# Almost Compact Foliations *

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**Abstract**

A foliation is em almost compact if the set of non-compact leaves is countable, and *essentially compact* if the set of non-compact leaves has Lebesgue measure zero. In this note, we consider the question whether an almost or an essentially compact foliation must have all leaves compact. Our first result is that if there exists at least one non-proper, non-compact leaf, then there are uncountably many non-compact leaves. Our second result extends the homology criteria of Edwrds, Millet and Sullivan, developed for compact foliations, to almost compact foliations. This is used to show that an almost compact foliation with a closed compact cross-section must be a generalized Seifert fibration. The third result is that an essentially compact foliation of codimension 2 must be a generalized Seifert fiber space with all leaves compact. This result has special interest in that we use the measurable Riemann mapping theorem, which gives a new proof of the previous result of Epstein and its extensions by Edwards, Millet and Sullivan and by E. Vogt. Several applications of these results are given; in particular, we show that a $C^1$-action of a finitely-presented group on a connected compact metric space either has a continuum of infinite orbits, or the action factors through the action of a finite group. Some elementary examples are given in the last section to illustrate the results of the paper.

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# 1 Main theorems

A *compact* foliation is one in which every leaf is a compact submanifold. We propose to call a foliation *almost compact* if there are at most a countable number of non-compact leaves, and *essentially compact* if the set of non-compact leaves has Lebesgue measure zero. In this paper we will address the following questions:

**PROBLEM 1.0.1** *Must an almost compact topological foliation of a compact manifold be a compact foliation?*

**PROBLEM 1.0.2** *Must an essentially compact $C^1$-foliation of a compact manifold be a compact foliation?*

Problem 1.0.2 has been attributed to Michel Herman, while Problem 1.0.1 is a variant of a question raised by Scot Adams: can a foliation of a compact manifold admit exactly one non-compact leaf?

In this paper we give a partial answer to Problem 1.0.1, as discussed below. Our main result is the following solution of Problem 1.0.2 for the case of group actions:

**THEOREM 1.0.3** *Let $\Gamma$ be a finitely-generated group acting via $C^1$-transformations on a compact manifold $X$. Then either there is a finite quotient group $\Gamma_f$ of $\Gamma$ such that the action factors through this quotient, or the set of non-closed orbits has positive Lebesgue measure.*

Compact foliations of compact manifolds have been extensively analyzed. In codimension one, a compact topological foliation has a double covering which fibers over a circle, and the fibers are double coverings of the leaves. In codimension two, the celebrated theorem of D. B. A. Epstein [7] for 3-manifolds and its generalization to higher dimensional manifolds by R. Edwards, K. Millet and D. Sullivan [6] and E. Vogt [23, 24], showed that a compact $C^1$-foliation must be a generalized Seifert space (that is, the leaf space is Hausdorff in the quotient topology). Each leaf then has finite holonomy and there is a global bound on the volumes of the leaves of the foliation. In contrast, in codimension three or greater, there exist compact, real analytic foliations of compact manifolds for which there is no upper bound on the volume of the leaves, and hence the transversal structure of the foliation is very complicated. D. Sullivan [22] gave the first such example of a compact (smooth) foliation, with each leaf a circle and there is no upper bound on the total lengths of the leaves. Sullivan's example was in codimension four; subsequently examples were constructed in codimension three, and also real analytic examples were produced.

The *Epstein hierarchy* [7, 6] is an order structure on the set of compact leaves which measures of the global complexity of the foliation in terms of the (transverse) asymptotic behavior of the leaf total-volume function. A compact foliation which is a generalized Seifert fibration has Epstein heierarchy with one level containing all of the leaves of the foliation. In general, a complicated Epstein heierarchy corresponds to more involved dynamical behavior of the leaves. E. Vogt developed in [25] a procedure for constructing compact foliations with arbitrarily complicated Epstein hierarchy. In particular, there is no upper bound on the volumes of the leaves in the foliations Vogt constructed.

For a codimension one topological foliation of a compact manifold $M$, the set of compact leaves is a closed subset, as a consequence of the well-developed structure theory for codimension one foliations (cf. Theorem 4.1.3, page 96, [12]). This implies that a codimension one foliation of a compact manifold is either compact, or there exists a non-empty open set of non-compact leaves.

