The Transverse Euler Class for Amenable Foliations*

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Abstract

We prove that the *transverse Euler class* of a foliation vanishes in the measurable leafwise cohomology theory for an amenable foliation. When there exists a holonomy invariant transverse measure for the foliation, we prove that the corresponding *average transverse Euler class* vanishes if the foliation is amenable with respect to this measure. These results generalize the Hirsch-Thurston vanishing theorem for the Euler class of foliated sphere bundles with amenable holonomy. The *average transverse Euler class* is also shown to vanish for a transverse invariant measure which satisfies one of the conditions: the measure is defined as the limit of an amenable averaging sequence; or, it is a discrete measure defined as the ω-limit of an averaging sequence; or, it is non-atomic with closed support in a tubular neighborhood of an *almost tangent submanifold* for the foliation.

Contents

1 Introduction and results

The purpose of this paper is to investigate the relationship between the values of the transverse Euler class $E(\nu, \mathcal{F})$ of a C^1-foliation $\mathcal{F}$ and the dynamics and geometry of $\mathcal{F}$. In particular, we give several vanishing theorems for the Euler class of the normal bundle to a C^1-foliation of a compact manifold. The first of these generalize a well-known vanishing result of Hirsch and Thurston [13] for the Euler class of flat vector bundles over a compact base. We then study the measured transverse Euler class, and its relationship to the dynamics of the foliation near the support of the measure.

The Euler class $E(\mathbf{E}) \in H^{2n}(M; \mathbf{R})$ of a $2n$-dimensional flat vector bundle $\mathbf{E}$ over a compact manifold M need not vanish, but there are strong restrictions on its values arising from geometric considerations. A famous result of J. Milnor [23] strictly bounds the Euler number of a flat oriented 2-plane bundle $\mathbf{E} \to \Sigma$ over a compact oriented surface Σ by one-half of the Euler characteristic $\chi(\Sigma)$ of Σ. J. Wood [37] generalized Milnor's inequality to foliated circle bundles $V \to \Sigma$ over a compact surface Σ, but with upper bound $\chi(\Sigma)$. Later work of D. Sullivan, M. Gromov and J. Smillie extended these results to flat oriented vector bundles with arbitrary even dimension, and odd-dimensional sphere bundles over a compact manifold with a C^1-foliation transverse to the fibers [9, 33].

There is a second type of restriction on the Euler class of a flat vector bundle, given in terms of the linear holonomy group $\Gamma \subset GL(2n, \mathbf{R})$ of the bundle. For example, the theorem of Hirsch and Thurston [13] cited previously implies that if the holonomy group Γ is *amenable*, then the Euler class $E(\mathbf{E}) \in H^{2n}(M; \mathbf{R})$ vanishes. There is also the remarkable result of P. Deligne and D. Sullivan [4], which states that if the linear holonomy group Γ is conjugate to a subgroup of the complex subgroup $GL(n, \mathbf{C}) \subset GL(2n, \mathbf{R})$, then a finite multiple of the bundle is trivial as a flat bundle. In particular, the real Euler class $E(\mathbf{E}) = 0$. Both of these results are special cases of a more general phenomenon which is a consequence of the van Est theorem for flat bundles [18, 30, 31]: Let $H(\Gamma) \subset GL^+(2n, \mathbf{R})$ be the *algebraic hull* of the holonomy group; if the universal Euler class $\tilde{E} \in H_c^{2n}(GL^+(2n; \mathbf{R}); \mathbf{R})$ in continuous cohomology restricts to zero in $H_c^{2n}(H(\Gamma); \mathbf{R})$, then $E(\mathbf{E}) = 0$.

In this paper, we will consider a C^1-foliation $\mathcal{F}$ of a compact oriented manifold V of dimension p, even codimension $2n$, and leaf dimension $m = p - 2n$. Let ν denote the normal bundle to $\mathcal{F}$. The *Bott connection* on ν is a natural, flat partial connection, ω^B, ([1]; see also Chapter 3, [19]) with covariant derivative operator denoted by $\nabla^{\mathcal{F}}$. The defining property of $\nabla^{\mathcal{F}}$ is that it equals the Lie derivative operator on smooth sections of ν when evaluated on vector fields along the leaves of the foliation. For the leaf $L_x \subset V$ through a point $x \in V$, ω^B restricts to an affine flat connection on $\nu|_{L_x} \to L_x$. (This is a consequence of the Jacobi identity for the Lie derivative operator.) Thus, a C^1-foliation yields a family of flat bundles, endowed with flat affine connections, over its leaves, parametrized by the points of V.

The leafwise deRham cohomology of $\mathcal{F}$ is the cohomology $H^*(\mathcal{F})$ of the complex of leafwise forms on V which are smooth when restricted to leaves, and continuous as global forms. Assume there are given orientations on the tangential distribution $T\mathcal{F}$ and the normal bundle ν. The *transverse Euler class* for $\mathcal{F}$ is defined as the "Euler characteristic class" of ν for this cohomology theory, and is denoted by $E(\nu, \mathcal{F}) \in H^{2n}(\mathcal{F})$.

Recall that a transverse measure μ for a foliation $\mathcal{F}$ is *quasi-invariant* if the property that a transversal $X \subset V$ has μ-measure zero is invariant under the transverse holonomy of $\mathcal{F}$.

Given a transverse measure μ to $\mathcal{F}$ which is quasi-invariant, we can also consider the complex of leafwise smooth, transversally μ-measurable forms on $\mathcal{F}$, with cohomology groups denoted $H^*_\mu(\mathcal{F})$ (cf. Zimmer [39]). The *measurable transverse Euler class* is the "Euler class" for ν in this cohomology theory, denoted by $\mathcal{E}_\mu(\nu, \mathcal{F})$. For the special case of transverse Lebesgue measure on V, we denote this class by $\mathcal{E}(\nu, \mathcal{F})$.

A *measured foliation* is a pair $(\mathcal{F}, \mu)$, where $\mathcal{F}$ is a C^1-foliation and μ is an invariant transverse locally finite measure. The existence of a non-trivial transverse invariant measure is a dynamical hypotheses about $\mathcal{F}$ (cf. [28, 32].) A measured foliation $(\mathcal{F}, \mu)$ with $T\mathcal{F}$ oriented has a *Ruelle-Sullivan foliation cycle* [29], C_μ, which is a closed current on V. Its cohomology class, $[C_\mu] \in H^{2n}(V; \mathbf{R})$, plays the role of the fundamental class $[M]$ of the base M for the study of the Euler characteristic of a flat bundle.

Assume that the leaf dimension equals the codimension, $m = 2n$, for a measured foliation $(\mathcal{F}, \mu)$ with $T\mathcal{F}$ oriented. The (real-valued) *average transverse Euler characteristic* of the measured foliation is obtained by pairing the foliation class $[C_\mu]$ with the μ-transverse Euler class $\mathcal{E}_\mu(\nu, \mathcal{F})$:

$$\chi_\mu(\nu) = \langle [C_\mu], \mathcal{E}_\mu(\nu, \mathcal{F}) \rangle = \langle [C_\mu] \setminus \mathcal{E}_\mu(\nu, \mathcal{F}), [V] \rangle$$

This yields a subtle invariant of the foliation, as it is formed from a primary class for the flat normal bundles over the leaves. The general principle we will develop is that $\chi_\mu(\nu)$ is a measure of the "complexity" of the transverse holonomy of the foliation in a neighborhood of the support of μ. For example, our results support the conjecture that if the transverse measure μ has no *atoms* (compact leaves of $\mathcal{F}$ with positive μ-measure), then $\chi_\mu(\nu) = 0$.

The definition of $\chi_\mu(\nu)$ is similar to that of the *average (leafwise) Euler characteristic*, $\chi_\mu(\mathcal{F})$, of a measured foliation [3, 27], which is obtained by evaluating the leafwise Euler class $E(T\mathcal{F})$ on the foliation class $[C_\mu]$. The number $\chi_\mu(\mathcal{F})$ measures the "average Euler characteristic of 'large' compact subsets of a 'μ-typical leaf' of $\mathcal{F}$".

The *average transverse Euler class* of an oriented measured foliation $(\mathcal{F}, \mu)$ is defined for even codimension, but without restriction on the leaf dimension. It is the the homology slant product of the Ruelle-Sullivan homology class $[C_\mu]$ with the Euler cohomology class $E(\nu)$:

$$E(\nu, \mu) = [C_\mu] \backslash E(\nu) = [C_\mu] \setminus \mathcal{E}_\mu(\nu, \mathcal{F}) \in H_{m-2n}(V; \mathbf{R})$$

For the case $m = 2n$, this reduces to the average transverse Euler invariant, $E(\nu, \mu) = \chi_\mu(\nu)$, via the identification $H_0(V; \mathbf{R}) \cong \mathbf{R}$.

Let us now state the theorems which relate the "Euler invariants" introduced above, $E(\nu, \mathcal{F})$, $\mathcal{E}(\nu, \mathcal{F})$, $\mathcal{E}_\mu(\nu, \mathcal{F})$, $E(\nu, \mu)$ and $\chi_\mu(\nu)$, to the geometry and dynamics of the foliation. Our first result, Theorem 1.1, generalizes the Hirsch-Thurston Theorem for amenable flat bundles, and is the counterpart for the Euler class of the vanishing theorems for the Weil measures of amenable foliations [15].

THEOREM 1.1 *Let $\mathcal{F}$ be a codimension 2n, C^1-foliation of a compact manifold V. Suppose that the measured equivalence relation $\mathcal{R}(\mathcal{F}) \subset V \times V$ determined by $\mathcal{F}$ is amenable with respect to the Lebesgue measure class on $V \times V$. Then the measurable transverse Euler class $\mathcal{E}(\nu, \mathcal{F})$ vanishes.*

The proof of Theorem 1.1 follows from Lemma 3.6 and Corollary 4.2 below.

COROLLARY 1.2 *Let $(\mathcal{F}, \mu)$ be a measured C^1-foliation with $\mathcal{R}(\mathcal{F})$ amenable with respect to Lebesgue measure, and assume μ is an absolutely continuous transverse measure. Then the average transverse Euler class $E(\nu, \mu) = 0$.*

Corollary 1.2 follows directly from Proposition 2.8 below.

Theorem 1.1 has a novel application which provides an effective obstacle to the amenability of foliations. The leafwise Euler class can be paired with any transverse closed form on V with maximal transverse rank (the residual forms of Heitsch and Hurder [12]) to yield a cohomology class on V. In particular, for a codimension 2n, C^2-foliation the n^{th}-Pontrjagin class, $P_n(\nu) \in H^{4n}(V; \mathbf{R})$, is represented by a residual form. Pairing the class $\mathcal{E}(\nu, \mathcal{F})$ with $P_n(\nu)$ yields the third power of the Euler class, $E(\nu)^3 \in H^{6n}(V; \mathbf{R})$.

COROLLARY 1.3 *Suppose that $\mathcal{F}$ is a C^2-foliation on a smooth manifold V with $E(\nu)^3 \neq 0$. Then the equivalence relation $\mathcal{R}(\mathcal{F})$ is not amenable for the Lebesgue measure class.* $\square$

There exist transversally C^1-foliations on compact oriented manifolds with $E(\nu)^3 \neq 0$, using the recent work of T. Tsuboi [36] and the Thurston realization theorem [35], but it is not known whether there exists a C^2-foliation with $E(\nu)^3 \neq 0$.

For a measured foliation $(\mathcal{F}, \mu)$, the natural measure class on the equivalence relation $\mathcal{R}(\mathcal{F})$ is that determined by the measure μ. In section 3 we introduce the definition of a *completely amenable* foliation.

THEOREM 1.4 *Let $(\mathcal{F}, \mu)$ be a measured C^1-foliation on a compact manifold V. If the equivalence relation $\mathcal{R}(\mathcal{F})$ is completely amenable with respect to the transverse measure class of μ, then the μ-measurable transverse Euler class $\mathcal{E}_\mu(\nu, \mathcal{F}) = 0$. Consequently, the average transverse Euler class $E(\nu, \mu) = 0$, and when $m = 2n$ the average transverse Euler characteristic $\chi_\mu(\mathcal{F}) = 0$.*

Theorem 1.4 follows from Lemma 3.5 and Corollary 4.2 below.

