Exceptional Minimal Sets of $C^{1+\alpha}$-Group Actions on the Circle

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Abstract

We prove two extensions of Sacksteder's Theorem for the action $A : \Gamma \times S^1 \to S^1$ of a finitely-generated group Γ on the circle by $C^{1+\alpha}$-diffeomorphisms. If the action A has an exceptional minimal set K with a gap endpoint of exponential orbit growth rate, or if the action A on K has positive topological entropy, then the exceptional set K is hyperbolic. That is, A has a linearly contracting fixed-point in K. A key point of the paper is to prove a foliation closing lemma using the foliation geodesic flow technique.

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1 Introduction

Let Γ be a finitely-generated group. In this paper we will study the differentiable dynamics of a $C^{1+\alpha}$-action $A : \Gamma \times S^1 \to S^1$ of Γ on the circle, for $0 < \alpha < 1$. The usual structure theorems for C^2-actions on the circle are based on two techniques, the "naive distortion lemma" and the "Koppell lemma" (cf. [7]), which do not hold for the actions considered in this paper. Our idea is to replace the "naive distortion lemma" with more sophisticated methods of smooth dynamical systems; in particular, we introduce the *foliation closing lemma* for transversally hyperbolic measures (Proposition 4.1 below). Our main results are two theorems that recover, in part, the known structure theory for C^2-actions.

A closed non-empty subset $K \subset S^1$ is *minimal* for the action A if K is invariant – $A_\gamma(K) = K$ for each $\gamma \in \Gamma$ – and there is no closed, proper subset of K with these properties. There is a trichotomy for the topological structure of K:

- K is a finite union of points;
- K has interior, hence $K = S^1$ and every point in K has dense orbit in S^1;
- K is a Cantor set in S^1 and every point in K has dense orbit in K.

In the third case, we say that K is an *exceptional minimal set.* The complement of K is then a countable union of open intervals,

$$S^1 - K = \bigcup_{n=1}^{\infty} I_n \ , \quad I_n = (a_n, b_n). \tag{1}$$

The open intervals I_n are called the *gaps* of K, and the points $\{a_n, b_n \mid n = 1, 2, \ldots\}$ are the *endpoints of gaps*, naturally.

The existence of an exceptional minimal set is a strong qualitative statement about the dynamics of A, and it is natural to try to characterize the geometry of such sets (cf. [1, 3, 15]). We consider the following property:

DEFINITION 1.1 *An exceptional minimal set K for A is said to be* hyperbolic *if there exists $x \in K$ and $\gamma \in \Gamma$ such that $A_\gamma(x) = x$ and $0 < A'_\gamma(x) < 1$.*

We also recall a definition from the theory of codimension-one foliations:

DEFINITION 1.2 *A point x is said to be* resilient *for the action A if:*

1. *the orbit $\{A_\gamma(x) \mid \gamma \in \Gamma\}$ contains x in the closure of $\{\{A_\gamma(x) \mid \gamma \in \Gamma\} - \{x\}\}$*
2. *there exists $\gamma \in \Gamma$ such that $A_\gamma(x) = x$ and A_γ is a contraction in an open neighborhood of x.*

The point x is linearly resilient *if the contraction A_γ is hyperbolic at x; that is, $0 < A'_\gamma(x) < 1$.*

The fixed-point x in Definition 1.1 is hyperbolic for the action of A_γ, and as x is also in the closure of its orbit under Γ, this implies that x is linearly resilient.

A basic question is when must an exceptional minimal set be hyperbolic, or even just contain a resilient point? When each diffeomorphism A_γ of a C^1-action A has bounded variation for its first derivative, then the celebrated theorem of Sacksteder [20] implies that every exceptional minimal set for A contains a resilient point. A corollary of Theorem 1.3 below is that the minimal set must also be hyperbolic.

The well-known examples of Denjoy are C^1-actions of $\Gamma = \mathbf{Z}$ on S^1 with exceptional minimal sets, but no periodic points, hence there can be no resilient points. Herman gave a refined construction in [9] of the Denjoy example of class $C^{1+\alpha}$, for any $\alpha < 1$. Recently, Sullivan has observed that renormalization methods show that there are Denjoy examples in the class $C^{1+\Lambda_*}$, where Λ_* denotes the Zygmund class (cf. section 2, [12]). These "Denjoy examples" show that for a $C^{1+\alpha}$-action with $0 < \alpha < 1$, it will be necessary to impose additional hypotheses on the action A to obtain that an exceptional minimal set is *a priori* hyperbolic.

Choose a finite subset $\Gamma_0 = \{g_1, \ldots, g_d\} \subset \Gamma$ such that Γ_0 generates Γ, the identity $e \in \Gamma_0$ and $\gamma \in \Gamma_0$ implies that $\gamma^{-1} \in \Gamma_0$. The set Γ_0 defines a word metric on Γ, where

$$|\gamma| \leq n \Leftrightarrow \gamma = \gamma_{i_n} \cdots \gamma_{i_1} \text{ for some n} - \text{tuple } \{\gamma_{i_1}, \ldots, \gamma_{i_n}\} \subset \Gamma_0 \tag{2}$$

We introduce the finite sets

$$\Gamma_n = \{\gamma \in \Gamma \mid |\gamma| \leq n\} \tag{3}$$

and for each $x \in S^1$, set

$$\Gamma_n(x) = \{A_\gamma(x) \mid \gamma \in \Gamma_0\} \ . \tag{4}$$

Define the growth rate of the group Γ with respect to the generating set Γ_0 to be:

$$gr(\Gamma; \Gamma_0) = \lim_{n \to \infty} \left\{ \frac{\log(\#\Gamma_n)}{n} \right\} \ . \tag{5}$$