Recall that a leaf $L$ of $\mathcal{F}$ is *proper* if the induced topology for the inclusion $L \subset V$ agrees with the manifold topology on $L$. Otherwise, we say that $L$ is *non-proper*. A foliation $\mathcal{F}$ is *proper* if every leaf is proper. K. Millett [14] generalized the Epstein hierarchy to proper foliations. However, for a non-compact proper leaf, examples of Inaba [13] show that triviality of the holonomy need not imply the leaf has a foliated open neighborhood, so the Epstein hierarchy is simply defined in terms of the local holonomy groups. This failure of local stability is at the heart of problems 1.0.1 and 1.0.2. The topological structure of proper foliations in higher dimensions with non-trivial Epstein hierarchy is not very well developed, and in this paper we investigate the special case of almost compact foliations.

We first describe some general results concerning the topological nature of non-compact leaves.

**THEOREM 1.0.4** *Let $L$ be a non-proper leaf of a topological foliation $\mathcal{F}$. Then the topological closure $\overline{L}$ contains uncountably many non-compact leaves.*

**COROLLARY 1.0.5** *An almost compact foliation is proper.*

We next address the basic question of whether there exist almost compact foliations which are not compact. Our results in this direction are based on extending the techniques of Edwards, Millet and Sullivan [6] to the case of almost compact foliations. The first result corresponds to Theorem 1 of [6].

**THEOREM 1.0.6** *Let $\mathcal{F}$ be a almost compact $C^1$-foliation of codimension $n$ of a compact manifold $M$ of dimension $m$, with $T\mathcal{F}$ an oriented subbundle of $TM$. Suppose there is an open half-space of $H_{m-n}(M;\mathbf{R})$ which contains all homology classes determined by a compact leaves of $\mathcal{F}$. Then $\mathcal{F}$ is a compact foliation, and hence is a generalized Seifert fibration.*

The following observation is due to Edwards, Millet and Sullivan in the compact case, and follows directly from Theorem 1.0.6 in the almost compact case.

**COROLLARY 1.0.7** *Suppose that every compact leaf of $\mathcal{F}$ has negative (respectively, positive) Euler characteristic. If $\mathcal{F}$ is almost compact, then $\mathcal{F}$ is a generalized Seifert fibration.*

A compact leaf of an almost compact foliation also has restrictions on the Euler class $E(\nu) \in H^{2n}(M;\mathbf{R})$ of its normal bundle. In a previous work, the self-intersection numbers of the compact leaves of a $C^1$-foliation were studied, in relation to the dynamics of the foliation $\mathcal{F}$. For the case of almost compact foliations, we have:

**THEOREM 1.0.8** *Let $\mathcal{F}$ be a $C^1$-foliation of even codimension $2n$, with orientable leaves of dimension $m$. Then for each compact leaf $L \subset M$, the self-intersection class*

$$[L] \cap [L] = \langle [L], E(\nu) \rangle \in H_{m-n}(M;\mathbf{R}).$$

*is always zero.*

**COROLLARY 1.0.9** *Let $L \subset TM$ be a $C^1$-embedded compact orientable submanifold with self-intersection number $[L] \cap [L] \neq 0$. Then $L$ is not homotopic to a leaf of an almost compact $C^1$-foliation $\mathcal{F}$ of $M$.*

A closed cross-section to $\mathcal{F}$ is an immersed compact manifold without boundary $N \hookrightarrow M$ which is tranverse to $\mathcal{F}$ and intersects each leaf of $\mathcal{F}$ at least once. We have the following partial answer to Problem 1.0.1:

**THEOREM 1.0.10** *Let $\mathcal{F}$ be an almost compact topological foliation of a compact manifold which admits a closed cross-section. Then $\mathcal{F}$ is a generalized Seifert fibration.*

Methods similar to those used to prove Theorem 1.0.10 apply also to continuous group actions, to yield a generalization of the theorem of Montgomery and Zippin that a pointwise periodic homeomorphism is periodic (page 224, [15]).

**THEOREM 1.0.11** *Let $\Gamma$ be a finitely generated group. Suppose that $\phi : \Gamma \times Y \to Y$ is a continuous action on a compact topological manifold $Y$, and all but a countable number of orbits of $\phi$ are finite. Then there exists an integer $K > 0$ such that every orbit of $\phi$ has order at most $K$.*

In summary, we observe that if $\mathcal{F}$ is an almost compact, but not compact, foliation of a compact manifold $M$, then:

- the codimension of $\mathcal{F}$ is at least 2, or at least 3 if $\mathcal{F}$ is $C^1$;
- $\mathcal{F}$ does not admit a closed cross-section;
- all leaves of $\mathcal{F}$ are proper;
- each compact oriented leaf $L$ has zero self-intersection class;
- the closure $\overline{E}$ of the set of all non-compact leaves $E$ of $\mathcal{F}$ is nowhere dense;
- the *bad set* $B$ is not empty.

