ is orientable and we choose a section $z \in \Gamma(Y, \Lambda^q Q)$ over Y satisfying $\omega(z) = 1$. Define a q-form on Y

$$\hat{c}_J = i(z)\Delta(c_J).$$

Locally, there are vector fields $z_1, \ldots, z_q$ which frame Q with $z = z_1 \wedge \ldots \wedge z_q$; then for vector fields $y_1, \ldots, y_q$ we have $\hat{c}_J(y_1, \ldots, y_q) = c_J(z_1, \ldots, z_q, y_1, \ldots, y_q)$. The assumption that c_J has degree $2q$ implies that the form $\Delta(c_J)$ belongs to $\Gamma(M, \Lambda^q Q^*) \wedge \mathscr{A}^q(M)$, whence $\Delta(c_J) = \omega \cdot \hat{c}_J$ on Y. The form $\hat{c}_J$ is not closed, but we observe that

$$0 = d\Delta(c_J) = d(\omega \cdot \hat{c}_J) = d\omega \cdot \hat{c}_J \pm \omega \cdot d\hat{c}_J = \pm \omega \cdot d\hat{c}_J.$$

This implies that $d\hat{c}_J$ vanishes when restricted to a leaf $L \subseteq Y$, and so $\hat{c}_J|_L \in \mathscr{A}^q(L)$ is a closed form.

Now consider

$$\int_Y \phi \cdot \Delta(y_I c_J) = \int_Y \phi \cdot \Delta(y_I) \cdot \omega \cdot \hat{c}_J = (-1)^q \int_{x \in T} \left\{ \int_{L_x} \phi \cdot \Delta(y_I) \cdot \hat{c}_J|_{L_x} \right\} \cdot \omega, \quad (3.1)$$

where the expression in (3.1) is the integral over the compact fibers $\{L_x | x \in T\}$ of the fibration $\pi : Y \to T$, and the integrand factors because ω is a basic form on Y of degree $q = \dim T$. The foliation $\mathscr{F}|_Y$ has no holonomy, and so for all leaves $L_x \subseteq Y$ the leaf class $\chi_{L_x}(y_I) \in H^{n-2q}(L)$ is zero. Since $\hat{c}_J|_{L_x}$ is a closed form, and L_x is closed, each integral $\int_{L_x} \phi \cdot \Delta(y_I) \cdot \hat{c}_J = 0$. The expression in (3.1) thus vanishes, and this proves our claim.

4. *When the bad set has positive measure*

The bad set X_1 is closed and nowhere dense in M; thus if the Lebesgue measure $\mu(X_1) > 0$, then the transverse structure of X_1, and hence of each $Y_\alpha = X_\alpha - X_{\alpha+1}$, is very complicated. We must show that $\int_{Y_\alpha} \phi \cdot \Delta(y_I c_J) = 0$ for each α, and special care is needed to make the techniques of §3 extend. We fix α. The set Y_α can be exhausted by closed saturated sets $K \subseteq Y_\alpha$ by Lemma 1.2, and so it is enough to show that the integral over such a K vanishes. The difficulty in extending the formula 3.1 is that the quotient space $T = K/\mathscr{F}$ is compact and Hausdorff but has no interior, and so we cannot use T to define a transverse invariant volume form for $\mathscr{F}$ on K. To circumvent this, we reduce the integral over K to a finite sum of integrals over compact saturated subsets $K_i \subseteq K$. Each set K_i is chosen to have an open neighborhood U_i on which there is a transverse volume form ω_i which is invariant when restricted to K_i. The existence of such an ω_i is then sufficient to make the method of §3 work for the integral over K_i.

For the reader's convenience and to fix our notation, we give the definition of a foliation chart.