The existence of the limit and other properties of the word metric are discussed in Milnor [16], where it is proven that the property $gr(\Gamma; \Gamma_0) > 0$ is independent of the choice of Γ_0. If $gr(\Gamma; \Gamma_0) > 0$ for some choice of generators, we write $gr(\Gamma) > 0$ and say that Γ has *exponential growth.* We write $gr(\Gamma) = 0$ when $gr(\Gamma; \Gamma_0) = 0$ for some choice of generators, and say that Γ has *sub-exponential growth.*

The growth rate of an orbit of the group action is similarly defined,

$$gr(A; \Gamma_0; x) = \limsup_{n \to \infty} \left\{ \frac{\log(\#\Gamma_n(x))}{n} \right\} \ , \tag{6}$$

except that the lim sup is required as the limit need not exist (cf. [6]). When $gr(A; \Gamma_0; x) = 0$, we say that the orbit has *subexponential growth.* The estimate $gr(A; \Gamma_0; x) \leq gr(\Gamma; \Gamma_0)$ always holds, so that $gr(\Gamma) = 0$ implies each orbit of Γ has subexponential growth.

A hyperbolic point $x \in K$ for an exceptional minimal set K must have $gr(A; \Gamma_0; x) > 0$. This follows from an elementary argument, using that the orbit of x intersects the domain of the contraction, so there exists a subset of the orbit which maps quasi-isometrically onto a free integer tree, and this forces the orbit growth to be greater than that of the tree, which is exponential (cf. [8]). In particular, Γ must also have exponential growth.

Ou first result is a converse to the above observation: exponential growth of an endpoint of an exceptional minimal set forces the set to be hyperbolic, and hence resilient.

THEOREM 1.3 *Let $A : \Gamma \times S^1 \to S^1$ be a $C^{1+\alpha}$-action, for $0 < \alpha < 1$, with an exceptional minimal set K. Suppose there exists a gap $I_0 \subset K$ with an endpoint x_0 such that $gr(A; \Gamma_0; x_0) > 0$. Then there exists a hyperbolic fixed-point for the action of A on K, and consequently K is hyperbolic.*

Our second result is based on growth estimates for arbitrary points in the exceptional minimal set, but needs these estimates on a net of points in the set. This is expressed in terms of the *geometric entropy* of a C^1-action, as defined by Ghys, Langevin and Walczak [4], which is based on the use of ϵ-separated sets. Let S^1 have the Riemannian metric with total length 2π, and metric distance function $d : S^1 \times S^1 \to [0, \pi]$. For each $\epsilon > 0$, we say that a finite subset $\{x_1, \ldots, x_\ell\} \subset S^1$ is (ϵ, n)-separated if for any pair $x_i \neq x_j$ there exists $\gamma_{ij} \in \Gamma_n$ such that

$$d(A_{\gamma_{ij}}(x_i), A_{\gamma_{ij}}(x_j)) > \epsilon.$$

Let $H(A, \epsilon, n)$ denote the maximal cardinality of an (ϵ, n)-separated subset of S^1. Then the geometric entropy of A is the limit

$$h(A) = \lim_{\epsilon > 0} h(A, \epsilon) \tag{7}$$

$$h(A, \epsilon) = \limsup_{n \to \infty} \frac{\log H(A, \epsilon, n)}{n} \tag{8}$$

The finiteness of (7) and (8) are consequences of the differentiability of the action A, and are well-known to fail if the action is only continuous. Thus, $h(A)$ is a geometric invariant of the action, depending on more than its topological dynamics. For a closed subset $X \subset S^1$, we similarly define the restricted geometric entropy $h(A; X)$ as above, but requiring that all of the sample points $\{x_1, \ldots, x_\ell\} \subset X$.

THEOREM 1.4 *Let $A : \Gamma \times S^1 \to S^1$ be a $C^{1+\alpha}$-action, for $0 < \alpha < 1$, with an exceptional minimal set K. If $h(A; K) > 0$, then there exists a hyperbolic fixed-point for the action of A on K, and consequently K is hyperbolic.*

Theorems 1.3 and 1.4 are proved in two steps, with the second step in common to both. First, we reduce the the growth hypotheses to a conclusion about the asymptotic behavior of the derivative cocycle for the action. For $X \subset S^1$, define the *absolute exponent* of A on X:

$$E(A; X) = \limsup_{n \to \infty} \frac{1}{n} \sup_{x \in X} \left(\max_{\gamma \in \Gamma_n} \left\{ \left| \log |A'_\gamma(x)| \right| \right\} \right). \tag{9}$$

When $E(A; X) > 0$, we say that A has *non-uniform hyperbolicity* on X. For a C^1-action A which satisfies either of the growth hypotheses of Theorems 1.3 or 1.4, we show in section 2 that A is non-uniformly hyperbolic on the exceptional minimal set K.

The second step in our proofs is to show that non-uniform hyperbolicity on a closed set implies the existence of a hyperbolic periodic orbit. This will be based on the technique of the foliation geodesic flow, using a foliation obtained by suspending the group action. We use the hypothesis $E(A; K) > 0$ to produce a hyperbolic, invariant transverse measure for the leafwise geodesic flow, then prove a leafwise closing lemma using the stable manifold theory of Pesin [17, 19]. The Hölder hypothesis is needed at this stage to obtain the regularity results for the stable manifolds used to prove the closing lemma. This second step is an application of what we have called the Pesin theory for foliations, developed in [10, 11].