Finally, we conclude this paper with three elementary examples. The first shows that the hypothesis $M$ is compact cannot be removed. The next example is a topological foliation with uncountaby many non-compact leaves, but for which the set of non-compact leaves has Lebesgue measure zero. The construction of this example yields a counter-example Theorem 1.0.3 in the topological category, showing that the $C^1$-hypothesis is necessary. The third example illustrates the possibility that a minimal set in the closure of a proper leaf can contain non-proper leaves.

# 2 Topological dynamics

## 2.1 Non-proper leaves

We examine the geometry of a topological foliation in a neighborhood of a non-proper leaf in a "foliated space". Let $L$ be a non-compact leaf, and denote:

- $\overline{L}$ is the closure of $L$;
- $\overline{L}_c$ is the union of the compact leaves in $\overline{L}$;
- $\overline{L}_{nc} = \overline{L} \setminus \overline{L}_c$ is the union of the non-compact leaves in $\overline{L}$.

A subset $X \subset M$ is *$\mathcal{F}$-saturated*, or just *saturated*, if for every point $x \in X$ the leaf of $\mathcal{F}$ through $x$ is contained in $X$. If $X$ is a closed set, the restriction of $\mathcal{F}$ to $X$ then defines a *foliated space* or *foliated set* in the sense of [8, 16]. That is, $X$ is a topological space with a covering by "flow boxes" for the "foliation" $\mathcal{F}|X$, which consists of a disjoint union of flatly embedded topological submanifolds.

The topological closure of a leaf is a saturated set, hence $\overline{L}$ with the restricted foliation $\mathcal{F}|\overline{L}$ is a foliated space.

Let $X$ be a foliated space with foliation $\mathcal{F}|X$. A leaf $L \subset X$ is said to be *without holonomy* for $\mathcal{F}|X$ if for every closed path $\gamma \subset L$, the holonomy of $\mathcal{F}_X$ along $\gamma$ is the identity map. Let $X_{woh} \subset X$ denote the subset of leaves which are without holonomy for $\mathcal{F}|X$. The main theorem of [8] implies that $X_{woh}$ is a dense $G_\delta$ in $X$ whenever $X$ is a compact space.

Theorem 1.0.4 of the Introduction follows immediately from the following.

**PROPOSITION 2.1.1** *Let $L$ be a non-proper leaf. Then $U \cap \overline{L}_{nc}$ is an uncountable set for every non-empty, saturated, relatively open $U \subset \overline{L}$.*

**Proof**. Consider an open transversal $T \subset M$ to $\mathcal{F}$ which intersects $L$ in a non-empty set $S = L \cap T$. The hypothesis that $L$ is non-proper implies that the closure $\overline{S}$ of $S$ is a perfect set, hence must be uncountable. Consequently, $\overline{L}$ consists of an uncountable set of leaves. Moreover, there is a dichotomy: either $\overline{S}$ is open, or it is locally homeomorphic to a Cantor set (cf. Proposition 2.2.2, [12]).

Next note that a compact leaf $L_0 \subset \overline{L}$ must have non-trivial holonomy for $\mathcal{F}|\overline{L}$; otherwise, the structure theorem for topological foliations (Corollary 2, [10]) implies that there is a relatively open neighborhood $L_0 \subset U \subset \overline{L}$ which consists of a union of leaves, and $\mathcal{F}|U$ is a product foliation. This implies no leaf of $\mathcal{F}|\overline{L}$ is asymptotic to $L_0$ contradicting that $L_0$ is in the closure of $L$. We conclude that $\overline{L}_{woh} \subset \overline{L}_{nc}$.

Let $U \subset \overline{L}$ be non-empty, saturated and relatively open, then $U \cap \overline{L}_{woh}$ is a dense $G_\delta$ in $U$. For the transversal $T$ above, $U \cap T$ will be second category, and so a dense $G_\delta$ subset must also be second category. If $U \cap \overline{L}_{nc}$ is a countable union of leaves, then the subset $\overline{L}_{woh} \cap T \subset \overline{L}_{nc} \cap U \cap T$ is a countable set, giving a contradiction. $\square$

## 2.2 Topological dynamics of proper leaves

In this section, we collect several observations about the "dynamics" of proper leaves, which address two aspects of their geometry: the structure of the $\omega$-limit set of the leaf, and the relation between the endset of the leaf and its closure. We first recall a basic property of proper leaves:

**LEMMA 2.2.1** *Let $X$ be a compact foliated set. Then $L$ is proper leaf if and only if $(\overline{L} \setminus L)$ is compact.*

**Proof**. Suppose that $L$ is proper. Every point $y \in L$ has an open neighborhood $U$ which intersects $L$ in a connected disc contained in the leaf through $y$, and $U$ contains no other points of $L$. Thus, $L$ is relatively open in the compact set $\overline{L}$ hence $\overline{L} \setminus L$ is compact.