DEFINITION 4.1. A *foliation chart* (U, ϕ) for $\mathscr{F}$ centered at $x \in M$ consists of an open neighborhood $U \subseteq M$ of x and a diffeomorphism

$$\phi = (\psi, f) : U \to D^{m-q} \times D^q \subseteq \mathbb{R}^m$$

such that $\phi(x) = 0$, and the second factor defines $\mathscr{F}|_U$ as the level sets of $f : U \to D^q \subseteq \mathbb{R}^q$. For each $y \in U$, the set

$$D_y = \phi^{-1}\big(\psi(y) \times D^q\big) \subseteq U$$

is the transversal to $\mathscr{F}$ through y associated to (U, ϕ).

Since Q is orientable, we choose an orientation, and require also that the local map $f_* : Q|_U \to T\mathbb{R}^q$ be orientation preserving, where $\mathbb{R}^q$ has the standard orientation.

The decomposition $\{K_1, \ldots, K_r\}$ of K will be defined after some preliminary constructions. Given a leaf $L \subseteq K$, choose a base point $x \in L$. Since L is compact, we can choose a finite set of open foliation charts $\{(V_j, \phi_j) \mid j = 1, \ldots, p\}$ with V_j centered at $y_j \in L$, $y_1 = x$, and $L \subseteq \bigcup_{j=1}^{p} V_j \equiv V$. Let β_j be a path in L from x to y_j. Recall that D_{y_j} is the transverse disc for (V_j, ϕ_j) centered at y_j. By shrinking V_1 in the transverse direction if necessary, we can assume that the holonomy along β_j defines a diffeomorphism into, denoted by $\gamma_{1j} : D_x \to D_{y_j}$.

Set $C \equiv M - V$, a compact set. Let $T = K/\mathscr{F}$; by the proof of Lemma 1.2, we know that $\pi : K \to T$ is an open proper map. Thus, $\pi(K \cap C)$ is compact in T, and $Z \equiv \pi^{-1}\big(\pi(K \cap C)\big)$ is a closed saturated set. For technical reasons, we shrink the cover $\{V_j\}$ once again by deleting the set Z. Define open sets $U_j \equiv V_j - Z$ for $j = 1, \ldots, p$ and let $U_L = \bigcup_{j=1}^{p} U_j$. Observe that $K - Z$ is saturated and we have $L \subseteq K - Z = K \cap U_L \subseteq U_L$.

The compact set K is covered by the open sets U_L for $L \subseteq K$. Choose a finite subcover $U_{L_1}, \ldots, U_{L_r}$. For each L_i let D_i be the transverse disc through a base point $x_i \in L_i$, as in the above construction. Each leaf $L \subseteq K$ must intersect some U_{L_i} and thus intersect the transverse disc D_i. Recall that the quotient $T = K/\mathscr{F}$ is a compact hausdorff space, and let $\pi : K \to T$ be the quotient map. Then the sets $\tilde{D}_i = \pi(D_i \cap K)$, $1 \leqslant i \leqslant r$, form an open cover for T. Choose a decomposition $T = T_1 \cup \ldots \cup T_r$ with $T_i \subseteq \tilde{D}_i$ a closed subset of T and such that, for $i \neq j$, $T_i \cap T_j$ has measure zero as a subset of the transversal disc D_i. We can now define the closed subsets of K:

$$K_i = \pi^{-1}(T_i), \qquad 1 \leqslant i \leqslant r.$$

It is immediate that $K = K_1 \cup \ldots \cup K_r$ and $\mu(K_i \cap K_j) = 0$ if $i \neq j$. Note that K_i is contained in the open set U_{L_i}.

The care taken above to construct the sets K_i and U_{L_i} seems to be necessary, because the set $K \subseteq X_1$ has no *a priori* restriction on its global topology. To make the constructions which follow, we need to localize to well-behaved pieces of K. For the rest of this section, we fix i and set $K = K_i$, $U = U_{L_i}$ and $L = L_i$.