2 Orbit growth, entropy and non-uniform hyperbolicity

In this section we assume that $A : \Gamma \times S^1 \to S^1$ is a C^1-action, and show how exponential orbit growth estimates imply non-uniform hyperbolicity. The techniques used in this section are typical for codimension-one dynamics, especially in the various proofs of Sacksteder's Theorem (cf. [5]). We first prove the following:

PROPOSITION 2.1 *Let A, Γ, x_0 and K be as in Theorem 1.3. Given $\epsilon > 0$, there exists constants $c_1, c_2 > 0$, where $gr(A; \Gamma_0, x_0) - \epsilon < c_2 < gr(A; \Gamma_0; x_0)$, and a sequence of elements $\gamma_n \in \Gamma$ with $|\gamma_n| \to \infty$, such that*

$$\log |A'_{\gamma_n}(x_0)| < c_1 - n \cdot c_2. \tag{10}$$

Proof. We can assume without loss of generality that Γ acts via orientation-preserving C^1-diffeomorphisms, so that $A'_\gamma > 0$ for all $\gamma \in \Gamma$. Fix $0 < \epsilon < gr(A; \Gamma_0; x_0)/2$. The proposition is a consequence of the following two lemmas.

LEMMA 2.2 *There exists constants $c_1, c_3 > 0$ with $c_3 > gr(A; \Gamma_0; x_0) - \epsilon/2$, and for each positive integer n a point $y_n \in I_0$ and element $\gamma_n \in \Gamma$, such that*

$$\log A'_{\gamma_n}(y_n) < c_1 - c_3 \cdot |\gamma_n| \tag{11}$$

Proof. Let $\delta = \epsilon/3$ and set $c_3 = gr(A; \Gamma_0; x_0) - \delta$ and $c_2 = gr(A; \Gamma_0; x_0) - 2\delta$. Choose an increasing subsequence $\{i_n \in \mathbf{Z}^+\}$ for which

$$\log \#\Gamma_n(x_0) > c_3 \cdot i_n.$$

There is a unique gap of K corresponding to each point in $\Gamma_{i_n}(x_0)$, so the number of gaps of K that can be reached from x_0 by a word in Γ of length at most i_n has an estimate from below by $\exp(c_3 \cdot i_n)$. Therefore, for each n there exists $\gamma_n \in \Gamma_{i_n}$ so that the image gap $J_n = A_{\gamma_n}(I_0)$ has length estimated above by

$$|J_n| \leq 2\pi \cdot \exp(-c_3 \cdot i_n).$$

We require for our later applications that γ_n have the least length of all words in Γ which map I_0 to J_n; if necessary, replacing γ_n with a word of shorter length. (Thus, γ_n is a *shortcut* in the notation of [2, 5].) The Mean-Value Theorem implies there exists $y_n \in I_0$ such that

$$0 < A'_{\gamma_n}(y_n) < \frac{|I_0|}{2\pi} \cdot \exp(-c_3 \cdot i_n).$$

Set $c_1 = -\log(|I_0|/2\pi)$ and observe that $|\gamma_n| \leq i_n$ to obtain (11). $\square$

For each pair (γ_n, y_n) chosen in the proof of Lemma 2.2, set $d_n = |\gamma_n|$. There then exists elements $\{\gamma_{\ell,d_\ell} | 1 \leq \ell \leq d_n\} \subset \Gamma_0$ such that

$$\gamma_n = \gamma_{n,d_n} \cdots \gamma_{n,1} . \tag{12}$$

Set $y_{n,0} = y_n$, and for each $1 \leq \ell \leq d_n$ set

$$\begin{aligned} A_{n,\ell} &= A_{\gamma_{n,\ell}} \circ \cdots \circ A_{\gamma_{n,1}} \\ y_{n,\ell} &= A_{n,\ell}(y_n) \end{aligned}$$

The choice of each γ_n to have shortest length implies that $y_{n,i} \neq y_{n,j}$ for $j > i$ – for if not, then we can delete the factor $\gamma_{n,j} \cdots \gamma_{n,i+1}$ from (12) to obtain a shorter word taking the gap I_0 to the gap J_n. (This gives intuitive meaning to the notation that γ_n is a shortcut.)

LEMMA 2.3

$$\log A'_{\gamma_n}(x_0) < c_1 - c_2 \cdot |\gamma_n| \ . \tag{13}$$

Proof. Let N be a positive integer such that for all $\gamma \in \Gamma_0$ and all $x, y \in S^1$ with $d(x, y) < 2\pi/N$,

$$\left|\log A'_\gamma(y) - \log A_\gamma(x)\right| < \epsilon/6 = \delta/2 \ . \tag{14}$$

Define a constant

$$c_4 = \max_{\gamma \in \Gamma_0} \left\{ \max_{x \in S^1} \left|\log A'_\gamma(x)\right| \right\} \ .$$

Then write

$$\log A'_{\gamma_n}(y_n) = \sum_{\ell=0}^{d_n - 1} \log A'_{\gamma_{n,\ell+1}}(y_{n,\ell}) \ . \tag{15}$$

As all of the points $\{y_{n,0}, \ldots, y_{n,d_n}\}$ are distinct, there can be at most N points in the set which correspond to endpoints of gaps of length greater than $2\pi/N$, and each such point contributes at most c_4 to the sum (15). For each point $y_{n,\ell}$ in a gap of length less than $2\pi/N$, the estimate (14) implies that

$$\left|\log A'_{\gamma_{n,\ell+1}}(y_{n,\ell}) - \log A'_{\gamma_{n,\ell+1}}(A_{n,\ell}(x_0))\right| < \delta/2 \ .$$

Thus, after summing we obtain the estimate

$$\left|\log A'_{\gamma_n}(y_n) - \log A'_{\gamma_n}(x_0)\right| \leq 2Nc_4 + \delta d_n/2$$

so that for n sufficiently large, Lemma 2.2 implies that

$$\begin{aligned} \log A'_{\gamma_n}(x_0) &\leq \log A'_{\gamma_n}(y_n) + 2Nc_4 + \delta d_n/2 \\ &\leq c_1 - c_3 \cdot d_n + \delta d_n \\ &\leq c_1 - c_2 \cdot |\gamma_n| \ . \quad \square \end{aligned}$$

The next result is used in the proof of Theorem 1.4. Note that it does not asume that K is exceptional, only that it is not a finite set.