Recall briefly the strategy of the the proof of the Hirsch-Thurston Theorem: The flat structure ∇ on the bundle E implies there exists a foliation $\mathcal{F}_\nabla$ on the unit sphere bundle $S(E) \to M$ of E, transverse to the S^{2n-1}-fibers. The leaves of $\mathcal{F}_\nabla$ are Galois coverings of the base M, associated to the holonomy homomorphism of the bundle. The amenability hypothesis implies there exist a holonomy invariant transverse measure for this foliation, which is a probability measure when restricted to the fibers. Hirsch and Thurston then observe that the existence of such a measure implies that the Euler class of E must vanish (cf. section 4 below.)

We wil adopt a similar approach for the case of the normal bundle to a foliation, which leads to the proofs of Theorems 1.1 and 1.4 in sections 3 and 4. Let $\pi : \hat{V} \to V$ denote the unit normal sphere bundle to $\mathcal{F}$. The Bott connection defines a C^0-foliation $\hat{\mathcal{F}}$ of $\hat{V}$, where the leaves of $\hat{\mathcal{F}}$ cover those of $\mathcal{F}$ (cf. Chapter 2, [19]). We are given a transverse measure class for $\mathcal{F}$, and to obtain the vanishing of the corresponding transverse Euler class it suffices to exhibit a measure $\hat{\mu}$ on the fibers of the projective normal bundle which is defined almost everywhere for μ, and is invariant under the holonomy of $\hat{\mathcal{F}}$. The idea of the proofs of both Theorems 1.1 and 1.4 is to formulate the appropriate amenability hypotheses which imply that such a measure exists, then follow the Hirsch-Thurston method to obtain vanishing in the appropriate cohomology theory.

Our considerations of transverse Euler invariants suggest formulating a concept of much weaker form of "amenability" for foliations, by specializing to the linear holonomy action on the projective normal bundle of the foliation groupoid. The following question is based on this notion, and by the Hirsch-Thurston method as applied in this paper to foliations, is the key dynamical concept for proving the vanishing of the transverse Euler class.

PROBLEM 1.5 *Find general criteria, in terms of the dynamics of $\mathcal{F}$, for when there exists a family of probability measures $\hat{\mu}$ on the fibers of the projective normal bundle $P^+(\nu)$ which is holonomy invariant for the foliation $\hat{\mathcal{F}}$.*

Our next two results address how the values of the transverse Euler invariants depend on the dynamics of the foliation in a neighborhood of a given transverse measure.

An *amenable sequence* for $\mathcal{F}$ is a collection $\{Z_i \mid i = 1, \ldots\}$ of closed subsets of leaves of $\mathcal{F}$ such that they are an *averaging sequence* for $\mathcal{F}$ (cf. [6], and section 5 below) , and for each i the holonomy group Γ_i of $\mathcal{F}$ generated by the closed paths in the subset Z_i is an amenable group.

THEOREM 1.6 *Let μ be an invariant transverse measure for $\mathcal{F}$ defined as a weak-* limit of an amenable sequence for $\mathcal{F}$. Then the average transverse Euler class $E(\nu, \mu) = 0$. Consequently, for $m = 2n$, the average transverse Euler characteristic $\chi_\mu(\nu) = 0$.*

Theorem 1.6 follows directly from Corollary 5.7.

No assumption is made about the support of the measure μ in Theorem 1.6. Let us introduce some standard terminology concerning the topological properties of the leaves in the support of a transverse measure μ. We say that it is:

- *atomic* if there is a compact leaf L of $\mathcal{F}$ with full μ-measure, and we then write $\mu = \mu_L$.
- *(countably) discrete* if it is a finite (countable) sum of atomic measures.
- *compact* if there is a Borel set in V consisting of a union of compact leaves with full μ-measure.
- *continuous* if every leaf of $\mathcal{F}$ has transverse measure equal to zero.

Every locally-finite invariant transverse measure can be decomposed into a disjoint sum of a countably discrete measure and a continuous measure, $\mu = \mu_a + \mu_c$ (cf. Introduction, [16]). The compact leaves in the support of μ_a are called the *atoms* of μ.

Now assume that μ is discrete, and also require that there exist an averaging sequence $\{Z_i \mid i = 1, \ldots\}$ which defines μ as a weak-* limit and is *disjoint* from μ. "Disjoint" means that for a compact leaf L in the support of μ, the intersection $L \cap \mathcal{Z}_i$ is empty for all i. Section 6 introduces a technique of "blowing-up" along compact leaves, or more generally along compact foliated submanifolds, and we use this to prove:

THEOREM 1.7 *Let μ be a discrete invariant transverse measure for $\mathcal{F}$, which is a weak-* limit of an averaging sequence for $\mathcal{F}$ that is disjoint from the support of μ. Then for each leaf L in the support of μ, the Euler class $E(\nu|L) \in H^{2n}(L; \mathbf{R})$ vanishes. In particular, the average transverse Euler class $E(\nu, \mu) = 0$.*

The proof of Theorem 1.7 follows from Proposition 6.4 below. There is a special case of this theorem, which generalizes the usual result that the Euler class of a bundle admitting a non-vanishing cross-section is zero. The following corollary also has applications to the vanishing of residues for foliations (cf. [11]):

COROLLARY 1.8 *Let L be a compact leaf of a C^1-foliation $\mathcal{F}$. Then the normal Euler class $E(\nu|L)$ for L vanishes if L is the ω-limit set of an averaging sequence.*

The main theorem of [16] states that the self-intersection class $[C_\mu] \cap [C_\mu]$ vanishes for a transverse measure with no atoms; and when there are atoms, it is equal to the average transverse Euler characteristic of the atomic part of the measure. This suggests the following:

CONJECTURE 1.9 *Let μ be an invariant transverse measure for $\mathcal{F}$. Suppose that μ is continuous; or more generally, that no atom of μ is isolated (for the ambient topology on V) in the support of μ. Then the average transverse Euler class $E(\nu, \mu) = 0$.*

Our last result gives a partial solution to this conjecture. A transverse measure μ is said to be *almost compact* if the closed support of μ is contained in a tubular neighborhood $\pi : U_N \to N$ of an *almost tangent* closed submanifold $N \subset V$. We say that the measure μ *covers* the submanifold N. (See section 7.)

THEOREM 1.10 *Let μ be an almost compact invariant transverse measure which covers an oriented submanifold $N \subset V$. Assume either that μ is not atomic, or μ is atomic with support on the compact leaf $L \subset U \to N$ which multiply covers the submanifold N under the projection π. Then*

$$E(\nu, \mu) = 0 \quad \text{and} \quad [C_\mu] \cap [C_\mu] = 0$$

As a corollary of the methods used to prove Theorem 1.10 in section 7, we obtain a vanishing theorem for the Euler class of vector bundles complementary to the Hirsch-Thurston results:

COROLLARY 1.11 *Let $\mathbf{E} \to M$ be an oriented $\mathbf{D}^{2n}$-bundle over a compact manifold M, with a C^1-foliation $\mathcal{F}$ that is transverse to the fibers. Assume there exists a non-trivial holonomy invariant transverse measure μ for $\mathcal{F}$ with compact closed support in $\mathbf{E}$, whose support does not consist of a single compact leaf that singly covers M, then the Euler class $E(\mathbf{E}) = 0$.*

Section 8 of the paper gives examples of foliated manifolds which illustrate the main results of this paper, and describe for the reader's convenience the known constructions of foliations relevant for understanding the properties of the transverse Euler class.

Finally, let us note that the extension of Milnor's inequality for flat bundles over surfaces to the average transverse Euler characteristic of a measured foliation has been considered by the second author in [24, 25]. For example, when the leaf dimension and codimension are both 2 and the transverse measure μ is defined by an averaging sequence of a leaf of subexponential growth (cf. [28]), then there is an estimate $|\chi_\mu(\nu)| \leq \frac{1}{2}|\chi_\mu(\mathcal{F})|$. Partial extensions of this estimate to arbitrary transverse measures and to foliations with leaves of dimension greater than two are given in [24], but the general extension remains an open problem.

2 Transverse Euler class

The Euler cohomology class, viewed as a characteristic class on real vector bundles, is characterized by a set of four axioms (cf. Chapter XII, section 5, [20]). There are several concrete realizations of the Euler class, each revealing a different aspect of its relation to geometry, and the equivalence of the various constructions is one of the basic exercises in differential topology (cf. sections 11 and 12, [2]). We will define the transverse Euler class using the differential form approach to this characteristic class. It is first necessary to introduce the various measurable cohomology theories associated to a foliated manifold, and for each cohomology theory there is a corresponding transverse Euler class. The main result of this section is that the transverse Euler class satisfies two "axioms", comparable to those for the usual Euler class. These axioms are the key to implementing the Hirsch-Thurston method of proving vanishing theorems for amenable holonomy.

Fix an oriented C^1-vector bundle $\mathbf{E} \to V$ with even fiber dimension $2n$. Let $\pi : P^+(\mathbf{E}) \to V$ denote the associated bundle of positive rays, with fibers diffeomorphic to S^{2n-1}.

DEFINITION 2.1 *A* global angular form *for* $\mathbf{E} \to V$ *is a* C^1*-exterior differential (2n-1)-form* ψ *on* $P^+(\mathbf{E})$ *such that:*

1. *For each* $x \in V$*, the cohomology class of the restriction of* ψ *to the sphere fiber* $\pi^{-1}(x)$ *equals the (integral) orientation cohomology class of the fiber.*
2. *The exterior differential* $d\psi$ *of* ψ *on the total space* $P^+(\mathbf{E})$ *is the lift of a closed* $2n$*-form* $e(\mathbf{E})$ *on* V*.*

The existence of the angular form can be shown by several methods. Chern-Weil theory gives an abstract construction when the unit sphere bundle has differentiability class at least C^2 (cf. Theorem 3.30, [19]). The angular form can also be directly constructed for sphere bundles using the transgression sequence for Cech cohomology of the fibration (cf. page 121, [2]).

PROPOSITION 2.2 *The cohomology class of the form* $e(\mathbf{E})$ *on* V *is independent of the choice of angular form* ψ*. The Euler class of* $\mathbf{E}$ *is defined to be its class* $E(\mathbf{E}) = [e(\mathbf{E})] \in H^{2n}(V; \mathbf{R})$*.* $\square$

A foliation $\mathcal{F}$ on a C^∞-manifold V without boundary is said to be *transversally* C^r if the tangent bundle $T\mathcal{F}$ is a C^r-subbundle of TV, *and* the leaves of $\mathcal{F}$ are smoothly immersed C^∞-submanifolds of V.

Assume that $\pi : \nu \to V$ is the normal bundle to a transversally C^r-foliation $\mathcal{F}$ on V, for $r > 0$. The Bott connection ω^B on ν defines a transversally C^{r-1}-foliation $\tilde{\mathcal{F}}$ on ν: the tangential distribution to $\tilde{\mathcal{F}}$ is defined as the horizontal vectors for ω^B which lie over $T\mathcal{F}$, so that $T\tilde{\mathcal{F}} = \pi^{-1}(T\mathcal{F}) \cap ker(\omega^B)$. The basic property of the Bott connection on ν is that its restriction to leaves is flat, which is equivalent to the integrability of the distribution $T\tilde{\mathcal{F}}$. The restriction of π to a leaf of $\tilde{\mathcal{F}}$ is local diffeomorphism, so is a Galois covering of a corresponding leaf of $\mathcal{F}$. The foliation $\tilde{\mathcal{F}}$ is invariant under the radial dilation on ν (cf. pages 25-27, [19]), and we define $\hat{\mathcal{F}}$ as the quotient transversally C^{r-1}-foliation on $P^+(\nu)$. Note that the leaves of $\hat{\mathcal{F}}$ also cover those of $\mathcal{F}$.

There is a second foliation on $P^+(\mathcal{F})$, denoted by $\pi^*\mathcal{F}$. The leaves of $\pi^*\mathcal{F}$ are the oriented projective normal bundles over the leaves of $\mathcal{F}$. Note that $\pi^*\mathcal{F}$ and $\mathcal{F}$ have the same codimension, but their leaf dimensions differ by 2n-1.

Let $T\mathcal{F}^*$ denote the dual vector bundle to the tangent bundle of $\mathcal{F}$, and let $\Lambda^{\cdot}(T\mathcal{F}^*)$ be the corresponding bundle of real exterior algebras. Observe that a vector $\omega \in \Lambda^q(T\mathcal{F}^*)$ is only defined on q-frames of vectors tangent to the leaves of $\mathcal{F}$. (A choice of a supplemental bundle $TM \cong T\mathcal{F} \oplus Q$ is necessary to define an embedding of graded algebra bundles, $\Lambda^{\cdot}(T\mathcal{F}^*) \subset \Lambda^{\cdot}(TV^*)$.)