LEMMA 4.2. *There is a smooth q-form ω on U which defines $\mathscr{F}$ on U and satisfies $\theta(v)\omega_x = 0$ for all $x \in K$ and vector fields v on U tangent to $\mathscr{F}$.*

Here, $\theta(v)\omega_x$ denotes the Lie derivative of ω along v evaluated at x.

Since $\omega(v) = 0$ for all v tangent to $\mathscr{F}$, the Cartan formula $\theta(v) = i(v) \circ d + d \circ i(v)$ applied to the form ω in (4.2) yields the vanishing condition

$$i(v)d\omega_x = 0 \qquad \text{for all } x \in K \text{ and } v \text{ tangent to } \mathscr{F}. \tag{4.3}$$

Assuming Lemma 4.2, we finish the proof of the theorem. Choose a section z of $\Lambda^q Q$ over U satisfying $\omega(z) = 1$, and then define $\hat{c}_J = i(z)\Delta(c_J)$ as in §3 so that $\Delta(c_J) = \omega \cdot \hat{c}_J$. On a neighborhood of a point $x \in K$, let $z_1, \ldots, z_q$ be a framing of Q with $z = z_1 \wedge \ldots \wedge z_q$ and let $y_1, \ldots, y_{q+1}$ be vector fields tangent to $\mathscr{F}$. Since $\omega \wedge d\hat{c}_J = \pm d\omega \wedge \hat{c}_J$ we have at x that

$$d\hat{c}_J(y_1, \ldots, y_{q+1}) = \omega \wedge d\hat{c}_J(z_1, \ldots, z_q, y_1, \ldots, y_{q+1}) = \pm d\omega \wedge \hat{c}_J(z_1, \ldots, z_q, y_1, \ldots, y_{q+1})$$

$$= \pm \sum_{l=1}^{q+1} (-1)^{q+l} \cdot d\omega(y_l, z_1, \ldots, z_q) \cdot \hat{c}_J(y_1, \ldots, \hat{y}_l, \ldots, y_{q+1}) = 0$$

because $i(y_l)d\omega_x = 0$ from (4.3). For a leaf $L \subseteq K$, this implies that $\hat{c}_J|_L$ is a closed form.

The q-form ω determines a transverse measure to $\mathscr{F}$ on K, and the property $\theta(v)\omega = 0$ on K implies that the measure is invariant. Then by [5, Lemma, p. 25], we have a decomposition

$$\int_K \phi \cdot \Delta(y_I c_J) = \int_K \phi \cdot \Delta(y_I) \cdot \hat{c}_J \cdot \omega = \int_{x \in T} \left\{ \int_{L_x} (\phi \cdot \Delta(y_I) \cdot \hat{c}_J)|_{L_x} \right\} \cdot \omega. \tag{4.4}$$

The restriction $\phi \cdot \hat{c}_J|_L$ is a closed form, and Proposition 2.3 implies that $\Delta(y_I)|_L$ is exact for every leaf $L \subseteq K$. Therefore, for each $x \in L$, the integral $\int_{L_x} \phi \cdot \Delta(y_I) \cdot \hat{c}_J = 0$ since L_x is a closed manifold. The integral in (4.4) thus is zero, as was to be shown.

We now prove Lemma 4.2. Let $\{(U_i, \phi_i) : 1 \leqslant i \leqslant p\}$ be the cover of L by foliation charts as defined earlier with $U = \bigcup_{i=1}^{p} U_i$. Then $f_i : U_i \to D^q \subseteq \mathbb{R}^q$ defines $\mathscr{F}$ on U_i, and $D_i = \phi_i^{-1}(0 \times D^q)$ is a transversal to $\mathscr{F}$ in U_i. Set $T = K \cap D_1$, a compact set. Recall that this cover of L was chosen so that, for each i, there is an open neighborhood V of T in D_1 and a diffeomorphism into, $\gamma_{1i} : V \to D_i$, the transition function from f_1 to f_i. Let W be an open set with $T \subseteq W \subseteq \overline{W} \subseteq V$.