Conversely, assume $\overline{L} \setminus L$ is compact then given $y \in L$ there are disjoint an open neigborhoods $(\overline{L} \setminus L) \subset V$ and $y \in U$. Let $P$ denote the open connected component of $L \cap U$ containing $y$. Choose an open subset $y \in U' \subset U$ which is disjoint from $(L \setminus (L \cap V)) \setminus P$. Then $U'$ intersects $L$ in the relatively open neighborhood $P$ of $y$. □

Define a partial ordering on leaves: $L_0 < L_1$ if $L_0 \subset \overline{L}_1$.

**COROLLARY 2.2.2** *The partial ordering restricted to the set of proper leaves is strict. That is, $L_0 < L_1 < L_0$ implies $L_0 = L_1$.*

**Proof**. Assume that $L_1 < L_0$ and $L_0 \neq L_1$. By Lemma 2.2.1, $\overline{L}_0 \setminus L_0$ is closed, so $L_1 \subset \overline{L}_0$ implies $L_1 \subset \overline{L}_1 \subset (\overline{L}_0 \setminus L_0)$, so that $L_0 \cap \overline{L}_1 = \emptyset$ and $L_0 \not< L_1$. □

For a proper foliation of codimension one, there is a close relation between the partial ordering on the proper leaves and the structure of the ends of the leaves. This is part of the theory of levels for proper leaves, as developed by Cantwell and Conlon [3, 2, 5] and Nishimori [18, 17, 19, 20, 21]. (These papers consider $C^2$-foliations mostly, but their arguments in many cases apply also to proper leaves in topological foliations.) One consequence of these works is that the partial ordering on the proper leaves can be of infinite length (except in the case of analytic foliations [5].)

The theory of levels is not very well developed for foliations of higher codimension. The basic strategy is to introduce the derived filtration on the endset $\epsilon(L)$ of the leaf, and compare that to the derived filtration of the $\omega$-limit set. Recall the construction of the topological space $\mathcal{E}(L)$. Consider the collection of open subsets

$$\{U \mid U \text{ is a connected component of } \; L - K \; \text{ for some compact } \; K \subset L\}$$

This is ordered by inclusion, and $\mathcal{E}(L)$ is the set of ultrafilters obtained from this set, with the natural topology. That is, a point $\epsilon \in \mathcal{E}(L)$ is determined by a proper descending chain

$$U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$$

where $\cap U_n = \emptyset$. Two chains $\{U_n\}$ and $\{V_m\}$ are equivalent, hence determine the same end, if for each $U_n$ there exists $V_{m(n)} \supset U_n$ and *vice versa*. We call $\{U_n\}$ a defining chain for $\epsilon$. Topologize $\mathcal{E}(L)$ as follows: an open set $U$ is a *neighborhood* of $\epsilon$ if there exists a defining chain $\{U_n\}$ for $\epsilon$ so that $U_n \subset U$ for $n$ sufficiently large. Each open set $U$ determines an open set $U_\epsilon \subset \mathcal{E}(L)$, which is just the collection of all ends for which $U$ is a neighborhood.

The end set $\mathcal{E}(L)$, and also $L$ itself, is said to have *finite type* if the $r^{th}$-derived set $\mathcal{E}(L)^{r+1}$ is empty for some $r < \infty$. We call the least such $r$ the *depth* of $L$. The endset is *perfect* if $\mathcal{E}(L) = \mathcal{E}(L)^1$.

Each end $\epsilon \in \mathcal{E}(L)$ has a well-defined $\omega$-limit set

$$\overline{\epsilon} = \bigcap_n \overline{U_n}$$

Here are some well-known topological properties of the limit sets $\overline{\epsilon}$. For example:

**LEMMA 2.2.3 (cf. 1.28, [9])** *$\overline{\epsilon}$ is connected.*

**Proof**: Let $\epsilon$ be defined by $\{U_n\}$. Suppose $U$ and $V$ are open sets in $M$ which intersect $\overline{\epsilon}$ non-trivially. Then both $U \cap L$ and $V \cap L$ must contain $U_n$ for some $n$ sufficiently large, hence cannot be disjoint. □

**LEMMA 2.2.4** *Let $X$ be a compact foliated set. Then $\overline{\epsilon}$ either contains a non-proper leaf, or each minimal set in $\overline{\epsilon}$ consists of a single compact leaf.*