Let $\Omega^q(\mathcal{F})$ denote the exterior algebra of smooth q-forms along the leaves of $\mathcal{F}$. That is, $\phi \in \Omega^q(\mathcal{F})$ is a section of the bundle $\Lambda^q(T\mathcal{F}^*)$ which is C^r on V, and smooth when restricted to the leaves of $\mathcal{F}$. The leafwise exterior differential defines a map $d_{\mathcal{F}} : \Omega^q(\mathcal{F}) \to \Omega^{q+1}(\mathcal{F})$ for each $0 \leq q \leq m$. The cohomology of this complex, $H^{\cdot}(\mathcal{F})$, is called the *smooth leafwise cohomology* .

The leafwise differential $d_{\mathcal{F}}$ is defined on any section of $\Lambda^{\cdot}(T\mathcal{F}^*)$ which is smooth when restricted to leaves of $\mathcal{F}$. Also, $d_{\mathcal{F}}$ preserves transverse measurability of sections. We can therefore introduce modifications of the smooth leafwise cohomology, based on allowing more general classes of sections with respect to their transverse behavior. Let $\Omega^{\cdot}_{r'}(\mathcal{F}) \stackrel{def}{=} \Gamma_{r'}(\Lambda^{\cdot}(T\mathcal{F}^*))$ denote the space of sections ϕ which are smooth along leaves, and for which both ϕ and $d_{\mathcal{F}}(\phi)$ have regularity class "r'" transverse to the foliation. When $r' = 0$, these are the sections for which ϕ and $d_{\mathcal{F}}\phi$ are continuous on V and C^∞ along leaves. The cohomology groups for these complexes are denoted by $H^{\cdot}_{r'}(\mathcal{F})$. When $r' = r$, we omit the subscript "r'" from the notation.

Given a transverse quasi-invariant measure μ, we write $\Omega^{\cdot}_{\mu}(\mathcal{F})$ to denote the sections which are smooth along leaves and such that both ϕ and $d_{\mathcal{F}}(\phi)$ are μ-measurable and bounded almost everywhere with respect to μ. Moreover, we identify two sections if they differ on a μ-measurable set of leaves with μ-measure zero. (A form ϕ in this complex is allowed to be undefined on a set which is a union of leaves of measure zero, by the device of declaring it equal to the zero section on this set.) The resulting cohomology theory is denoted $H^{\cdot}_{\mu}(\mathcal{F})$. This measurable cohomology theory for foliations is suggested by the works of G. Mackey [22], and was introduced by R. Zimmer in the preprint [39] for the study of foliation ergodic theory. For a C^1-foliation, Lebesgue measure is always quasi-invariant. We denote by $H^{\cdot}_{(\infty)}(\mathcal{F})$ the cohomology theory for Lebesgue measure.

LEMMA 2.3 *For every quasi-invariant transverse measure μ for $\mathcal{F}$, there exist natural maps*

1. $H^{\cdot}(\mathcal{F}) \to H^{\cdot}_0(\mathcal{F}) \to H^{\cdot}_{\mu}(\mathcal{F})$;
2. $\pi^* : H^{\cdot}_{\mu}(\mathcal{F}) \longrightarrow H^{\cdot}_{\mu}(\pi^*\mathcal{F})$.

Proof. (1) There are well-defined inclusions of complexes $\Omega^{\cdot}(\mathcal{F}) \subset \Omega^{\cdot}_0(\mathcal{F}) \subset \Omega^{\cdot}_{\mu}(\mathcal{F})$ which commute with the differentials. (2) The differential of the fibration map π induces a map on leafwise differential forms for any μ. □

DEFINITION 2.4 *Let $\mathcal{F}$ be a C^1-foliation of the smooth manifold V, with oriented normal bundle ν. A* global angular $\mathcal{F}$-form *for ν is a (2n-1)-form $\psi_{\mathcal{F}} \in \Omega^{2n-1}_0(\pi^*\mathcal{F})$ such that*

1. *$\psi_{\mathcal{F}}$ restricts to each fiber of π to yield the orientation class of ν;*
2. *$d_{\pi^*\mathcal{F}}(\psi_{\mathcal{F}}) = \pi^*(e_{\mathcal{F}}(\nu))$ for a $d_{\mathcal{F}}$-closed continuous 2n-form $e_{\mathcal{F}}(\nu) \in \Omega^{2n}_0(\mathcal{F})$.*

The construction of a global angular $\mathcal{F}$-form on $P^+(\nu)$ follows by the same methods as for bundles over V. The proof of Proposition 2.2 is also local, so equally applies to the form $e_{\mathcal{F}}(\nu)$.

DEFINITION 2.5 (Transverse Euler Class) *Let $\mathcal{F}$ be a C^1-foliation on V with oriented normal bundle ν.*

- *The* transverse Euler class $E(\nu, \mathcal{F}) \in H^{2n}(\mathcal{F})$ *is the cohomology class of the closed form $e_{\mathcal{F}}(\nu)$ considered as an element of the complex $\Omega^{2n}(\mathcal{F})$.*
- *The* measurable transverse Euler class $\mathcal{E}(\nu, \mathcal{F})$ *is the image of $E(\nu, \mathcal{F})$ in $H^{2n}_{(\infty)}(\mathcal{F})$*
- *The* μ-transverse Euler class $\mathcal{E}_\mu(\nu, \mathcal{F})$ *of a measured foliation $(\mathcal{F}, \mu)$ is the image of $E(\nu, \mathcal{F})$ in $H^{2n}_\mu(\mathcal{F})$.*

There are two key properties of the transverse Euler class that are used in the proof of the vanishing theorems of section 1.

PROPOSITION 2.6 (Naturality) *Let $\mathcal{F}$ be a C^1-foliation on V with oriented normal bundle ν, and $\mathcal{F}'$ a C^1-foliation on V' with oriented normal bundle ν'. Let $f : V' \to V$ be a C^1-mapping sending leaves of $\mathcal{F}'$ to those of $\mathcal{F}$, and which is transverse to $\mathcal{F}$. Then $E(\nu', \mathcal{F}') = \pm f^*(E(\nu, \mathcal{F}))$, with sign according to whether f is orientation preserving or reversing.*

Proof. Transversality of the map f implies that there is an induced map of the foliated normal sphere bundles, $\hat{f} : P^+(\mathcal{F}') \to P^+(\mathcal{F})$ which covers the map f. Therefore, the pull-back form $\psi'_{\mathcal{F}'} = f^*(\psi_{\mathcal{F}}) \in \Omega^{2n-1}_0(\pi^*\mathcal{F}')$ is a global angular $\hat{\mathcal{F}}'$-form for ν', and thus $E(\nu')$ is represented by $f^*(e_{\mathcal{F}}(\nu))$. □

PROPOSITION 2.7 (Vanishing) *Let $\mathcal{F}$ be a C^1-foliation with oriented normal bundle ν of dimension $2n$. If the normal bundle ν admits a continuous non-vanishing vector field on V which is C^∞ along leaves of $\mathcal{F}$, then $E(\nu, \mathcal{F}) = 0$.*

Proof. On the total space $P^+(\mathcal{F})$ the form $\pi^*(e(\nu)) = d_{\pi^*\mathcal{F}}(\psi_{\mathcal{F}})$. The existence of a non-vanishing vector field implies that there is a cross-section $\sigma : V \to P^+(\mathcal{F})$ to π, and thus $e_{\mathcal{F}}(\nu)$ is exact in the complex $\Omega^{2n}_0(\mathcal{F})$. □

The cohomology groups $H^{\cdot}_\mu(\mathcal{F})$ can be difficult to compute, so the main application of the leafwise cohomology theory is via its rôle in pairing with other geometric data. There are two examples of this which arise in this paper: pairing with a Ruelle-Sullivan class to obtain invariants of measured foliations [14]; and pairing with primary classes of the normal bundle, as in the definition of Weil measures for a foliation [12].

PROPOSITION 2.8 *Let $(\mathcal{F}, \mu)$ be a measured C^1-foliation of the compact manifold V with both $T\mathcal{F}$ and ν oriented bundles, where the leaves of $\mathcal{F}$ have dimension m. Then there exists a bilinear pairing*

$$\times_\mu : H^q_\mu(\mathcal{F}) \times H^{m-q}(V; \mathbf{R}) \longrightarrow \mathbf{R} \tag{1}$$

Proof. For a closed form $\psi \in \Omega^{m-q}(V)$, the exterior product $\phi \wedge \psi \in \Omega^m_\mu(\mathcal{F})$ is again a closed leafwise smooth form. The integral of $\phi \wedge \psi$ against the transverse measure μ is then well-defined (using the partition-of unity method [29, 14]), and we set

$$\times_\mu(\phi, \psi) = \int_V \phi \wedge \psi \; d\mu \; . \tag{2}$$

The leafwise Stokes' Theorem then implies that the pairing (2) does not depend on the representatives ϕ and ψ chosen from their respective cohomology classes (cf. proof of Theorem 1.6, [14] or Proposition 2.6, [12]). □

The cohomology theory $H^{\cdot}_\mu(\mathcal{F})$ is a module over the algebra $B^\infty(V/\mathcal{F})$ of Borel functions on V which are constant on leaves, so the method of (section 2, [12]) yields:

COROLLARY 2.9 *For each class $[\phi] \in H^q_\mu(\mathcal{F})$, there is a well-defined, countably additive measure on the Σ-algebra $B^\infty(V/\mathcal{F})$ of measurable $\mathcal{F}$-saturated sets in V, represented by the linear functional*

$$\begin{array}{rcl} [\phi]_\mu : B^\infty(V/\mathcal{F}) & \longrightarrow & H_{m-q}(V; \mathbf{R}) \\ {[\phi]_\mu(f)} & = & \times_\mu(f \cdot \phi, \psi) \; , \quad f \in B^\infty(V/\mathcal{F}). \quad \Box \end{array} \tag{3}$$

COROLLARY 2.10 *Let $(\mathcal{F}, \mu)$ be a codimension-2n measured C^1-foliation of the compact manifold V with both $T\mathcal{F}$ and ν oriented bundles. Then there exists a well-defined* average transverse Euler class*,*

$$E(\nu, \mu) = [E(\nu, \mathcal{F})]_\mu(1) \in H_{m-2n}(V; \mathbf{R}). \quad \Box \tag{4}$$

For the second application of leafwise cohomology, let $A^{\cdot}(V/\mathcal{F}) \subset \Omega^{\cdot}_1(V)$ denote the ideal of differential forms on V which vanish when restricted to the leaves of $\mathcal{F}$. Define $\mathcal{R}(V/\mathcal{F}) = \wedge^{2n} A^{\cdot}(V/\mathcal{F})$ to be the $2n$-power of this ideal. Recall from (section 2, [14]; and also [12]) that a form $\psi \in \mathcal{R}(V/\mathcal{F})$ is said to be *residual.* Locally, such ψ has a factorization $\psi = \psi_0 \wedge \omega_0$, where ω_0 is a transverse $2n$-form. The Frobenius theorem implies that $d = d_{\mathcal{F}}$ on $\mathcal{R}(V/\mathcal{F})$, so that this is a differential ideal in $\Omega^{\cdot}_1(V)$ for the full exterior differential d. Its cohomology groups are denoted by $H^{\cdot}(V, \mathcal{F})$.