The standard volume form on $\mathbb{R}^q$ is denoted by $d\mu = dx_1 \wedge \ldots \wedge dx_q$. We begin by defining $\omega_1 = f_1^*(d\mu)$, a closed q-form on U_1 which restricts to a volume form on $V \subseteq D_1 \subseteq U_1$. Let V_i (respectively W_i) denote the image of V (respectively W) under the diffeomorphism into,

$$V \xrightarrow{\gamma_{1i}} D_i \xrightarrow{f_i} \mathbb{R}^q,$$

and let $\tilde{\omega}_{1i}$ denote the q-form on V_i which is the push-forward of $\omega_1|_V$. We extend $\tilde{\omega}_{1i}$ to a volume form defined on all of $\mathbb{R}^q$: choose a partition of unity $\{\lambda_i, (1-\lambda_i)\}$ for the cover $\{V_i, \mathbb{R}^q - \overline{W}_i\}$ of $\mathbb{R}^q$, and set $\tilde{\omega}_i = \lambda_i \tilde{\omega}_{1i} + (1-\lambda_i) \cdot d\mu$. Define $\omega_i = f_i^*(\tilde{\omega}_i)$, a transverse invariant volume form for $\mathscr{F}$ on U_i. Note that the restriction of ω_i to the transversal $\phi_i^{-1}(0 \times W_i)$ agrees with the translation $(\gamma_{1i}^{-1})^*(\omega_1|_W)$.

Choose a partition of unity $\{\alpha_1, \ldots, \alpha_{p+1}\}$ subordinate to the open cover $\{U_1, \ldots, U_p, M - X\}$ of M, where $X = \bigcup_{i=1}^{p} \overline{\phi_i^{-1}(D^{m-q} \times W_i)}$. Let ω_{p+1} be a q-form on $M - X$ which defines $\mathscr{F}$ and has the same orientation as ω_1 on U_1. Set $\omega = \sum_{i=1}^{p+1} \alpha_i \omega_i$ and restrict to U to get the q-form of 4.2. The assumption that the foliation charts are compatibly oriented implies that ω defines $\mathscr{F}$ on U.

It remains to show that $\theta(v)\omega_x = 0$ for $x \in K$ and v a vector field tangent to $\mathscr{F}$. This is equivalent to proving that $\omega_x = (\omega_i)_x$ for all $x \in U_i \cap K$, as $\omega_i = f_i^*(\tilde{\omega}_i)$ is invariant under the flow Φ_t of v, and Φ_t preserves the fibers of $f_i : U_i \to \mathbb{R}^q$.

The function α_{p+1} vanishes on X, and so $\omega = \sum_{i=1}^{p} \alpha_i \omega_i$ on $K \subseteq X$. If we show that $(\omega_i)_x = (\omega_j)_x$ for $x \in K \cap U_i \cap U_j$, then we can conclude that $\omega_x = (\omega_i)_x$ as desired. Because ω_i and ω_j are pull-backs from $\mathbb{R}^q$, we need only show, for the transverse slice $V_x = \phi_i^{-1}(\psi_i(x) \times V_i)$, that the restrictions $\omega_i|_{V_x}$ and $\omega_j|_{V_x}$ agree at x. Both of these restricted forms are defined as the push-forward via $(\gamma_{1i}^{-1})^*$ and $(\gamma_{1j}^{-1})^*$ of $\omega_1|_V$, and so

$$\omega_j|_{V_x} = (\gamma_{1i} \circ \gamma_{1j}^{-1})^* \omega_i|_{V_x}.$$

At x, the action of $(\gamma_{1i} \circ \gamma_{1j}^{-1})^*$ on the q-forms $\Lambda^q Q_x^*$ is induced from the action of the linear holonomy along the leaf L containing x. This action is trivial by the following result.