**Proof**. Zorn's Lemma implies that every closed saturated subset of the compact space $M$ contains a minimal saturated closed subset. Every leaf in such a minimal set $Z \subset \overline{\epsilon}$ must be dense in $Z$. Thus, either $Z$ consists of a single compact leaf or each leaf in $Z$ must contain itself in its closure, hence cannot be proper. □

**COROLLARY 2.2.5** *Suppose that $\mathcal{F}$ is a proper topological foliation of a compact manifold $M$, then every minimal set of $\mathcal{F}$ consists of a single compact leaf.* □

Even for a proper foliation, the structure of the foliated set $\overline{\epsilon}$ can be complicated. For example, $\overline{\epsilon}$ can consist of a compact foliation on a proper submanifold (see Example 5.0.2 below). We single out the special case when $\overline{\epsilon}$ reduces to the simplest case:

**DEFINITION 2.2.6** *A proper leaf $L$ has the* Poincaré-Bendixson property *if it has finite type of depth $r$, and for each $\epsilon \in \mathcal{E}(L)^r$ the limit set $\overline{\epsilon}$ consists of a compact leaf.*

One of the main theorems of codimension one proper foliations is that every leaf has the Poincaré-Bendixson property [2, 3, 4, 11], and there is a close relation between the dynamics of $L$ and the derived filtration of $\overline{L}$. For higher codimensions, it is an open problem to develop a similar relation, even for a leaf with the Poincaré-Bendixson property.

Note that if $\mathcal{F}$ has only a finite number of non-compact leaves, then all leaves must have finite type as the compact leaves have type 0. This raises the question:

**PROBLEM 2.2.7** *Does every leaf of an almost compact foliation have finite type?*

relation between the endset topology (intrinsic) and the derived set topology. for example:

**LEMMA 2.2.8** *Let $L_0 < L_1$ then there is an inclusion $\mathcal{E}(L_0) \subset \mathcal{E}(L_1)$.* □ *???*

another example: if $\epsilon_i \to \epsilon_0$ then $\cup \overline{\epsilon_i}^1 \subset \overline{\epsilon_0}$ That is, the derived operation is preserved by realization

Then go into compact asympts in next section.

finally look into homology questions

## 2.3 Asymptotics in almost compact foliations

In this section, we study the asymptotic properties of compact leaves in an almost compact foliation. This leads to a type of structure theorem for almost compact foliations, in terms of the possible asymptotic cycles defined by sequences of compact leaves.

We will assume given a "volume" function on the space of compact leaves, which assigns to a leaf $L$ its total volume $V(L)$. For a $C^1$-foliation, this can be defined by choosing a Riemannian metric on $TM$. Then $V(L)$ is the total volume of $L$ for the induced Riemannian metric on $TL$.

For a topological foliation, the volume function $V(L)$ is integer valued, and defined via a combinatorial approach. Cover $M$ by a locally-finite collection of relatively compact, open flow boxes for $\mathcal{F}$. We moreover assume that the closure of each flow box is properly contained in a larger (open) flow box. We obtain an open transversal $T \subset M$ by taking $T$ as the union of the local transversals defined by each flow box of the covering of $\mathcal{F}$. Then define $V(L)$ as the number of points in the intersection of $L$ with the open transversal $T$. As a matter of practice, one can define $V(L)$ for any transversal $T$ to $\mathcal{F}$, where we require only that $T$ be locally flat, intersect every leaf of $\mathcal{F}$ and be open in the sense of transversals. Equivalently, $T$ intersects each leaf of $\mathcal{F}$, and for each point $x \in T$ there is an open subset $U_x \subset T$ which is an open subset of a local transversal to $\mathcal{F}$ defined by a local flow box. We will use the notation $V_T(L)$ when necessary to indicate the choice of tranversal used in the definition of the volume function.

The volume function $V(L)$ defined as above enjoys several nice properties which relate it to the behavior of the compact leaves of a foliation. For a point $x \in M$, let $L_x$ denote the leaf of $\mathcal{F}$ through $x$. The following combination of Propositions 4.1 and 8.1 of [6] remains true for the case of proper foliations. See sections 4 and 8 of that paper for details of the notation.