PROPOSITION 2.4 *Let* $A : \Gamma \times S^1 \to S^1$ *be a* C^1*-action with minimal set* K*. If the geometric entropy* $h(A;K) > 0$*, then* A *is non-uniformly hyperbolic on* K*. More precisely, for a fixed generating set* Γ_0*, we show that*

$$E(A;K) \geq h(A;K) \tag{16}$$

Proof. We can again assume that the action A is via orientation preserving C^1-diffeomorphisms. Choose $\delta > 0$ and set $c_5 = h(A;K) - \delta/2$. From the definition of $h(A;K)$, we can find $\epsilon > 0$ and $N > 0$ so that for each $n > N$ there is a subset $\Delta_n = \{x_{n,1}, \ldots, x_{n,d_n}\} \subset K$ with $d_n \geq \exp(n \cdot c_5)$, and for each $i \neq j$ and element $\gamma_{n,i,j} \in \Gamma_n$ so that

$$d(A_{\gamma_{n,i,j}}(x_{n,i}), A_{\gamma_{n,i,j}}(x_{n,j})) > \epsilon \ . \tag{17}$$

Thus, for each $n > N$ we can find a pair $y_n, z_n \in \Delta_n$ and an element $\gamma_n \in \Gamma_n$ so that

$$\begin{aligned} d(y_n, z_n) &\leq 2\pi \cdot \exp(-n \cdot c_5) \\ d(A_{\gamma_n}(y_n), A_{\gamma_n}(z_n)) &\geq \epsilon \end{aligned}$$

By the Mean-Value Theorem, for $\delta_n = \gamma_n^{-1}$, there exists a point w_n between $A_{\gamma_n}(y_n)$ and $A_{\gamma_n}(z_n)$ such that

$$\log A'_{\delta_n}(w_n) \leq \frac{2\pi}{\epsilon} \cdot \exp(-n \cdot c_5) \ . \tag{18}$$

If an infinite subset of $\{w_n\}$ lies in K, then (18) directly yields the estimate

$$E(A;K) \geq c_5 = h(A;K) - \delta/2 \ .$$

This is the case, for example, if K is all of S^1. As $\delta > 0$ was arbitrary, this yields (16).

Suppose that K is exceptional, and all but a finite number of the points $\{w_n\}$ lie in the gaps of K. The proof of Lemma 2.2 did not use that I_0 was a fixed gap in any essential way, so we can repeat its proof for w_n in place of the points y_n of Lemma 2.2, x_n the endpoint of the gap containing w_n in place of x_0, and then replacing each δ_n by its shortcut if necessary. We then obtain

$$E(A;K) \geq h(A;K) - \delta$$

and as δ was arbitrary, this implies (16). $\square$

3 Geodesic flow of a group action

Non-uniform hyperbolicity on a set K implies the existence of a hyperbolic invariant transverse measure on K. We prove this result by introducing the foliation geodsesic flow associated to a group action. The importance of hyperbolic measures was brought out by the work of Katok [13] and Ruelle [19] for actions of the group $\mathbf{Z}$. The results of this section have extensions to more general foliated manifolds and group actions, as discussed in [10, 11]

Let us first construct the foliation geodesic flow associated to a C^1-action of a finitely-generated group Γ on S^1. Choose a compact orientable manifold B without boundary such that there is a surjection of the fundamental group $\rho : \pi_1 = \pi_1(B, b) \to \Gamma$. For example, if Γ has a generating set with d elements, then the surface Σ_{2d} of genus $2d$ maps onto a free group with d generators, and hence onto Γ. We can therefore always assume that B is a closed surface, although this is not used in the following constructions.

Let the group π_1 act via deck translations on the left on the universal cover $\tilde{B}$ of B, and on S^1 via $A \circ \rho$. For the product action of π_1 on $\tilde{B} \times S^1$, let M denote the quotient manifold $\pi_1 \backslash (\tilde{B} \times S^1)$. The horizontal product foliation of $\tilde{B} \times S^1$, with leaves $\{\tilde{B} \times \{x\} | \ x \in S^1\}$, is invariant under the action of π_1, so descends to a foliation on M denoted by $\mathcal{F}_A$. Note that M is an S^1-fibration over B, $p : M \to B$, and each leaf of $\mathcal{F}_A$ is transverse to the fibers of p. The differentiability of $\mathcal{F}_A$ is exactly that of the action A, which is $C^{1+\alpha}$.

Endow TM with a Riemannian metric such that:

1. each fiber of p has constant length 2π;
2. the leaves of $\mathcal{F}_A$ are orthogonal to the fibers of p;
3. the metric on the tangential distribution $T\mathcal{F}_A$ is the lift of a Riemannian metric from TB.

Fix an identification $S^1 \cong p^{-1}(b)$, then we obtain a holonomy representation $h_A : \pi_1 \to \mathrm{Diff}^{1+\alpha}(S^1)$. Recall that for a closed loop $c(t)$ representing an element $\delta \in \pi_1$ and a point $x \in S^1$, we lift this to a leafwise path starting at x, and $y = h_A(\delta)(x)$ is the endpoint of the lift. The basic property of the suspension construction is that the holonomy action h_A and the original action $A \circ \rho$ of π_1 are $C^{1+\alpha}$-conjugate, so that the transverse dynamics of $\mathcal{F}_A$ determines the dynamics of the original action A.