Observe that $H^q(V, \mathcal{F}) = 0$ for $0 \leq q < 2n$, and that $H^{2n}(V, \mathcal{F})$ consists of invariant transverse measures for $\mathcal{F}$ which are represented by closed C^1-forms on V. The interest in this cohomology theory is that for a C^2-foliation and for every Chern form C of degree $2n$ on $GL(2n, \mathbf{R})$, the Chern-Weil construction of characteristic classes using the Bott connection for ν ([1]; or see Chapter 4, [19]) yields a well-defined invariant $\Delta(C) \in H^{4n}(V, \mathcal{F})$ of the foliation $\mathcal{F}$. It is straightforward to show:

PROPOSITION 2.11 *For a C^1-foliation $\mathcal{F}$ on V, there is a well-defined pairing*

$$H^q_{(\infty)}(\mathcal{F}) \times H^{q'}(V, \mathcal{F}) \longrightarrow H^{q+q'}(V; \mathbf{R}). \quad \Box \tag{5}$$

COROLLARY 2.12 *Let $\mathcal{F}$ be a C^2-foliation with oriented normal bundle ν. Let $E(\nu) \in H^{2n}(V; \mathbf{R})$ denote the Euler class of ν. Then $E(\nu)^3 = E(\nu, \mathcal{F}) \wedge \Delta(C_{2n})$, where $C_{2n} = P_n$ is the n^{th}-Pontrjagin symmetric polynomial. Consequently, $E(\nu)^3$ vanishes if $\mathcal{E}(\nu, \mathcal{F}) = 0$.* □

3 Amenability for foliations

Amenability for foliations is formulated in terms of properties of the affine cocycles over measurable equivalence relations. We will introduce a related notion of *complete amenability*, which is based on the affine cocycles for the fundamental groupoid of a foliation, and depends also on the topological structure of the equivalence relation defined by a foliation. The definitions of this section set-up the framework for the proofs of "Hirsch-Thurston vanishing theorems", which are discussed in the following section.

We begin with a discussion of cocycles and amenability for topological group actions. First, suppose that G is locally compact second countable topological group. G is *amenable* if for every continuous G-action on a compact metrizable space, X, there is a G-invariant probability measure on X. Given a group action, Zimmer defines amenability for the action (cf. section 4.2, [40]) in terms of fixed-point properties of *measurable cocycles* over the action. Amenability for actions has the effect of extending the classical notion of amenability to the "virtual subgroups" of G, in the sense of Mackey [22] (section 1, [38]; cf. also section 4.3, [40]).

Let (S, μ) be a G-space. That is, S is a standard Borel space, μ is a σ-finite measure on S, and the action $S \times G \to S$ is Borel with the measure μ quasi-invariant under the action of G. Given a second countable topological group, H, a Borel function $\alpha : S \times G \to H$ is a *cocycle* over the G-action if for μ-almost all $s \in S$,

$$\alpha(s, gh) = \alpha(s, g)\alpha(sg, h) \quad \text{for all} \quad g, h \in G.$$

Two cocycles $\alpha, \beta : S \times G \to H$ are *cohomologous*, and we write $\alpha \sim \beta$, if there is a Borel function $\phi : S \to H$ such that for μ-almost all $s \in S$,

$$\beta(s, g) = \phi(s)\alpha(s, g)\phi(sg)^{-1} \quad \text{for all} \quad g \in G.$$

We say that α is a coboundary if it is cohomologous to the trivial cocycle into H.

Let E be a separable Banach space and *Iso(E)* the group of isometric isomorphisms of E. Let E_1^* denote the unit ball in the dual space E^*, and $H(E_1^*)$ the group of homeomorphisms of E_1^* with the topology of uniform convergence. The induced map $Iso(E) \to H(E_1^*)$ is continuous and Borel (Lemma 1.3, [38]).

A *Borel field of compact convex sets over* S is an assignment $A_s \subset E_1^*$ for each $s \in S$ such that each A_s is a non-empty convex compact subset, and the fiber product $\{(s, \lambda) | \lambda \in A_s\}$ is a Borel subset of $S \times E_1^*$. Given a Borel cocycle $\alpha : S \times G \to Iso(E)$, there is an induced Borel adjoint cocycle $\alpha^* : S \times G \to H(E_1^*)$ defined by the rule $\alpha^*(s, g) = (\alpha(s, g)^{-1})^*$. A Borel field of compact convex sets $\{A_s | s \in S\}$ is said to be *α-invariant* if

$$\alpha^*(s, g)A_{sg} = A_s \quad \text{for each} \ \ g \in G \ \ \text{and for} \ \ \mu - \text{almost all} \ \ s \in S.$$

DEFINITION 3.1 (Amenable group action) *(S, μ) is an* amenable *G-space if for every separable Banach space E, cocycle $\alpha : S \times G \to Iso(E)$, and α-invariant Borel field $\{A_s | s \in S\}$, there is a Borel function $\phi : S \to E_1^*$ such that for μ-almost all $s \in S$, $\phi(s) \in A_s$ and $\alpha^*(s, g)\phi(sg) = \phi(s)$ for all $g \in G$.*

When S is a singleton space, the trivial action of G on S is amenable if and only if the group G is amenable (Proposition 1.5, [38]). More generally, for an amenable topological group G, every G-space (S,μ) is an amenable G-space. Conversely, if μ is a G-invariant probability measure on S, then (S,μ) is an amenable G-space only if G is amenable.

We next discuss two generalizations of amenability for actions to amenability of foliations. The first is a standard extension of the above ideas to the case of measured equivalence relations (cf. section 4, [26]). Let $\mathcal{R}(\mathcal{F}) \subset V \times V$ be the equivalence relation determined by the leaves of $\mathcal{F}$, where two points in V are equivalent if they lie on the same leaf. The smooth structure on the leaves determines a natural Lebesgue measure class on the leaves denoted by $\widetilde{dvol_{\mathcal{F}}}$. Each quasi-invariant transverse measure μ has a locally-defined product measure with $\widetilde{dvol_{\mathcal{F}}}$ to yield a measure class $\tilde{\mu}$ on the set $\mathcal{R}(\mathcal{F})$. A *cocycle* over $\mathcal{R}(\mathcal{F})$ is a Borel function $\alpha : \mathcal{R}(\mathcal{F}) \to H$ such that for $\tilde{\mu}$-almost all $x \in V$,

$$\alpha(y,x)\alpha(x,z) = \alpha(y,z) \quad \text{for all} \quad (z,x),(x,z) \in \mathcal{R}(\mathcal{F}).$$

Given a separable Banach space E, a *Borel field of compact convex sets over* V is an assignment $A_x \subset E_1^*$ for each $x \in V$ such that each A_x is a non-empty convex compact subset, and the fiber product set $\{(x,\lambda) | \lambda \in A_x\} \subset V \times E_1^*$ is a Borel subset. Given a Borel cocycle $\alpha : \mathcal{R}(\mathcal{F}) \to Iso(E)$, the Borel field $\{A_x | x \in V\}$ is said to be α-*invariant* if for $\tilde{\mu}$-almost all $x \in V$, $\alpha^*(y,x)A_y = A_x$ for all $(y,x) \in \mathcal{R}(\mathcal{F})$. We call the induced action of $\mathcal{R}(\mathcal{F})$ on the fibers of this field the α-*holonomy* of $\mathcal{F}$.

DEFINITION 3.2 (Amenable foliation) $(\mathcal{F},\mu)$ *is an* amenable foliation *if for every separable Banach space E, cocycle* $\alpha : \mathcal{R}(\mathcal{F}) \to Iso(E)$*, and* α*-invariant Borel field* $\{A_x | x \in V\}$*, there is a Borel function* $\phi : V \to E_1^*$ *such that for* $\tilde{\mu}$*-almost all* $x \in V$*,* $\phi(x) \in A_x$ *and* $\alpha^*(y,x)\phi(y) = \phi(x)$ *for all* $(y,x) \in \mathcal{R}(\mathcal{F})$*.*

The geometric interpretation of the amenability condition is straightforward. The data given is a field of compact convex sets over each point in V. The cocycle data consists of "transfer operators" between the fiber sets over the points. (This is exactly analogous to the holonomy operators usually encountered for transverse structures to foliations, which is the reason for the notation "α-holonomy".) Amenability then states that for every such structure, there exists a Borel section of the Borel field which is invariant under the α-holonomy.

The α-holonomy defined by a cocycle is via isomorphisms, so while the compact convex fiber sets may not be constant, their isomorphism type is constant on leaves, and so will be constant on the ergodic components of the equivalence relation $\mathcal{R}(\mathcal{F})$.

Recall the definition of the normal linear holonomy cocycle of $\mathcal{F}$. Given $x \in V$ and an arbitrary $y \in L_x$, for each leafwise path $\gamma : [0,1] \to L_x$ from y to x, there is a linear transformation $h([\gamma]) : \nu(y) \to \nu(x)$ which depends only upon the homotopy class of the path γ. This is defined by integrating the Bott connection on ν along the curve (cf. chapter 2, [21]). The fact that the Bott connection is flat when restricted to leaves implies that the holonomy transport depends only on the (leafwise) homotopy class of the path.

For any bundle with Riemannian metric over a manifold, there always exists an everywhere defined, *Borel* orthonormal framing of the bundle. In particular, for ν we can choose such a framing which has a positive orientation, and then the normal linear holonomy takes values in the matrix group $GL^+(2n,\mathbf{R})$.

If the linear holonomy along a path γ in a leaf L_x depends only on the endpoints of the path, then we say that L_x has *trivial linear holonomy.* A result of Hurder and Katok

(Lemma 7.1, [15]) states that for a C^1-foliation, Lebesgue almost every leaf has trivial linear holonomy. We can therefore define, Lebesgue almost everywhere, the transverse linear holonomy cocycle over $\mathcal{R}(\mathcal{F})$.

The definition of amenability for foliations above applies only to α-holonomy which depends strictly upon the pair of points $(y, x) \in \mathcal{R}(\mathcal{F})$ for $\tilde{\mu}$-almost all $x \in V$, and not on any other geometric data. For the transverse Lebesgue measure class, it applies to the transverse linear holonomy cocycle. However, if the transverse measure class in consideration is not absolutely continuous with respect to Lebesgue measure, then on a set of positive measure the transverse linear holonomy can depend also on the leafwise path between the two endpoints, and therefore the usual notion of amenability for the equivalence relation of a foliation is not applicable. This motivates introducing the following definition of *complete amenability* for a foliation.

The *fundamental groupoid* $\mathcal{P}(\mathcal{F})$ of a foliation is the set

$$\mathcal{P}(\mathcal{F}) = \{(y, x, [\gamma]) \ \mid (y, x) \in \mathcal{R}(\mathcal{F}), [\gamma] \in \pi_1^{\mathcal{F}}(V, x, y)\}.$$

The set $\pi_1^{\mathcal{F}}(V, x, y)$ consists of the homotopy classes of paths $\gamma : [0, 1] \to L_x$ into the leaf L_x of $\mathcal{F}$ through x with $\gamma(0) = x$ and $\gamma(1) = y$. Composition is defined by concatenation of paths. This definition is comparable to the that of the *holonomy groupoid* $\mathcal{G}(\mathcal{F})$ of a foliation (cf. [10]), except that we do not impose any condition involving the transverse holonomy of the path γ. There are natural "forgetful maps" of groupoids

$$\mathcal{P}(\mathcal{F}) \to \mathcal{G}(\mathcal{F}) \to \mathcal{R}(\mathcal{F}).$$

The groupoid $\mathcal{P}(\mathcal{F})$ is a countable-to-one space over the equivalence relation $\mathcal{R}(\mathcal{F})$, so that an equivalence class of measures $\tilde{\mu}$ on $\mathcal{R}(\mathcal{F})$ determines such a class of $\mathcal{P}(\mathcal{F})$, which we again denote by $\tilde{\mu}$.

A Borel cocycle $\alpha : \mathcal{P}(\mathcal{F}) \to H$ is defined as before, and will naturally act on a given field of compact convex sets over V.

The normal linear holonomy naturally defines a Borel cocycle $h_{\mathcal{F}} : \mathcal{P}(\mathcal{F}) \to GL^+(2n, \mathbf{R})$. A different choice of orthonormal framing results in a cohomologous cocycle, so that the cohomology class of $h_{\mathcal{F}}$ is a well-defined invariant of $\mathcal{F}$.

DEFINITION 3.3 (Completely amenable foliation) $(\mathcal{F}, \mu)$ *is a* completely amenable *foliation if for every separable Banach space E, cocycle* $\alpha : \mathcal{P}(\mathcal{F}) \to Iso(E)$*, and* α*-invariant Borel field* $\{A_x | x \in V\}$*, there is a Borel function* $\phi : V \to E_1^*$ *such that for* $\tilde{\mu}$*-almost all* $x \in V$*,* $\phi(x) \in A_x$ *and* $\alpha^*(y, x, [\gamma])\phi(y) = \phi(x)$ *for all* $(y, x, [\gamma]) \in \mathcal{P}(\mathcal{F})$*.*

Observe that complete amenability implies amenability, as every cocycle over $\mathcal{R}(\mathcal{F})$ lifts to a cocycle over $\mathcal{P}(\mathcal{F})$. In both definitions, we are given a field of compact convex sets over each point in V. The cocycle data for complete amenability also consists of "transfer operators" between the fiber sets over the points. However, the dependence of the transfer data on the leafwise path chosen makes complete amenability a more subtle property of the foliated manifold.