**PROPOSITION 2.3.1** *Suppose that $\mathcal{F}$ is a proper topological foliation of a compact manifold $M$ without boundary, and suppose that $X$ is any locally compact saturated subset of $M$. Then the restricted volume function $V|X : X \to (0, \infty]$ has the following properties, at any $x \in X$:*

**Compactness.** *$V(L_x)$ is finite if and only if $L_x$ is compact.*

**Semi-continuity.** *If $L_x$ is compact, then $V|X$ is lower semi-continuous at $x$, as follows: for any integer $n > 0$ and $\epsilon > 0$, then for any $y$ in a sufficiently small neighborhood $U_\epsilon$ of $x \in X$, either:*

- *$V(L_y) > nV(L_x)$, or*
- *there exists an integer $j$, $1 \leq j \leq n$, such that $|V(L_y) - j \cdot V(L_x)| < \epsilon$*

**Continuity.** *For $L_x$ compact, the following conditions are equivalent:*

- *the restricted volume function $V|X$ continuous at $x$;*
- *the restricted holonomy group $\mathcal{H}_x|X$ of $\mathcal{F}|X$ at $x$ is trivial;*
- *the local holonomy pseudogroup at $x$ is stable and trivial.*

**Boundedness.** *For $L_x$ compact, the following conditions are equivalent:*

- *the restricted volume function $V|X$ is bounded on some neighborhood of $x \in X$;*
- *the restricted holonomy group $\mathcal{H}_x|X$ of $\mathcal{F}|X$ at $x$ is finite.*
- *the local holonomy pseudogroup at $x$ is stable and finite.* □

The key concept for the study of compact foliations is Epstein's idea of the *bad set.* We extend this definition to almost compact foliations as follows:

**DEFINITION 2.3.2** *Let $\mathcal{F}$ be an almost compact foliation.*

- *The* bad set *$B$ of $\mathcal{F}$ is the union of all leaves $L$ such that there exists a sequence of* compact *leaves $\{L_i \mid i = 1, 2, \ldots\}$ such that*

$$L \subset \bigcap_{p>0} \left\{ \overline{\bigcup_{i=p}^{\infty} L_i} \right\} \quad \text{and} \quad \lim_{i\to\infty} V(L_i) = \infty \tag{1}$$

- *The* good set*, $G$, of $\mathcal{F}$ is the union of the leaves not in the bad set;*
- *The* exceptional set*, $E$, is the union of the non-compact leaves.*

**LEMMA 2.3.3** *1. the bad set $B$ is closed;*

*2. $B$ contains all non-compact leaves of $\mathcal{F}$;*

*3. the set of leaves without holonomy $G_{woh} \subset G$ is open and dense;*

**Proof**. (1) Let $\{L_j\} \subset B$ be a sequence of leaves, with $L$ a leaf in their closure. For each $j$ there exists a sequence of compact leaves $\{L_{jk}\}$ satisfying (1). Then there exists a subsequence $\{\text{Ł}_{jk(j)}\}$ where $k(j) \to \infty$ with $L$ in its closure, so $L$ satisfies (1).

(2) Every leaf is in the closure of its nearby leaves, so a non-compact leaf in an almost compact foliation is in the closure of the nearby compact leaves. The boundedness property implies that every leaf in the closure of a sequence of leaves with uniformly bounded volumes must have bounded volume, hence is compact. Thus, $E \subset B$.

(3) is a consequence of the genericity of the set of leaves without holonomy and the fact that the set $E$ is countable. □

**COROLLARY 2.3.4** *Let $\mathcal{F}$ be an almost compact foliation of a compact manifold M, with a non-compact leaf L. Then $\overline{L} \subset B$, and in particular, $B$ is not empty.*

**Proof**. $L \subset B$ which is a closed subset of $M$. □

One of the key ideas in the Edwards, Millet and Sullivan [6] study of compact foliations is the *Moving Leaf Proposition*, a version of which holds for almost compact foliations.

**PROPOSITION 2.3.5 (Moving Leaf Proposition)** *Let $\mathcal{F}$ be an almost compact topological foliation of a compact manifold M. Assume that the bad set $B$ is not empty. Then there exists a one parameter path $\{L_t \mid 0 \le t < \infty\}$ where:*

- *$L_t$ is compact without holonomy for all $0 \le t < \infty$;*
- *$L_t$ depends continuously on $t$;*
- *$V(L_t)$ tends to $\infty$ with $t$;*
- *For each saturated open neighborhood $U$ of $B \cup E$, there is a number $t_U$ such that $t > t_U$ implies $L_t \subset U$.*

**Proof**. The good set $G$ is a countable disjoint union of connected open sets, $G = \bigcup_{\ell \in \mathcal{I}} G_\ell$. The fixed-point set of a homeomorphism of finite order does not separate if the homeomorphism is not the identity. Therefore, the intersections $N_\ell = G_\ell \cap G_{woh}$ are again open connected sets, and in particular are the path components of $G_{woh}$.