Let T_1B denote the unit tangent bundle of B, and $V = T_1\mathcal{F}_A$ be the bundle of unit vectors in TM which are tangent to the leaves of $\mathcal{F}_A$. Note that our choice of metric on TM yields a natural map $Dp : T_1\mathcal{F}_A \to T_1B$. Let $\hat{\mathcal{F}}_A$ denote the foliation of the compact manifold V, whose leaves $\hat{L}$ are the unit tangent bundles to the leaves of $\mathcal{F}_A$. For $x \in M$ and $v \in T_1M_x$ we obtain a typical point $(x, v) \in V$.

For $x \in M$, let L_x denote the leaf of $\mathcal{F}_A$ through x, and endow TL_x with the restricted Riemannian metric from TM. Note this metric is the lift of the metric on TB. Given a unit vector $v \in T_1L_x$, we can form the geodesic $f_t(x, v)$ in the complete Riemannian manifold L_x, defined for all $t \in \mathbf{R}$. Note that the curve $f_t(x, v) \subset L_x \subset M$ is not necessarily a geodesic for the metric on TM; for example, when L_x is a totally geodesic submanifold, $f_t(x, v)$ will be a geodesic in M. The projected curve $p(f_t(x, v))$ is a geodesic in B.

The *foliation geodesic flow* , $F^V : V \times \mathbf{R} \to V$, is defined by letting $F^V(x, v, t_0) \in V$ be the endpoint of the geodesic $f_t(x, v)$ at time $t = t_0$ (cf. [10, 21]). We summarize the basic properties of this flow:

LEMMA 3.1 *1. F^V is a $C^{1+\alpha}$-mapping;*

2. F^V covers the geodesic flow F^B on the base T_1B;

3. F^V maps the leaves of $\hat{\mathcal{F}}_A$ into themselves, so preserves the foliation $\hat{\mathcal{F}}_A$;

4. For fixed $t = t_0$, the differential $Df_{t_0} = DF^V_{t_0} : TV \to TV$ presevres the orthogonal decomposition $TV \cong T\hat{\mathcal{F}}_A \oplus Tp$. $\square$

Let $\partial/\partial x$ denote the positively oriented, unit length vector field along the fibers of π. By Lemma 3.1.4, there exists a *positive* scalar function f^{v}_t on V so that

$$Df_t\left(\partial/\partial x|_{(x,v)}\right) = f^{\mathrm{v}}_t(x,v) \cdot \partial/\partial x|_{f_t(x,v)} \; , \tag{19}$$

and this function satisfies the multiplicative cocycle law over the leafwise geodesic flow

$$f^{\mathrm{v}}_{t+s}(x,v) = f^{\mathrm{v}}_s(f_t(x,v)) \cdot f^{\mathrm{v}}_t(x,v) \; . \tag{20}$$

Let $\hat{X} \subset V$ be a closed set. Define the *absolute exponent* of f^{v}_t along $\hat{X}$ to be

$$E(f^{\mathrm{v}}_t;\hat{X}) = \sup_{(x,v)\in\hat{X}} \; \limsup_{t\to\infty} \; \frac{|\log\{f^{\mathrm{v}}_t(x,v)\}|}{t} \; . \tag{21}$$

The exponent $E(f^{\mathrm{v}}_t;\hat{X})$ is independent of the choice of Riemannian metric on TV, and we adopt the notation $E^t(\hat{X}) = E(f^{\mathrm{v}}_t;\hat{X})$.

Return now to the group action A whose suspension yields the foliation $\mathcal{F}_A$, and let $X \subset S^1$ be a Borel subset invariant under the action of A. We let $\hat{X} \subset V$ denote the saturation of $X \subset S^1 \cong p^{-1}(b)$ under *the leafwise geodesic flow f_t*.

LEMMA 3.2 *There exists a constant $c_6 > 0$ so that for any Borel invariant set $X \subset S^1$,*

$$\frac{1}{c_6} \cdot E(A;X) \leq E^t(\hat{X}) \leq c_6 \cdot E(A;X) \; . \tag{22}$$

Proof. The proof of this is standard, so we just indicate the steps involved. The choice of a Riemannian metric on TB defines a distance function on π_1, declaring the distance between two words $\{\delta_1,\delta_2\}$ as the geodesic length in the universal cover $\tilde{B}$ between the translates $\delta_1 \cdot b_0$ and $\delta_2 \cdot b_0$, for a basepoint $b_0 \in \tilde{B}$. This distance function on π_1 is eqivalent to the word metric (cf. [18, 14]). Given a sequence of points $\{y_n \in X\}$ and words $\{\gamma_n \in \Gamma\}$ with

$$\lim_{n\to\infty} \frac{-\log\{A'_{\gamma_n}(x_n)\}}{n} = E(A;K) \; ,$$

we can find leafwise geodesics in the leaves of $\hat{X}$ which begin and end on the fiber $p^{-1}(b)$ with the same exponents, and whose lengths are related to the word lengths of the γ_i by a constant depending only on the Riemannian metric on TB. This implies the left estimate in (22). The right estimate follows similarly, noting that a leafwise path in $\hat{X}$ beginning and ending in the fixed fiber $p^{-1}(b)$ can be replaced by a leafwise geodesic path having the same endpoints and holonomy, without incresing the pathlength. This geodesic path projects to a closed path in B, and yields an element of π_1 with the same exponent, whose word length is related to the geodesic length by the same constant as before. $\square$