There is an alternative description of the transfer maps associated to a cocycle over the homotopy groupoid. Choose a covering for V by foliation charts, so that in each chart the foliation is diffeomorphic to a foliation by discs (the *plaques*) with a transversal disc. Moreover, require that the intersection of adjacent charts be either empty, or a contactable set. The bundle $\{A_x | x \in V\}$ can then be "trivialized" over each such open chart. The holonomy data, for x and y are in adjacent open sets with intersecting plaques, will consist of an identification map between the sets A_y and A_x, and is required to be locally constant in x and y. For arbitrary $(y, x) \in \mathcal{R}(\mathcal{F})$ and a leafwise path γ between them, we compose the adjacent trivializations along the path from y to x to obtain the α-holonomy. This description is the same as the construction used for the transverse linear holonomy cocycle of the normal bundle to a foliation, in which case we take A_x to be the normal vector space to $\mathcal{F}$ at x.

For a foliation of a compact manifold V, with V as the only leaf, the equivalence relation $\mathcal{R}(\mathcal{F}) \cong V \times V$ is trivial and $(\mathcal{F}, \mu)$ is trivially amenable. On the other hand, the homotopy groupoid $\mathcal{P}(\mathcal{F})$ fibers over $V \times V$ with fiber isomorphic to the fundamental group $\pi_1(V)$, so that $(\mathcal{F}, \mu)$ is completely amenable only if $\pi_1(V)$ is a discrete amenable group. This distinction between the two notions of amenability persists whenever the transverse measure μ for $\mathcal{F}$ has non-trivial atoms, as the example above can be easily embedded into higher codimension foliations. This raises an interesting open question:

PROBLEM 3.4 *Let μ be a transverse quasi-invariant measure for a C^1-foliation $\mathcal{F}$ on a smooth manifold V. Suppose that $(\mathcal{F}, \mu)$ is amenable and μ has no non-trivial atoms, then is $(\mathcal{F}, \mu)$ completely amenable?*

We now apply the above definitions to the normal linear holonomy cocycle. $P^+(\mathbf{R}^{2n})$ will denote the space of positive rays in $\mathbf{R}^{2n}$, which is diffeomorphic to the unit sphere S^{2n-1}. Let E denote the Banach space of continuous functions on $P^+(\mathbf{R}^{2n})$ equipped with the sup-norm. Then the projective action of $GL(2n, \mathbf{R})$ on $P^+(\mathbf{R}^{2n})$ is norm-preserving, and we obtain a cocycle $\alpha_{\mathcal{F}} = Ph_{\mathcal{F}} : \mathcal{P}(\mathcal{F}) \to Iso(E)$. The unit ball in the dual space E^* is identified with the space of probability measures on $P^+(\mathbf{R}^{2n})$.

LEMMA 3.5 *Let $(\mathcal{F}, \mu)$ be a completely amenable foliation. Then there exists a Borel family of probability measures $\{\hat{\mu}_x | \ \mu - \text{almost all } x \in V\}$ which are invariant under the projective action of the normal linear holonomy of $\mathcal{F}$, where μ_x is defined on the fiber $P^+(\nu(x))$ over x of the projective normal bundle $P^+(\nu)$.* □

Proof. Follows from the definition of complete amenability. □

LEMMA 3.6 *Let $(\mathcal{F}, \mu)$ be an amenable foliation, with μ absolutely continuous. Then there exists a Borel family of probability measures $\{\hat{\mu}_x | \text{ almost all } x \in V\}$ which are invariant under the projective action of the normal linear holonomy of $\mathcal{F}$.*

Proof. It was shown in (Proposition 7.1, [15]) that for Lebesgue almost every leaf L_x of $\mathcal{F}$, the linear holonomy $h_{\mathcal{F}}(y, x, [\gamma])$ depends only on the endpoints $(y, x) \in \mathcal{R}(\mathcal{F})$, and not on the path between them. The cocycle $\alpha_{\mathcal{F}}$ therefore descends to a cocycle $\tilde{\alpha}_{\mathcal{F}} : \mathcal{R}(\mathcal{F}) \to Iso(E)$, and the amenability of $\mathcal{R}(\mathcal{F})$ implies there exists a family of holonomy invariant measures, defined on almost every fiber. □

The existence of these holonomy invariant measures on the transverse projective bundle will be shown to imply vanishing of the normal Euler class in the next section. It would be interesting to better understand, in terms of the geometry of the foliation $\mathcal{F}$, when such measures exist.

4 Hirsch-Thurston vanishing criterion

The purpose of this section is to show that the existence of a Borel family of probability measures $\{\hat{\mu}_x | \ \mu - \text{almost all } x \in V\}$ on the fibers $P^+(\nu(x))$ which are invariant under the projective action of the normal linear holonomy of $\mathcal{F}$, implies that the transverse Euler class $\mathcal{E}_\mu(\nu, \mathcal{F}) \in H^{2n}_\mu(\mathcal{F})$ vanishes. We follow the outline of the proof of the Hirsch-Thurston theorem, except that the ideas are formulated for leafwise cohomology. There are two steps, beginning with a technical result:

PROPOSITION 4.1 *Let $\mathcal{F}$ be a C^1-foliation on V with quasi-invariant transverse measure μ. Suppose that there exists a Borel family of probability measures $\{\hat{\mu}_x | \ \mu -$ almost all $x \in V\}$ on the fibers $P^+(\nu(x))$ which are invariant under the action of the projective action $\alpha_{\mathcal{F}}$ of the normal linear holonomy. Then the natural map*

$$\pi^* : H^{\cdot}_\mu(\mathcal{F}) \longrightarrow H^{\cdot}_\mu(\pi^*\mathcal{F})$$

of Lemma 2.3 is injective.

Proof. We use the family of measures $\{\hat{\mu}_x\}$ to construct a map $\mathrm{Av}(\hat{\mu}) : H^{\cdot}_\mu(\pi^*\mathcal{F}) \to H^{\cdot}_\mu(\mathcal{F})$ such that $\mathrm{Av}(\hat{\mu}) \circ \pi^* = Id$. Given a form $\psi \in \Omega^q_0(\pi^*\mathcal{F})$ and exterior frame $\vec{v} = v_1 \wedge \cdots \wedge v_q$ of tangent vectors to $\mathcal{F}$ at $x \in V$, we must define $\mathrm{Av}(\hat{\mu})(\psi)(\vec{v})$. The leaves of the foliation $\hat{\mathcal{F}}$ are submanifolds of the leaves of $\pi^*\mathcal{F}$ and cover those of $\mathcal{F}$, so for each point on the fiber $\theta \in P^+(\nu(x))$ over x, we can lift $\vec{v}$ to an exterior frame $\vec{v}(x, \theta)$ of tangent vectors to $\hat{\mathcal{F}}$. Then define

$$\mathrm{Av}(\hat{\mu})(\psi)(\vec{v}) = \int_{P^+(\nu(x))} \psi(\vec{v}(x, \theta)) \, d\hat{\mu}_x(\theta) \tag{6}$$

To show that $\mathrm{Av}(\hat{\mu})$ is a well-defined map of complexes, note first that the integral in (6) is defined for μ-almost all $x \in V$, and the averaging over a probability measure has sup-norm one on leafwise forms. It is seen to commute with the leafwise differentials, by expressing the integral (6) in local foliation charts, and then represent the transverse measures $\hat{\mu}_x$ by coordinate measures which are constant along the leaves of $\hat{\mathcal{F}}$ (this is possible by holonomy invariance). The map $\mathrm{Av}(\hat{\mu})$ in these coordinates is integration of ψ over a transverse parameter, which is seen to commute with the leafwise derivatives. □

COROLLARY 4.2 *With the hypotheses of Proposition 4.1, $\mathcal{E}_\mu(\nu, \mathcal{F}) \in H^{2n}_\mu(\mathcal{F})$ is zero.*

Proof. The pull-back vector bundle $\pi^*(\nu) \to P^+(\nu)$ has a canonical non-vanishing leafwise smooth section, given by choosing a representative vector in ν for each point in the fiber over $x \in V$. (If we identify $P^+(\nu)$ with the unit sphere bundle of ν, then we are simply choosing the unit vector in ν over the point in $P^+(\nu)$ corresponding to that vector.) It follows from Proposition 2.7 that the Euler class $E(\pi^*(\nu), \pi^*\mathcal{F}) = 0$, as the global angular form for the bundle $\pi^*(\nu)$ is defined on $P^+(\nu)$. Naturality of the Euler class, Proposition 2.6, then implies that $\pi^*(E(\nu), \mathcal{F}) = E(\pi^*(\nu), \pi^*\mathcal{F}) \in H^{2n}(\pi^*\mathcal{F})$. The inclusion maps (2.3.1) are natural, so commute with the map π^*. From these remarks we observe that

$$\pi^*(\mathcal{E}_\mu(\nu, \mathcal{F})) = \mathcal{E}_{\hat{\mu}}(\pi^*(\nu), \pi^*\mathcal{F}) = 0$$

and so the injectivity of π^* on this cohomology theory yields the corollary. □

5 Amenable sequences

In this section we develop geometric conditions on a foliation which ensure that the average transverse Euler class vanishes. We first recall two well-known geometric constructions of invariant transverse measures for foliations, Fölner sequences (cf. Plante [28]), and their extension to the averaging sequences of Goodman and Plante [6]. We then introduce *amenable sequences*, which are Fölner sequences with the additional hypotheses that the transverse holonomy groups of the leafwise sets in the averaging sequence are also amenable. The main technical result of the section is that an amenable sequence gives rise to an averaging sequence *in the unit normal sphere bundle* to $\mathcal{F}$, whose associated closed current pushes forward to the current in the ambient manifold V defined by considering the the amenable sequence simply as an averaging sequence. We use this to conclude that the average Euler class $E(\nu, \mu) = 0$ for a measure μ defined by an amenable sequence.

Let $\mathcal{F}$ be a C^0-foliation of V. Endow TV with a Riemannian metric, g_V, which induces Riemannian metrics on each leaf $L_x \subset V$.

DEFINITION 5.1 (Plante [28]) *A leaf L_x is* Fölner *for $\mathcal{F}$ if there exists an increasing sequence of compact connected submanifolds with piecewise smooth boundary, $Z_1 \subset \cdots \subset Z_n \subset \cdots \subset L_x$ so that the ratio of the Riemannian volume of the boundary ∂Z_n relative to the volume of the interior of Z_n satisfies*

$$\frac{\mathrm{vol}_{g_V}(\partial Z_n)}{\mathrm{vol}_{g_V}(Z_n)} \to 0 \quad as \ \ n \to \infty. \tag{7}$$

A leaf L_x has *subexponential growth* if the condition (7) is satisfied for some $x_0 \in L_x$, with $Z_n = B(x_0, n) \subset L_x$ the leafwise metric ball of radius n centered at x_0. A mild generalization of this is to allow leaves with *non-exponential growth*, where (7) is satisfied for some subsequence of metric balls, $\{Z_n = B(x_0, k_n) | k_n \to \infty\}$. Compactness of V implies that these definitions are independent of the choices of metric g_V and of basepoint $x_0 \in V$. The notation "*L_x is Fölner*" is suggested by the analogy with amenable Lie groups, which always admit a Fölner sequence $Z_n \subset G$ which satisfies the conditions (7) for the left-invariant metric on G.