LEMMA 4.5. *Let L be a leaf in a compact foliation. Then the linear holonomy of L is unipotent, and if the normal bundle Q restricted to L is orientable, then every element has determinant one.*

Proof. Let $dh : \pi_1(L, x) \to \mathrm{Gl}_q\mathbb{R}$ denote the linear holonomy of L. Assume that there exists $\gamma \in \pi_1(L, x)$ for which $dh(\gamma)$ has an eigenvalue of modulus not equal to 1; we can assume it to be less than one. By the stable manifold theorem, a local diffeomorphism representing the holonomy element $h(\gamma)$ has a stable contracting manifold of dimension at least one. This implies that there is a leaf of $\mathscr{F}$ asymptotic to L, which is impossible if all leaves are compact.

For each γ the determinant of $dh(\gamma)$ is real with modulus one, and so must be ± 1. For $Q|_L \to L$ orientable the only possibility is that $\det(dh(\gamma)) = 1$.

Under our assumptions, the linear holonomy of each leaf in K has determinant 1; thus the action of $(\gamma_{1i} \circ \gamma_{1j}^{-1})^*$ on $\Lambda^q Q_x^*$ is trivial, and hence $\omega_j|_{V_x} = \omega_i|_{V_x}$ on $K \cap V_x$. This finishes the proof of the theorem.

References

1. R. BOTT, 'Lectures on characteristic classes and foliations', *Lectures on algebraic and differential topology*, Lecture Notes in Mathematics 279 (Springer, Berlin, 1972), pp. 1–94.
2. R. BOTT and A. HAEFLIGER, 'On characteristic classes of Γ-foliations', *Bull. Amer. Math. Soc.*, 78 (1972), 1038–1044.
3. J. CANTWELL and L. CONLON, 'The dynamics of open foliated manifolds and a vanishing theorem for the Godbillon–Vey class', preprint, Washington University, St. Louis, Missouri, 1982.

4. G. DUMINY, 'L'invariant de Godbillon–Vey d'un feuilletage se localise dans les feuilles ressort', preprint, Université de Lille, 1982.
5. R. EDWARDS, K. MILLET and D. SULLIVAN, 'Foliations with all leaves compact', *Topology*, 16 (1977), 13–32.
6. D. B. A. EPSTEIN, 'Periodic flows on three-manifolds', *Ann. of Math.* (2), 95 (1972), 66–82.
7. A. HAEFLIGER, 'Varietés feuilletés', *Ann. Scuola Norm. Sup. Pisa* (3), 16 (1962), 367–379.
8. J. HEITSCH, 'Flat bundles and residues for foliations', preprint, University of Illinois at Chicago, 1981.
9. F. KAMBER and P. TONDEUR, *Foliated bundles and characteristic classes*, Lecture Notes in Mathematics 493 (Springer, Berlin, 1975).
10. K. C. MILLETT, 'Compact foliations', *Differential topology and geometry, Dijon* 1974, Lecture Notes in Mathematics 484 (Springer, Berlin, 1975), pp. 277–287.
11. T. MIZUTANI, S. MORITA and T. TSUBOI, 'The Godbillon–Vey classes of codimension one foliations which are almost without holonomy', *Ann. of Math.* (2), 113 (1981), 515–527.
12. H. SHULMAN and D. TISCHLER, 'Leaf invariants for foliations and the Van Est isomorphism', *J. Differential Geom.*, 11 (1976), 535–546.
13. D. SULLIVAN, 'A counterexample to the periodic orbit conjecture', Publications Mathématiques 46 (Institut des Hautes Etudes Scientifiques, Paris, 1976), pp. 5–14.
14. E. VOGT, 'The first cohomology group of leaves and local stability of compact foliations', *Manuscripta Math.*, 37 (1982), 229–267.

Department of Mathematics,
Princeton University,
Fine Hall,
Box 37,
Princeton,
New Jersey 08544,
U.S.A.