A probability measure $\mathbf{m}$ on V, viewed as a linear functional on $C^0(V)$, is f_t-invariant if for all $g \in C^0(V)$ and any $t \in \mathbf{R}$, $\mathbf{m}(g) = \mathbf{m}(g \circ f_t)$. We say that $\mathbf{m}$ has support in $\hat{X} \subset V$ if $\mathbf{m}(g) = 0$ for all g vanishing on $\hat{X}$. Introduce the logarithmic derivative function

$$\mathcal{D}f^{\mathrm{v}}(x,v) = \frac{d}{dt}\left[\log\{f_t^{\mathrm{v}}(x,v)\}\right]_{t=0}$$

The main result of this section is then:

PROPOSITION 3.3 *Let $\hat{X}$ be a closed invariant saturated set for the leafwise geodesic flow f_t. Then for all $\epsilon > 0$, there exists an invariant probability measure $\mathbf{m}_\epsilon$ supported in $\hat{X}$ such that*

$$\mathbf{m}_\epsilon(\mathcal{D}f^{\mathrm{v}}) \leq -\left\{E^t(\hat{X}) - \epsilon\right\}. \tag{23}$$

Proof. Fix $\hat{X}$ and $\epsilon > 0$, then choose a sequence of points $\{(x_n, v_n)\} \subset \hat{X}$ and times $\{t_n \in \mathbf{R}\}$ tending to $+\infty$ such that

$$\log\{f_{t_n}^{\mathrm{v}}(x_n, v_n)\} \leq -t_n \cdot \{E^t(\hat{X}) - \epsilon\}\ . \tag{24}$$

Define probability measures $\{\mathbf{m}_n\}$ on V by setting

$$\mathbf{m}_n(g) = \frac{1}{t_n} \cdot \int_0^{t_n} g(f_t(x_n, v_n))\, dt\ . \tag{25}$$

We have that $\mathbf{m}_n(\mathcal{D}f^{\mathrm{v}}) = \frac{1}{t_n} \cdot \log\{f_{t_n}^{\mathrm{v}}(x_n, v_n)\}$ by the cocycle law and the Fundamental Theorem of Calculus. Choose a weak-* limit $\mathbf{m}_*$ of the sequence of measures $\{\mathbf{m}_n\}$. Then $\mathbf{m}_*$ is flow-invariant and supported in the closed set $\hat{X}$, as each measure $\mathbf{m}_n$ has support in $\hat{X}$. The estimate (24) and continuity of the logarithmic derivative function $\mathcal{D}f^{\mathrm{v}}$ implies the estimate

$$\mathbf{m}_*(\mathcal{D}f^{\mathrm{v}}) \leq -\{E^t(\hat{X}) - \epsilon\}\ . \tag{26}$$

Thus, $\mathbf{m}_*$ is a flow-invariant probability measure on V satisfying all of the conclusions of the proposition, except possibly ergodicity with respect to the flow. However, the estimate (26) is linear in the measure $\mathbf{m}_*$, so for a flow-ergodic decomposition of $\mathbf{m}_*$, there must exist some ergodic component (an flow-ergodic probability measure on V) which also satisfies (26), which we then let equal to $\mathbf{m}_\epsilon$. $\square$

COROLLARY 3.4 *Let $\hat{X}$ be a closed, geodesic flow invariant set in V with $E(A; \hat{X}) > 0$. Then there exists an ergodic probability measure $\mathbf{m}_X$ on V, invariant under the foliation geodesic flow and supported in $\hat{X}$, such that for $\mathbf{m}_X$-almost every $(x, v) \in \hat{X}$, the limit*

$$e^t(\mathbf{m}_X) = \lim_{t \to \infty} \frac{\log\{f_t^{\mathrm{v}}(x,v)\}}{t} \tag{27}$$

exists and satsifies $e^t(\mathbf{m}_X) \leq -E^t(\hat{X})$.

Proof. Let $\mathbf{m}_*$ be a weak-* limit of the family of invariant probability measures $\{\mathbf{m}_\epsilon \mid \epsilon > 0\}$ given by Proposition 3.3. This will satisfy (27), so there exists a flow-ergodic component of $\mathbf{m}_*$ which satisfies the same estimate. We take this to be $\mathbf{m}_X$. The existence and uniqueness of the limit (26) follows from the Birkhoff Ergodic Theorem. □.

We say that the flow-ergodic measure $\mathbf{m}_X$ with $e^t(\mathbf{m}_X) < 0$ produced in Corollary 3.4 is an $\mathcal{F}_A$-*transversally hyperbolic* measure.

In summary, we have shown that for a closed invariant set $K \subset S^1$ with $E(A; K) > 0$, there exists an $\mathcal{F}_A$-transversally hyperbolic measure $\mathbf{m}_K$ supported on the foliation geodesic flow saturation $\hat{K}$ of K. Along the orbits of this flow contained in the support of $\mathbf{m}_K$, the transverse linear holonomy of $\mathcal{F}_A$ is non-uniformly contracting in positive time with exponent $e^t(\mathbf{m}_K)$. Thus, we have converted the non-uniform hyperbolicity of the group action on S^1 into non-uniform hyperbolicity along certain orbits for the foliation geodesic flow.

4 The foliation closing lemma

The main result of this section is a leafwise closing lemma for transversally hyperbolic measures. This is the 1-dimensional version of the closing lemma for non-uniformly hyperbolic measures for $\mathbf{Z}$-actions (cf. Theorem 4.1, [13]). Define the $\mathcal{F}_A$-support of a measure $\mathbf{m}$ on the unit tangent bundle $V = T_1 M$ to be the union of all leaves in M which intersect the pushdown $p(spt(\mathbf{m})) \subset M$ of the support of $\mathbf{m}$.