DEFINITION 5.2 (Goodman-Plante [6]) *An* averaging sequence *for $\mathcal{F}$ is a sequence of compact connected submanifolds $\{Z_n | n = 1, 2, \ldots\}$ with piecewise smooth boundaries, where each $Z_n \subset L_n$ (for some leaf L_n that need not be fixed,) which satisfies (7).*

Averaging sequences were introduced by Goodman & Plante as a very general mechanism to construct invariant transverse measures for foliations. Let us recall a primary result of [6], and then an application of interest:

PROPOSITION 5.3 (Proposition 1.1,[6]) *Let $\mathcal{Z} = \{Z_n | n \to \infty\}$ be an averaging sequence for $\mathcal{F}$. Then there exists a non-trivial invariant transverse measure $\mu_{\mathcal{Z}}$ for $\mathcal{F}$ with support contained in the ω-limit set of the sequence $\mathcal{Z}$.* □

COROLLARY 5.4 *Let $\mathcal{Z} = \{Z_n | n \to \infty\}$ be an averaging sequence for $\mathcal{F}$. Then $\mathcal{Z}$ determines a foliation cycle $C_{\mathcal{Z}}$ for $\mathcal{F}$, which induces a linear mapping $[C_{\mathcal{Z}}] : H^m(V; \mathbf{R}) \to \mathbf{R}$, where m is the leaf dimension of $\mathcal{F}$.*

Proof. We recall the proof of this corollary for later application. Let $\{\psi_i | i = 1, 2, \ldots\}$ be a countable set of m-forms on V which are dense in the set of all continuous m-forms. The integration map

$$\langle C_{Z_n}, \psi \rangle = \mathrm{vol}_{g_V}(Z_n)^{-1} \int_{Z_n} \psi$$

is a continuous linear map on the space of continuous m-forms ψ on V, with an estimate on its norm independent of n. We can therefore choose an infinite subsequence $\{Z_k | k \in I_1\}$, where $I_1 \subset \mathbf{Z}^+ = \{1, 2, \ldots\}$, such that for each fixed i, $\langle C_{Z_k}, \psi_i \rangle$ converges as $k \to \infty$. The current $C_{\mathcal{Z}}$ is determined by its values on the set $\{\psi_i | i = 1, 2, \ldots\}$ as the limiting value of this subsequence. Stokes' Theorem and the Fölner condition (7) implies that the resulting current vanishes on coboundaries. □

Note that the homology class $[C_{\mathcal{Z}}] \in H_m(V; \mathbf{R})$ obtained in the above proof may depend upon the subsequence chosen, and hence $[C_{\mathcal{Z}}]$ is not necessarily an invariant of the averaging sequence. The main point of the method of the proof above is to show that *some* non-trivial foliation cycle exists. The same process applied to transverse functions shows that some invariant transverse measure exists, and the cycle constructed above is the corresponding homology class.

We come now to the main definition of this section. Assume that $\mathcal{F}$ is a C^1-foliation, and let $\{Z_n | n \to \infty\}$ be an averaging sequence for $\mathcal{F}$. Select a base point $x_n \in Z_n \subset L_n$ for each n. Consider the maps of fundamental groups,

$$\pi_1(Z_n, x_n) \to \pi_1(L_n, x_n) \to \mathcal{P}(\mathcal{F}).$$

The linear holonomy cocycle $h_{\mathcal{F}} : \mathcal{P}(\mathcal{F}) \to GL(2n, \mathbf{R})$ restricts to the image of $\pi_1(Z_n, x_n)$ to define the *normal linear holonomy homomorphism* of the set Z_n. The image group $\Gamma_n \subset GL(2n, \mathbf{R})$ is called the *normal linear holonomy* of Z_n.

DEFINITION 5.5 *An* amenable sequence *for $\mathcal{F}$ is an averaging sequence $\{Z_n | n \in \mathbf{Z}^+\}$ such that the normal linear holonomy group Γ_n of each Z_n is amenable as a discrete group.*

For example, if the foliation $\mathcal{F}$ is without holonomy (the holonomy is trivial along every closed leafwise path), then every averaging sequence is an amenable sequence. Or if the holonomy group of each leaf of $\mathcal{F}$ is amenable, then every averaging sequence is an amenable sequence.

The main technical result is now:

PROPOSITION 5.6 *Let $\{Z_n | n \in \mathbf{Z}^+\}$ be an amenable sequence for $\mathcal{F}$, with $\{Z_k | k \in I_1\}$ a subsequence so that the m-currents C_{Z_k} converge to an m-current $C_{\mathcal{Z}}$. Then there exists an averaging sequence $\hat{\mathcal{Z}} = \{\hat{Z}_k | k \in I_1\}$ for the foliation $\hat{\mathcal{F}}$ on $P^+(\nu)$ such that:*

1. *The interior of each $\hat{Z}_k$ is a finite covering of the interior of Z_k*

2. *There is an infinite subsequence $\{\hat{Z}_j \mid j \in I_2 \subset I_1\}$ so that the m-currents $C_{\hat{Z}_j}$ converge to an m-current $C_{\hat{\mathcal{Z}}}$, and $\pi_*(C_{\hat{\mathcal{Z}}}) = C_{\mathcal{Z}}$.*

Proof. Let $\tilde{Z}_n$ denote the holonomy cover of Z_n. For $x \in V$ and $\hat{x} \in P^+(\nu)$, the leaf $\hat{L}(\hat{x})$ of $\hat{\mathcal{F}}$ through $\hat{x}$ is the covering of $L(x)$ associated to the isotropy subgroup of the projective action of the linear holonomy homomorphism of $L(x)$ on the ray in $P^+(\nu(x))$

determined by $\hat{x}$. Therefore, the holonomy covering $\tilde{L}(x)$ of $L(x)$ is a covering of $\hat{L}(\hat{x})$. It follows that the preimage of Z_n in a leaf $\hat{L}_n$ of $\hat{\mathcal{F}}$ which covers L_n consists of disjoint connected components each isometric to $\tilde{Z}_n$. We use amenability of the holonomy group Γ_n to produce a compact connected submanifold with corners $\tilde{Y}_n \subset \tilde{Z}_n$, so that the sequence $\{\tilde{Y}_n | n \in I_1\}$ satisfies (7), and then let $\hat{Z}_n$ be the push-forward of $\tilde{Y}_n$ into $\hat{L}_n \subset P^+(\nu)$.

Let $K_n \subset \tilde{Z}_n$ be a fundamental domain for the covering $\tilde{Z}_n \to Z_n$. By this we mean that K_n is a compact submanifold with piecewise-smooth boundary, so that for any element $\delta \in \pi_1(Z_n)$, the intersection of $\delta K_n \cap K_n$ is either a subset of the boundary of K_n, or is all of K_n, and the union of the translates of K_n by elements of $\pi_1(K_n)$ equals $\tilde{Z}_n$.

Let $\Delta_n(0) = \{\delta_1, \ldots, \delta_d\} \subset \pi_1(Z_n)$ be the finite subset of elements such that $\delta K_n \cap K_n \neq \emptyset$. It is elementary to check that $\Delta_n(0)$ is a symmetric generating set for $\pi_1(Z_n)$.

By hypotheses, the group $\pi_1(Z_n)$ is amenable, hence by the Fölner criterion for amenability [7], there exists an increasing sequence of finite subsets

$$\Delta_n(0) \subset \cdots \subset \Delta_n(k) \subset \cdots \subset \pi_1(Z_n)$$

whose union is all of $\pi_1(Z_n)$, and such that the symmetric differences

$$\frac{\#\left(\delta_i \Delta_n(k) \setminus \Delta_n(k)\right) \ + \ \#\left(\Delta_n(k) \setminus \delta_i \Delta_n(k)\right)}{\#(\Delta_n(k))} \longrightarrow \infty \ \ as \ \ k \to \infty \tag{8}$$

for all $1 \leq i \leq d$.

The boundary of the set K_n has two components, $\partial K_n = W_n^+ \cup W_n^-$, where W_n^+ maps into a component of the boundary of Z_n, and W_n^- maps into the interior of Z_n. The restriction of the covering map, $\pi : \tilde{Z}_n \to Z_n$, to the domain K_n is one-to-one on the interior $\overset{\circ}{K}_n$ and generically two-to-one on ∂K_n. Thus, the averaging sequence criterion (7) on the sequence $\{Z_n\}$ implies there is an estimate

$$\frac{\mathrm{vol}_{g_V}(W_n^+)}{\mathrm{vol}_{g_V}(K_n)} \to 0 \quad as \ \ n \to \infty. \tag{9}$$

For the other component of the boundary, define constants

$$C_n = \frac{\mathrm{vol}_{g_V}(\partial K_n)}{\mathrm{vol}_{g_V}(K_n)} \geq \frac{\mathrm{vol}_{g_V}(W_n^-)}{\mathrm{vol}_{g_V}(K_n)} \tag{10}$$

For each n we choose k_n so that the quotient in (8) is less than $(nC_n)^{-1}$ and define

$$\tilde{Y}_n = \bigcup_{\delta \in \Delta_n(k_n)} \delta K_n$$

Observe that there are two types of boundary components for the set $\tilde{Y}_n$: the first is the union over $\Delta_n(k_n)$ of the translates of the boundary sets W_n^+, and by (9) these have volume asymptotically small relative to the mass of $\tilde{Y}_n$. The second consists of translates of the boundary sets W_n^-, say $\delta' W_n^-$. However, for a component of this type to arise, it is necessary for there to exist some $\delta \in \Delta_n(0)$ for which $\delta(\delta' W_n^-) \not\subset \tilde{Y}_n$, for otherwise this boundary component of $\delta' W_n^-$ would be in the interior of $\tilde{Y}_n$. We conclude that the number of such boundary components arising from translates of W_n^-

are estimated by C_n times the quotient (8) so that this contribution also tends to zero by the choice of k_n. The sequence $\{\tilde{Y}_n\}$ therefore satisfies (7) and hence the image $\{\hat{Z}_n\} \subset P^+(\nu)$ is an averaging sequence for $\hat{\mathcal{F}}$.

Let $\{Z_k | k \in I_1\}$ be the subsequence so that C_{Z_k} converges to the invariant current $C_{\mathcal{Z}}$. Choose a countable dense set of continuous m-forms on $P^+(\nu)$ and an infinite subsequence $I_2 \subset I_1$ so that $C_{\hat{Z}_i}$ converges on this dense set for $i \in I_2$. By the Fölner condition, this will converge to a closed current $C_{\hat{\mathcal{Z}}}$ on $P^+(\nu)$. It remains to observe that the boundary of each set $\hat{Z}_n$ integrates to zero on any continuous m-form along leaves, so the pushforward of the current $C_{\hat{Z}_n}$ under π_* is equal to C_{Z_n}. Continuity of π_* implies that $\pi_*(C_{\hat{\mathcal{Z}}}) = C_{\mathcal{Z}}$. $\square$

COROLLARY 5.7 *Let $(\mathcal{F}, \mu)$ be a C^1-measured foliation with μ defined as the asymptotic limit of an amenable sequence. Then for any closed $(m-2n)$-form ϕ on V,*

$$\langle [C_\mu], [\phi] \cup E(\nu) \rangle = \lim_{k \in I_1} \left\{ \mathrm{vol}_{g_V}(Z_k)^{-1} \int_{Z_k} \phi \wedge e(\nu) \right\} = 0 \tag{11}$$

Proof. With notation as in Definition 2.4 and the proof of Proposition 5.6, the right side of (11) equals

$$\lim_{j \in I_2} \left\{ \mathrm{vol}_{g_V}(\hat{Z}_j)^{-1} \int_{Z_j} \pi^*(\phi \wedge e(\nu)) \right\} = \lim_{j \in I_2} \left\{ \mathrm{vol}_{g_V}(\hat{Z}_j)^{-1} \int_{Z_j} \pi^*(\phi) \wedge d(\psi_{\mathcal{F}}) \right\} = 0 \quad \square$$

6 Blowing-up along foliated submanifolds

In this section we introduce the technique of "blowing-up" along compact foliated submanifolds. The blow-up of a C^1-foliation on V about a finite set of compact leaves $\{L_1, \ldots, L_d\}$ yields a C^0-foliation $\tilde{\mathcal{F}}$ on $\tilde{V}$, where each compact leaf L_i is replaced with the special fiber $W_i = P^+(\nu|L_i)$. The important property of this process is that it converts asymptotic sequences for $\mathcal{F}$ with ω-limit $\{L_1, \ldots, L_d\}$, into asymptotic sequences for $\tilde{\mathcal{F}}$ with ω-limit in the special submanifolds $\{W_1, \ldots, W_d\}$. We then use this technique to prove prove Theorem 1.7 of the introduction.

Let $\mathcal{F}$ be a codimension q C^r-foliation of the closed manifold V. Let $X \subset V$ be a codimension s closed proper submanifold, with $0 < s \leq q$. We do not assume that X is connected. We assume that $\mathcal{F}$ restricts to a C^r-foliation of X. Introduce the normal $\mathbf{R}^s$-bundle $NX \to X$ to X, the associated sphere bundle $S(NX)$ and projective bundle $P(NX)$.