**LEMMA 2.3.6** *The restricted function $V|N_\ell$ is unbounded for some $\ell$.*

**Proof**. Suppose first that the bad set $B$ is not a subset of $\overline{E}$. Then the proof of the moving leaf proposition (pages 22-23, [6]) shows that for each point $x \in B \setminus (\overline{E} \cap B)$, there is a connected component $N_{\ell_x}$ containing $x$ in its closure and $V|N_{\ell_x}$ is unbounded.

Next, assume that $B \subset \overline{E}$. The set $E$ is countable, so the complement $M \setminus E$ is a connected set. Therefore, the open connected components of $G = M \setminus \overline{E} = M \setminus (B \cup E)$ are in one-to-one correspondence (via the obvious inclusions) with the open connected components of $M \setminus B$. It follows that for each leaf $L$ contained in $E$, there is an open connected component $N_L$ of $M \setminus B$ containing $L$. We can then follow the procedure of the proof of Proposition **??** to choose a sequence of compact leaves $\{L_i\} \subset N_L$, with $V(L_i)$ tending to infinity. By our remarks, the sequence $\{L_i\}$ is contained in $N_L \cap G_{woh}$ which is one of the components $N_\ell$. $\square$

Now to complete the proof of the Proposition, choose a sequence of leaves $\{L_i\} \subset N_\ell$ for some fixed $\ell$, with $V(L_i)$ tending to $\infty$. The function $V|N_\ell$ is continuous by Proposition 2.3.1, so we can join the leaves in this sequence by a continuous path of leaves in $N_\ell$ for which $V(L_t)$ tends to $\infty$ also. It only remains to observe that the leaf space of the restriction of $\mathcal{F}$ to $N_\ell$ is Hausdorff, so for every closed saturated subset of $N_\ell$ we can choose $T$ sufficiently large so that $L_t$ is disjoint from this set for $t > T$. $\square$

## 2.4 Foliations admitting a closed cross-section

In this section we begin the study of the rôle of homology obstacles to the existence of non-compact leaves in an almost compact foliation. We use an adaptation of the methods of Epstein [7] and Edwards, Millet and Sullivan [6]. The case when there exists a closed cross-section is simplest, and this is considered first.

Let $Y \subset M$ be a closed cross section to the topological foliation $\mathcal{F}$. We assume $Y$ is locally a transversal as discussed in section 4. We can also assume without loss of generality, by passing to appropriate finite covers as necessary, that the topological tangent bundle to $M$ and the normal microbundle to $\mathcal{F}$ are oriented (and hence each compact leaf of $\mathcal{F}$ is endowed with a consistent orienteation), and that the transversal $Y$ is an oriented compact manifold.

The proof of Theorem 1.0.10 follows from the next Lemma and Proposition 2.3.5.

**LEMMA 2.4.1** *Suppose that $Y \subset M$ is a closed cross-section to $\mathcal{F}$ as above. Define the volume function $V_Y$ using the transversal $Y$. Then for each open connected component $N_\ell$ of the set $G_{woh}$, the restriction $V_Y|N_\ell$ is constant.*

**Proof**. We briefly recall the proof from [6]. The compact leaves of $\mathcal{F}$ and the submanifold $Y$ are oriented and transverse, so the value of $V_Y(L)$ is just the absolute value of the algebraic intersection number of $L$ with $Y$. The algebraic intersection is determined by their homology classes, so that $V_Y(L_t)$ is constant on a continuous family $\{L_t\}$. In particular, the set $N_\ell$ is path connected, so $V_Y$ is constant there. □

Theorem 1.0.11 is a direct corollary of Theorem 1.0.10 if $\Gamma$ is assumed to be finitely presented. This is based on the observation that there then exists a compact oriented manifold $M_\Gamma$ with fundamental group $\Gamma$, and the suspension construction yields a topological foliation $\mathcal{F}_\phi$ on the total space of a fibration $M \to M_\Gamma$, such that the holonomy action $h_{\mathcal{F}_\phi} : \Gamma \to \mathrm{Homeo(Y)}$ is conjugate to the representation $\phi$ (cf. Chapter 5, [1]). The orbit structure of the action $\phi$ is mirrored in that of the leaves of $\mathcal{F}$, so the conclusions of Theorem 1.0.10 imply those of Theorem 1.0.11.