PROPOSITION 4.1 *Let $\mathcal{F}_A$ be a $C^{1+\alpha}$-foliation of a compact manifold M obtained by the suspension construction of an action A, with $\alpha > 0$. Given an ergodic invariant probability measure $\mathbf{m}$ for the foliation geodesic flow f_t with $e^t(\mathbf{m}) < 0$, there exists a closed leafwise path γ_* contained in the closure of the $\mathcal{F}_A$-support of $\mathbf{m}$, with linearly contracting transverse holonomy.*

Proof. The multiplicative ergodic theorem of Oseledec (cf. Ruelle [19]), applied to the leafwise geodesic flow f_t on V, asserts the existence of:

1. A Borel subset $Y \subset V$ which is flow-invariant, and for every f_t-invariant probability measure μ on V, $\mu(Y) = 1$;
2. Y is the union of disjoint Borel subsets Y_σ indexed by the f_t-ergodic measures σ, such that $f_t(Y_\sigma) \subset Y_\sigma$, and for each continuous function $g : V \to \mathbf{R}$ we have
$$\sigma(g) = \lim_{T\to\infty} \frac{1}{n} \int_{t=0}^{T} g(f_t(y))dt \quad \text{for all } y \in Y_\sigma;$$
3. For each $y \in Y$, the spectrum of the flow
$$\{-\infty < \lambda_y^{(1)} < \ldots < \lambda_y^{(s)} < +\infty\}$$
of Df_t and the associated filtration $V_y^{(1)} \subset \cdots \subset V_y^{(s)} = TV_y$ are defined, are Borel functions of y and constant on each Y_σ.

The tangent bundle Tp to the fibers of the fibration $p : M \to B$ lifts to a flow-invariant sub-bundle $Dp^*(Tp) \subset TV$, so the hypotheses $e(\mathbf{m}) < 0$ implies that for $\epsilon = e(\mathbf{m})/2$ and $y \in Y_{\mathbf{m}}$, there is an inclusion of subbundles

$$Dp^*(Tp)_y \subset V_y^\epsilon \stackrel{def}{=} \bigcup_{\lambda_y^{(i)} \leq \epsilon} V_y^{(i)} \ .$$

The non-uniformly hyperbolic stable manifold theorem (Pesin [17]; or Theorem 6.1, Ruelle [19]) implies that for each point $y \in Y_{\mathbf{m}}$, there is a stable manifold ν_y^ϵ for the flow, such that

1. There are Borel functions $\beta > \alpha > 0$ and $\gamma > 1$ defined on $Y_{\mathbf{m}}$ such that
$$\nu_y^\epsilon(\alpha(x)) = \left\{ w \in \overline{B}(y, \alpha(y)) \mid d(f_t(w), f_t(y)) \leq \beta(y) e^{t \cdot \epsilon} \text{ for all } t > 0 \right\}$$
is a $C^{1+\alpha}$-submanifold of the closed ball $\overline{B}(y, \alpha(y))$, tangent at 0 to V_y^ϵ
2. If $w, z \in \nu_y^\epsilon(\alpha(y))$, then
$$d(f_t(w), f_t(z)) \leq \gamma(y) d(w,z) e^{t \cdot \epsilon} \text{ for all } t > 0. \tag{28}$$

The reference [19] proves this for the discrete case $f = f_1$ with the flow direction as the central stable direction. The flow case is easily deduced from this.

For $\mathbf{m}$-almost every point $y \in V$, the forward orbit $\{f_t(y) \mid t > 0\}$ is recurrent through every open ball about y. We can therefore choose a recurrent point $y_* \in Y_{\mathbf{m}}$.

Fix a constant c_7 such that the ball $B(y_*, c_7 \cdot \alpha(y_*))$ is contained in a regular foliation chart $\phi : U \to (-\mathbf{1}, \mathbf{1})^{n-1} \times (-\mathbf{1}, \mathbf{1})$ with $y_* \in B(y_*, c_7 \cdot \alpha(y_*))$ such that $\phi(y_*) = (0,0)$. (A chart is *regular* if it admits an extension to a larger chart containing the closure of U.) Let $\phi_{\mathrm{v}} : U \to (-\mathbf{1}, \mathbf{1})$ be the projection onto the second factor, so that the leaves of $\mathcal{F}_A$ in U are the pre-images of points, $P_x = \phi_{\mathrm{v}}^{-1}(x)$ for $x \in (-\mathbf{1}, \mathbf{1})$. (The level sets P_x are called the *plaques* of $\mathcal{F}_A$ for this coordinate chart.)

The tangent space to the C^1-stable manifold $\nu_{y_*}^\epsilon(\alpha(y))$ contains the sub-bundle $Dp^*(Tp)$ which is transverse to the lifted foliation $\hat{\mathcal{F}}_A$ on V. Therefore, for c_7 sufficiently small and the foliation chart (ϕ, U) sufficiently small, we can assume that $\nu_{y_*}^\epsilon(\alpha(y))$ is everywhere transverse to $\hat{\mathcal{F}}_A$ in the closure of U. Choose a C^1-curve $\psi : (-\mathbf{1}, \mathbf{1}) \to \nu_{y_*}^\epsilon(\alpha(y))$ such that $\psi(x)$ and $\phi^{-1}(0,x)$ lie on the same plaque for all $-1 \leq x \leq 1$. This implies that the composition $\phi_{\mathrm{v}} \circ \psi$ is the identity on $[-\mathbf{1}, \mathbf{1}]$. Then there exists a constant $c_8 > 0$ such that

$$d(\psi(x), \psi(x')) \leq c_8 \cdot |x - x'| \text{ for all } -1 \leq x, x' \leq 1; \tag{29}$$
$$|\phi_{\mathrm{v}}(w) - \phi_{\mathrm{v}}(z)| \leq c_8 \cdot d(w,z) \text{ for all } z, w \in U. \tag{30}$$