PROPOSITION 6.1 *Let $(V, \mathcal{F})$ as above, with $X \subset V$ a closed foliated submanifold of codimension s. Then there exists:*

1. *a closed manifold $\overline{V}$ and embedding $P(NX) \hookrightarrow \overline{V}$;*
2. *a codimension q C^{r-1}-foliation $\overline{\mathcal{F}}$ on $\overline{V}$;*
3. *a C^{r-1}-mapping $\Pi : \overline{V} \to V$*

such that we have:

1. *the restriction $\Pi : \overline{V} - P(NX) \to V - X$ is a C^r-diffeomorphism;*

2. *the restriction* $\Pi : P(NX) \to X$ *is the fibration projection;*
3. $\overline{\mathcal{F}}$ *restricts to a foliation of* $P(NX)$*;*
4. Π *sends leaves of* $\overline{\mathcal{F}}$ *to leaves of* $\mathcal{F}$*, and the restriction of* Π *to the open manifold* $\overline{V} - P(NX)$ *is a quasi-isometry along leaves of* $\overline{\mathcal{F}}$*.*
5. *If* $s = q$ *and* X *is a finite union of compact leaves of* $\mathcal{F}$*, then the restricted foliation* $(P(NX), \overline{\mathcal{F}})$ *is* C^{r-1}*-diffeomorphic to the foliated* $P(\mathbf{R}^q)$*-bundle over* X *with the projectified linear holonomy foliation from the Bott connection on the restriction of the foliation normal bundle* $\nu|X$*.*

Proof. Let $\overline{V}$ be the manifold obtained by blowing-up along the submanifold X (cf. page 603, [8]). The map Π is the canonical map which is the identity off of X, and along the fiber $W = P(NX)$ is the fibration map collapsing the fiber $P(NX(x))$ to $x \in X$. The foliation $\overline{\mathcal{F}}$ is simply $\mathcal{F}$ away from the special fibers $P(NX)$. The content of the Proposition is that the C^r-foliation $\mathcal{F}$ on $\overline{V} - W$ limits in the C^{r-1}-topology to a linear foliation on W. This will follow from:

LEMMA 6.2 *The normal bundle* $NX \to X$ *admits a "Bott" connection, associated to the foliation* $\mathcal{F}$*.*

Proof. The restriction of the normal bundle to $\mathcal{F}$ to the submanifold X, $\nu|X \to X$, has a direct sum decomposition, $\nu|X \cong N_X(\mathcal{F}) \oplus NX$, where the first bundle consists of the normal vectors to $\mathcal{F}$ which are tangent to the submanifold X. The holonomy along leaves of $\mathcal{F}$ leaves the bundle $N_X(\mathcal{F})$ invariant, so the Bott covariant derivative operator $\nabla^{\mathcal{F}}$ on $\nu|X$ also leaves this subbundle invariant. It therefore induces a partial connection, defined along the leaves of $\mathcal{F}|X$, on the quotient bundle NX and which is flat when restricted to leaves. $\square$

Give TV a Riemannian metric, and choose $\epsilon > 0$ sufficiently small so that the exponential map on NX is a diffeomorphism into, when restricted to the ϵ-disc bundle about the 0-section $0_X \subset NX$. For each $\epsilon \geq \delta > 0$, let $N_\delta X$ denote the sphere subbundle of NX consisting of vectors of length δ. The blow-up topology on $\overline{V}$ is defined (via the normal exponential map) on the punctured bundle $NX - 0_X$ by rescaling the fiber metric on each submanifold $N_\delta X$ so that it is isometric with $N_\epsilon X$. At radius 0, we identify antipodal points in the limiting sphere bundle, to obtain a bundle over X with fibers $P(\mathbf{R}^q)$.

The normal holonomy for $\mathcal{F}$ along leafwise paths contained in X has a well-defined germinal action around $0_X \subset NX$. It is straightforward to calculate that the rescaling of the fibers $N_\delta X$ in the blowing-up process converts the germinal action into the projective linear action associated to the Bott connection on NX. We therefore obtain a C^0-foliation $\overline{\mathcal{F}}$ on $\overline{V}$, which agrees with $\mathcal{F}$ on $\overline{V} - W$.

The transverse structure of $\mathcal{F}$ depends C^r on the transverse coordinate on V. This translates into local Taylor series, in terms of the transverse coordinates, for the infinitesimal holonomy of $\mathcal{F}$. The blowing-up process scales the transverse coordinates in the normal directions to X, which results in decreasing the exponents of the Taylor series by one in the transverse variables. Thus, the Taylor series for the infinitesimal holonomy of $\overline{\mathcal{F}}$ about W is defined, but of one degree lower, which implies that $\overline{\mathcal{F}}$ is transversally of class C^{r-1}.

The leaves of $\overline{\mathcal{F}}$ on W cover those of $\mathcal{F}$ on X, so we can give the tangential distribution $T\overline{\mathcal{F}}$ a Riemannian metric induced from that on V restricted to $T\mathcal{F}$. Complete this to a Riemannian metric on $T\overline{V}$, and we observe that Π is actually an isometry when restricted to leaves of $\overline{\mathcal{F}}$. □

Recall that the ω-limit set of a leaf L of $\mathcal{F}$ is a closed union of leaves of $\mathcal{F}$, defined as the intersection over compact subsets of L:

$$\omega(L) = \bigcap_{K \subset L} \overline{L - K}$$

where $\overline{L - K}$ is the point-set closure in V. A similar definition can be made for an averaging sequence $\mathcal{Z} = \{Z_n | n \in \mathbf{Z}^+\}$:

$$\omega(\mathcal{Z}) = \bigcap_{k \geq 1} \overline{\bigcup_{n > k} (Z_n - Z_k)} \tag{12}$$

We use the technique of blowing-up to prove the vanishing of the normal Euler class for compact leaves. The next lemma is the key step for the general result that follows.

LEMMA 6.3 *Let $(\mathcal{F}, \mu)$ be a codimension $2n$ C^1-foliation with $\mu = \mu_L$ an atomic measure. Suppose that $L = \omega(\mathcal{Z})$ for an averaging sequence $\mathcal{Z}$, then the Euler class $E(\nu|L) \in H^{2n}(L; \mathbf{R})$ vanishes.*

Proof. First note that $L = \omega(\mathcal{Z})$ implies that there exists n_0 so that for $n > n_0$, $Z_n \cap L = \emptyset$ for all i. The property of being an averaging sequence is a quasi-isometry invariant, so the sequence $\overline{\mathcal{Z}} = \{\overline{Z}_n \mid n_0 < n < \infty\}$ obtained by considering each Z_n as a set in $\overline{V}$ will again be an averaging sequence.

Let $\overline{\mu}$ be an invariant transverse measure for $\overline{\mathcal{F}}$ defined as a weak-* limit of $\overline{\mathcal{Z}}$. $\omega(\mathcal{Z}) = L$ implies that the support of $\overline{\mu}$ is contained in $\omega(\overline{\mathcal{Z}}) = W$. Therefore, $\overline{\mu}$ is an invariant transverse measure for the foliated projective bundle $W \to L$. The transverse measure lifts to the double covering of W, which is the normal sphere bundle over L with induced linear foliation. The method of the Hirsch-Thurston Theorem then shows that the Euler class of $\nu|L$ vanishes in real cohomology. □

PROPOSITION 6.4 *Let $(\mathcal{F}, \mu)$ be a codimension $2n$ C^1-foliation with $\mu = \sum_{i=1}^{d} a_i \cdot \mu_{L_i}$ a finite sum of atomic transverse measures, with each $a_i \neq 0$. If μ is the weak-* limit of an averaging sequence $\mathcal{Z}$ and each L_i is disjoint from $\mathcal{Z}$, then for each $1 \leq i \leq d$, the Euler class $E(\nu|L_i) \in H^{2n}(L_i; \mathbf{R})$ is zero.*

Proof. We follow the "blowing-up" method of proof as used above for Lemma 6.3, applied to each leaf L_i in the support of μ. Fix a leaf L_i and fix an ϵ-tubular neighborhood $U_i \supset L_i$ which is disjoint from the other leaves in the support of μ. Define $\mathcal{Z}^i = \{Z^i_n = Z_n \cap U_i \mid n \in \mathbf{Z}^+\}$. It is obvious that $L_i = \omega(\mathcal{Z}^i)$, so to apply the blowing-up method we need only check that there is a subsequence of $\mathcal{Z}^i$ which satisfies (7).

Fix a sequence of nested compact transverse discs $\{\mathbf{D}^i_k \subset V \mid k \in \mathbf{Z}^+\}$ to the leaf L_i, with radius ϵ/k. For each $k > 1$, choose n_k so that the transverse measure ν_{n_k} determined by Z_{n_k} gives mass at least $|a_i|(1 - 1/n)$ to the disc $\mathbf{D}^i_k$, and mass at most $|a_i|/n$ to the annulus $\mathbf{D}^i_1 - \mathbf{D}^i_k$. It is immediate that the sequence $\{Zn_k\}$ satisfies (7). □

7 Almost compact transverse measures

In this section we consider the average Euler class for non-atomic transverse measures, and show that Conjecture 1.9 is true with an additional topological hypothesis on the support of the transverse measure.

Let $\mathcal{F}$ be a C^1-foliation of the compact Riemannian manifold V with leaf dimension m and oriented tangent bundle $T\mathcal{F}$. The normal bundle ν is identified with the orthogonal complement of $T\mathcal{F}$ in TV. For each $\epsilon > 0$, let ν_ϵ denote the $2n$-disc subbundle of ν whose fibers are the open discs of radius ϵ. For an embedded submanifold $N \subset V$, let $\pi : \mathbf{E} \overset{def}{=} \nu|N \to N$ denote the restriction of ν to N, and $\nu_\epsilon|N$ the corresponding disc sub-bundle. The geodesic exponential map $\exp : \nu \to V \times V$ is projected onto the second factor of V to obtain a smooth map $\exp^r : \nu \to V$, whose value at $\vec{v} \in \nu$ is the time one point of the geodesic starting at $\pi(\vec{v})$ with initial velocity $\vec{v}$.

An embedded closed submanifold $N \subset V$ is said to be *almost tangent* to $\mathcal{F}$ if there exists an $\epsilon > 0$ such that the restricted exponential map $\exp^r : \nu_\epsilon|N \to V$ is a diffeomorphism onto an open tubular neighborhood $U_\epsilon \supset N$, and sends the fibers of $\nu_\epsilon|N \to N$ to transversal discs to the foliation $\mathcal{F}$.

The *closed support* of a transverse measure μ is the smallest closed, saturated (i.e., a union of leaves of $\mathcal{F}$) subset $Z \subset V$ such that the restriction of μ to the complement $V \setminus Z$ is the zero measure.

A transverse measure μ is said to be *almost compact* if the closed support of μ is contained in an open tubular neighborhood U_ϵ of an almost tangent closed submanifold N as above.