For the case where $\Gamma$ is only assumed to be finitely generated, we prove Theorem 1.0.11 by following through the steps of the above lemmas and propositions, replacing leaves with the orbits of points of $Y$. The hypotheses of Theorem 1.0.11 imply that there is a countable set of points $I \subset Y$ whose orbits under the action $h_{\mathcal{F}}$ are infinite, and all points in the set $F = Y \setminus I$ have finite orbit. The "volume" function will be defined by $V : Y \to [1, \infty]$, where $V(y) = \infty$ for $y \in I$ and $V(y)$ is the number of points in the orbit of $y$ for $y \in F$. This function has the properties discussed in Proposition 2.3.1.

The good, the bad and the exceptional sets are similarly defined, and we observe that the proofs of the lemmas of section 3 hold more generally for topological group actions. We leave to the reader the task of checking the remaining details. It remains to show:

**LEMMA 2.4.2** *Let $\phi : \Gamma \to \mathrm{Homeo(Y)}$ be a representation of a finitely generated group $\Gamma$ and $K > 0$ a constant such that every orbit of $\phi$ has order at most $K$. Then there is a finite group $\tilde{\Gamma}$, a representation $\tilde{\phi} : \tilde{\Gamma} \to \mathrm{Homeo(Y)}$ and a homomorphism $\rho : \Gamma \to \tilde{\Gamma}$ such that $\phi = \tilde{\phi} \circ \rho$.*

**Proof**. Let $\Gamma_0$ be the normal subgroup of $\Gamma$ generated by the $p^{th}$ powers of the elements of $\Gamma$, where $p = K!$. By assumption, $\phi(\Gamma_0)$ acts trivially on $Y$, and therefore $\phi$ induces an action $\tilde{\phi}$ of $\tilde{\Gamma} = \Gamma/\Gamma_0$ on $Y$. We then observe that $\tilde{\Gamma}$ is a finitely generated $p$-group, which must be finite. □

# 3 $C^1$-foliations with a homology condition

tototo

# 4 Measurable theory

## 4.1 Essentially compact foliations

This is the hardest section.

# 5 Examples

We give examples to illustrate some of the behavior of non-compact proper leaves and their limit sets.

**EXAMPLE 5.0.1** Consider the foliation of $\mathbf{R}^3$ obtained by deleting one point from the Hopf foliation of $S^3$ by circles. The resulting foliation has all leaves circles, except for the one exceptional line corresponding to the circle containing the deleted point. This trivial example shows that compactness, either of $M$ or of a closed transversal to $\mathcal{F}$, is absolutely required in the results of this paper. □

**EXAMPLE 5.0.2** Construct any compact foliation of N then extend to N times [0,1] so that the leaves are proper and converge to all of N. □

**EXAMPLE 5.0.3** This example shows that the set of non-compact leaves need not have positive Lebesgue measure for a topological foliation, even though there are uncountably many non-compact leaves.

Construct the "Cantor function" $f : [0,1] \to [0,1]$ which is continuous, has range in the dyadic rationals on the complement of the "middle third" Cantor set $\mathbf{K} \subset [0,1]$, and is increasing on the complement. The range of $f$ is all of $[0,1]$, and the image of the Cantor set $\mathbf{K}$ (which has Lebesgue measure zero) is a set of full measure 1. In particular, the Cantor function $f$ is not absolutely continuous.

Define an "exotic" action $\phi$ of $\mathbf{Z}$ on the cylinder $[0,2] \times \mathbf{S}^1$ using the Cantor function to define a homeomorphism: For $[r, \theta] \in [0,2] \times \mathbf{S}^1$, set

$$\phi[r,\theta] = \left\{ \begin{array}{cc} [r, \theta + 2\pi f(r)] & \text{for } \ 0 \leq r \leq 1 \\ [r, \theta + 2\pi f(2-r)] & \text{for } \ 1 \leq r \leq 2 \end{array} \right\}$$

Clearly, the set of periodic orbits for $\phi$ has full measure, and the set of non-periodic orbits is non-empty. The circles $\{0\} \times \mathbf{S}^1$ and $\{2\} \times \mathbf{S}^1$ are fixed by the action, so we can identify them to points to obtain homeomorphism $\hat{\phi} : \mathbf{S}^2 \to \mathbf{S}^2$ which is a counter-example to Theorem 1.0.3 in the topological category.

The suspension of this example yields a 1-dimensional foliation of the manifold

$$\mathbf{S}^1 \times \mathbf{S}^2 \cong (\mathbf{R} \times \mathbf{S}^2)/(x, [r,\theta]) \sim (x+1, \hat{\phi}[r,\theta])$$

where almost every leaf is a circle. □

**EXAMPLE 5.0.4** We give an elementary example to show that the closure of a
□

**EXAMPLE 5.0.5** This example takes the Reeb foliation of $T^2$ and suspends it to get leaves spiraling in on $T^2$ with the circle foliation. □

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