Choose a time $t_* >> 0$ sufficiently large so that

$$f_{t_*}(y_*) \in B(y_*, c_7 \cdot \alpha(y_*))$$
$$\gamma(y_*) \alpha(y_*) e^{t_* \cdot \epsilon} < \frac{1}{10 (c_8)^2} \tag{31}$$

Let us now construct a C^1-map $G : (-\mathbf{1}, \mathbf{1}) \to (-\mathbf{1}, \mathbf{1})$ from the above data. For each point $x \in (-\mathbf{1}, \mathbf{1})$, the image $f_{t_*}(\psi(x)) \in B(y_*, c_7 \cdot \alpha(y_*))$, so there is a unique point $G(x) \in (-\mathbf{1}, \mathbf{1})$ such that $f_{t_*}(\phi^{-1}(0,x)) \in P_x$. This defines the map G, with $G(0) = z_* \in (-\mathbf{1}, \mathbf{1})$ corresponding to the plaque containing $f_{t_*}(y_*)$.

LEMMA 4.2 *The C^1-map G is a strict contraction.*

Proof. For two points $x, x' \in (-\mathbf{1}, \mathbf{1})$, we observe that

$$
\begin{aligned}
|G(x) - G(x')| &\leq c_8 \cdot d(f_{t_*}(\psi(x)), f_{t_*}(\psi(x'))) \ \text{ by (30)} \\
&< \frac{1}{10c_8} \cdot d(\psi(x), \psi(x')) \ \text{ by (31) and (28)} \\
&\leq \frac{1}{10} \cdot |x - x'| \ \text{ by (29).} \quad \square
\end{aligned}
$$

It follows that there exists a unique fixed-point $x_* \in (-\mathbf{1}, \mathbf{1})$ for the map G. The map G is C^1, so the estimate $|G(x) - G(x')| < \frac{1}{10}|x - x'|$ obtained above implies that the derivative $|G'(x_*)| < \frac{1}{10}$ (cf. proof of Lemma 6.2, [19]). It remains to observe that the point x_* is a hyperbolic fixed-point for the transverse holonomy along a closed leafwise path in $\mathcal{F}_A$. To see this, note that $z_* = \phi^{-1}(0, x_*)$ and $\psi(x_*)$ lie on the same plaque, P_{x_*}, so can be joined by a short geodesic path in the plaque. The geodesic segment $f_t(\psi(x_*)$ is certainly a path in a leaf of $\mathcal{F}_A$. The image $f_{t_*}(\psi(x_*))$ also lies on the plaque, P_{x_*}, so can be joined by a short leafwise geodesic segment to z_*, and we then concatenate these leafwise paths to get a leafwise, piecewise geodesic γ_* starting and ending at z_*, whose transverse holonomy is given by the map G.

Finally, note that the point z_* is in the closure of the $\mathcal{F}_A$-saturation of the pushdown into M of the forward orbit $\{f_t(y_*) \mid t > 0\}$. These orbits are contained in the support of $\mathbf{m}$ by ergodicity, so the hyperbolic fixed-point z_* lies in the closure of the $\mathcal{F}_A$-saturation of the pushdown of the support of $\mathbf{m}$. $\square$

We call Proposition 4.1 the $\mathcal{F}_A$-closing lemma, as the transverse hyperbolicity $e^t(\mathbf{m}) < 0$ is used to produce a linear contraction on the space of leaves in the ball $B(y_*, c_7 \cdot \alpha(y_*))$. Note that we only show above that the *plaque* P_{z_*} is fixed by the geodesic flow f_{t_*}. To get an actual closed loop, we have to abandon the geodesic flow construction, and resort to closing up the path by adding on short geodesics. Thus, we do not produce closed orbits for the flow f_t, but rather for the holonomy of the foliation that gave rise to the geodesic flow.

The proofs of Theorems 1.3 and 1.4 are now completed by observing:

COROLLARY 4.3 *Let $A : \Gamma \times S^1 \to S^1$ be a $C^{1+\alpha}$-action, for $\alpha > 0$, with $E(A; K) > 0$ on a minimal set $K \subset S^1$. Then there exist an element $\gamma_* \in \Gamma$ and $x_* \in K$ such that $A_{\gamma_*}(x_*) = x_*$ and $0 < A'_{\gamma_*}(x_*) < 1$.*

Proof. Let $\mathcal{F}_A$ be a $C^{1+\alpha}$-foliation of a compact manifold M obtained by the suspension construction of the action A. By Corollary 3.4, there exists an ergodic invariant probability measure $\mathbf{m}$ for the foliation geodesic flow f_t with $e^t(\mathbf{m}) < 0$. By Proposition 4.1, there exists a closed leafwise path γ_* with linearly contracting transverse holonomy, and γ_* is contained in the closure of the $\mathcal{F}_A$-support of $\mathbf{m}$, all of which is contained in the closed set $\hat{K}$. It therefore remains to observe that the path γ_* determines a closed path in the base space B, and hence a class denoted the same in $\pi_1(B, b_0)$ for appropriate basepoint $b_0 = p(z_*)$. The holonomy of this element is $C^{1+\alpha}$-conjugate to the map G in a neighborhood of the fixed-point on the fiber $p^{-1}(b_0)$, so has a linearly contracting fixed-point in the set K. $\square$

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