LEMMA 7.1 *Let μ be an almost compact invariant transverse measure for $\mathcal{F}$. Then there is:*

1. *a codimension $2m$-foliation $\mathcal{F}_{\mathbf{E}}$ of the restriction of the normal bundle $\mathbf{E} = \nu|N$;*
2. *a transverse invariant measure $\mu_{\mathbf{E}}$ for $\mathcal{F}_{\mathbf{E}}$ with closed support $\mathbf{K} \subset \mathbf{E}$ a compact set;*
3. *a diffeomorphism of fiber bundles, $\Phi : \mathbf{E} \cong \nu_\epsilon$ such that $\exp^r \circ \Phi$ maps*
    - *the foliation $\mathcal{F}_{\mathbf{E}}$ to the restriction of $\mathcal{F}$ to U_ϵ*
    - *the transverse measure $\mu_{\mathbf{E}}$ to the transverse measure μ.*

Proof. The closed support of μ is compact and contained in the open set U_ϵ by hypothesis, so there exists an $0 < \epsilon' < \epsilon$ such that the closed support of μ is contained in the properly contained subset $U_{\epsilon'} = \exp^r(\nu_{\epsilon'}|N)$ of U_ϵ. Choose a diffeomorphism Φ of bundles from $\mathbf{E}$ to ν_ϵ which is the identity on the subbundle $\nu_{\epsilon'}|N$. The transversality of $\exp^r$ on $\nu_\epsilon|N$ implies that $\mathcal{F}$ pulls back to a foliation $\mathcal{F}_{\mathbf{E}}$ on $\mathbf{E}$. The measure μ determines a transverse measure $\mu_{\mathbf{E}}$ for $\mathcal{F}_{\mathbf{E}}$, which by our choice of Φ has closed support $\mathbf{K} \subset \nu_{\epsilon'}|N$, hence $\mathbf{K}$ is compact. Clearly, with these constructions the conclusions of the lemma are all satisfied. □

Note that the foliated bundle $(\mathbf{E}, \mathcal{F}_{\mathbf{E}})$ is not complete, for the lift of a path in N to an arbitrary leaf of $\mathcal{F}_{\mathbf{E}}$ can diverge to infinity in the fibers of $\mathbf{E}$. This corresponds to the "covering path" in the open set U_ϵ drifting outside of this open set. However, the lift of a path in N to a leaf contained in $\mathbf{K}$ will stay in $\mathbf{K}$, as it is a saturated. As $\mathbf{K}$ is compact, such a path can be continued indefinitely, and therefore the restriction $\mathcal{F}_{\mathbf{K}}$ defines a foliation of the topological bundle $\pi : \mathbf{K} \to N$ which is transverse to the fibers and complete. Choose a basepoint $x_0 \in N$ and set $\mathbf{K}_0 = \mathbf{K} \cap \pi^{-1}(x_0)$. Then the holonomy of $\mathcal{F}_{\mathbf{K}}$ induces a representation $h_{\mathcal{F}_{\mathbf{K}}} : \pi_1(N, x_0) \to \mathrm{Homeo}(\mathbf{K}_0)$. The invariant measure $\mu_{\mathbf{E}}$ restricts to an invariant measure on $\mathbf{K}_0$ under this representation.

After these preliminaries, the proof of Theorem 1.10 will follow from the next three lemmas.

LEMMA 7.2 *Let μ be an almost compact invariant transverse measure for $\mathcal{F}$. Then the Ruelle-Sullivan class of μ is proportional to the homology class of the cycle N:*

$$[C_\mu] \sim [N] \in H_m(V; \mathbf{R})$$

Consequently, $E(\nu, \mu)$ is proportional to the cap product $E(\nu) \setminus [N]$.

Proof. The tangent bundle to N receives an orientation from that of $T\mathcal{F}$ by the transversality hypothesis. The open set $U_\epsilon \subset V$ retracts onto N, and the fundamental class of N is a generator of $H_m(U_\epsilon; \mathbf{R})$. The closed support of μ is contained in U_ϵ, so the Ruelle-Sullivan class of μ is in the image of $H_m(U_\epsilon; \mathbf{R}) \to H_m(V; \mathbf{R})$. □

By Lemma 7.2 and functoriality of the Euler class, to prove that $E(\nu, \mu) = 0$ it suffices to show that the Euler class $E(\mathbf{E}) = 0$ for the bundle $\pi : \mathbf{E} = \nu | N \to N$. Recall that a *Thom form* for $\mathbf{E}$ is a closed, compactly supported $2n$-form τ on $\mathbf{E}$ whose compactly supported cohomology class is the Poincaré dual to that of the zero section $N_0 \subset \mathbf{E}$. Then the Euler class $E(\mathbf{E}) \in H^{2n}(N; \mathbf{R})$ is obtained by restricting the Thom form τ to the zero section $N_0 \subset \mathbf{E}$ (see for example Proposition 12.4, [2]). Alternatively, $E(\mathbf{E}) = \pi_*([N_0] \cap [N_0])$ via Poincaré duality.

LEMMA 7.3 *Suppose that $\mu_{\mathbf{E}}$ is a countably discrete invariant transverse measure for $\mathcal{F}_{\mathbf{E}}$ with compact closed support. Then either $E(\mathbf{E}) = 0$, or the support of $\mu_{\mathbf{E}}$ consists of a single compact leaf L and $\pi | L : L \to N$ is a diffeomorphism.*

Proof. Suppose first that the support of $\mu_{\mathbf{E}}$ contains at least two disjoint compact leaves, $L_0, L_1 \subset \mathbf{E}$. Each homology class $[L_i] \in H_m(\mathbf{E}; \mathbf{R})$ is proportional to a non-zero multiple of $[N_0]$, so we can use these (disjoint) foliation cycles for $\mathcal{F}_{\mathbf{E}}$ to calculate

$$E(\mathbf{E}) = \pi_*([N_0] \cap [N_0]) \sim \pi_*([L_0] \cap [L_1]) = \pi_*(0) = 0$$

.

Assume the support of $\mu_{\mathbf{E}}$ consists of a single compact leaf L and the map $\pi | L : L \to N$ has degree $p > 1$. Then $[L] = p \cdot [N_0]$, and $E(\mathbf{E}) = p^{-2} \cdot \pi_*([L] \cap [L])$. However, the self-intersection $[L] \cap [L]$ can be calculated as the sum of the homology classes of the connected $(m - 2n)$-cycles obtained by taking a small perturbation of L which is transverse to L. This self-intersection cycle increases by a factor of only p under covers, so that $[L] \cap [L] = p \cdot [N_0] \cap [N_0]$, hence $E(\mathbf{E}) = p^{-1} \cdot \pi_*([L] \cap [L])$, which for $p > 1$ forces $E(\mathbf{E}) = 0$. □

The decomposition of the transverse measure μ into countably discrete and continuous parts, $\mu = \mu_a + \mu_c$, also decomposes the Ruelle-Sullivan class, $[C_\mu] = [C_{\mu_a}] + [C_{\mu_c}]$. The proof of Theorem 1.10 is completed by the next lemma:

LEMMA 7.4 *Suppose that μ is an almost compact transverse measure, and the summand μ_c is not the zero measure. Then $E(\nu, \mu) = 0$.*

Proof. By our previous remarks, it will suffice to show that $E(\mathbf{E}) = 0$. The Ruelle-Sullivan class for the continuous part, $[C_{\mu_c}]$, is again proportional to the class $[N]$, so that we can use $[C_{\mu_c}]$ to calculate the average Euler class as an intersection product, up to a non-zero constant:

$$E(\mathbf{E}) = \pi_*([N_0] \cap [N_0]) \sim \pi_*([C_{(\mu_{c\mathbf{E}})}] \cap [C_{(\mu_{c\mathbf{E}})}]) \tag{13}$$

Theorem 1 of [16] can be applied to the invariant measure $\mu_{c\mathbf{E}}$, as the measure has compact support. Theorem 1 implies that the intersection product $[C_{(\mu_{c\mathbf{E}})}] \cap [C_{(\mu_{c\mathbf{E}})}]$ is determined by the discrete summand for μ_c. As this is zero, we obtain that (13) vanishes, as was to be shown. $\square$

8 Examples

In this section we give a set of examples to illustrate our main theorems. The transverse Euler class is a delicate invariant, as it is a characteristic class for the transversal space to the foliation, and we are restricting it to leaves which are in a sense perpendicular to this transverse space. Therefore, its non-triviality requires a form of "dynamical transfer" from the leaf space $V/\mathcal{F}$ to the leaves of $\mathcal{F}$. Our first example illustrates how this can happen.

All of the examples with non-trivial transverse Euler class arise from the normal bundles to compact leaves. The basic idea is to take a standard example of a flat vector bundle with non-zero Euler class and embed this as the normal bundle to a compact leaf. Conjecture 1.9 is that these are the only possible examples.

EXAMPLE 8.1 (Foliated sphere bundles) We begin with a general remark. Suppose that M is a compact manifold with fundamental group Γ, and there is given a representation $\rho : \Gamma \to GL^+(2n, \mathbf{R})$ so that the associated flat vector bundle $\mathbf{E}(\rho) \to M$ has non-zero Euler class, $E(\mathbf{E}(\rho)) \in H^{2n}(M; \mathbf{R})$. The group $GL^+(2n; \mathbf{R})$ acts on the space $\mathbf{R}^{2n+1}$ via the natural action on the first $2n$-coordinates, and as the identity on the last coordinate. This induces an action of $GL^+(2n; \mathbf{R})$ on the space of positive rays in $\mathbf{R}^{2n+1}$, which is identified with S^{2n}. We compose the representation ρ with this action to obtain an action $\hat{\rho} : \Gamma \times S^{2n} \to S^{2n}$. This action leaves invariant the rays $x_\pm \in S^{2n}$ corresponding to the vectors $(0, \ldots, 0, \pm 1) \in \mathbf{R}^{2n+1}$, and the isotropy representation about these points is just ρ.

Construct a foliated manifold from $\hat{\rho}$ via the suspension construction. That is, for the universal cover $\tilde{M}$ with the left action of Γ via deck transformations, we have a foliation $\tilde{\mathcal{F}}$ of $\tilde{M} \times S^{2n}$ by the leaves $\tilde{M} \times \theta$, for $\theta \in S^{2n}$. The group Γ acts on the left on $\tilde{M} \times S^{2n}$ via $\gamma \cdot (x, \theta) = (\gamma x, \hat{\rho}(\gamma)(\theta))$, and this leaves the foliation $\tilde{\mathcal{F}}$ invariant. Let $\mathcal{F}_\rho$ denote the quotient foliation of $V = \Gamma \backslash (\tilde{M} \times S^{2n})$.

The fixed-points $\{x_\pm\}$ of the action $\hat{\rho}$ give rise to two compact leaves, $L_\pm \subset V$, each diffeomorphic to M. The normal bundle $\nu | L_\pm$ to each leaf $L_\pm$ is identified with the bundle $\mathbf{E}(\rho)$, so the Euler class $E(\nu | L_\pm)$ is non-zero. We therefore obtain the general result:

PROPOSITION 8.2 *Given a representation $\rho : \Gamma \to GL^+(2n, \mathbf{R})$ for which the associated flat vector bundle $\mathbf{E}(\rho) \to M$ has non-zero Euler class, there exists a codimension $2n$ real analytic foliation $\mathcal{F}_\rho$ of an S^{2n}-fibration $V \to M$, transverse to the fibers, which has two compact leaves, each with non-zero normal Euler class.* $\square$

There is a well-developed theory of the construction of representations satisfying the hypotheses of Proposition 8.2 (cf. Goldman [5]). We recall here the most basic example. Let $\Gamma \subset SL(2,\mathbf{R})$ be a torsion-free cocompact discrete subgroup. $SO(2) \subset SL(2,\mathbf{R})$ will denote the maximal compact subgroup of matrices A such that $AA^t = I$, where A^t denotes the matrix transpose. The double quotient $\Sigma = \Gamma\backslash SL(2,\mathbf{R})/SO(2)$ is a compact Riemann surface of negative curvature with fundamental group Γ. Take the representation ρ to be the inclusion $\Gamma \subset SL(2,\mathbf{R}) \subset GL^+(2,\mathbf{R})$.

Form the quotient manifold, $V = \Gamma\backslash\{(SL(2,\mathbf{R})/SO(2)) \times S^2\}$, with the foliation $\mathcal{F}_\rho$ defined as the image of the product foliation on $\{(SL(2,\mathbf{R})/SO(2)) \times S^2\}$ with leaves $(SL(2,\mathbf{R})/SO(2)) \times \theta$, for $\theta \in S^2$. The fixed points $\{x_+, x_-\}$ give rise to two compact leaves, denoted by $\{L_+, L_-\}$. The flat normal bundle to each leaf has holonomy associated given by the inclusion $\Gamma \subset SL(2,\mathbf{R})$, and this bundle is well-known to have non-zero Euler class (cf. [23, 5]).

The Euler class has a multiplicative property , so that given a collection of subgroups $\{\Gamma_1, \ldots, \Gamma_d\}$ in $SL(2,\mathbf{R})$ as above, the product representation of $\Gamma = \Gamma_1 \times \cdots \times \Gamma_d$ on $\mathbf{R}^{2d}$ has non-zero Euler class. This shows that non-trivial examples exist satisfying the hypotheses of Proposition 8.2 for all even codimensions.

Note that the examples constructed above involve a choice of the lattice subgroups, so this construction yields many-parameter families of examples. There are also more general constructions of representations of surface groups into $SL(2,\mathbf{R})$, having non-zero Euler class [5] which is distinct from the value of the Euler class produced by discrete embeddings of Γ in SL(2, $\mathbf{R}$). These alternate representations give rise to foliations which are not topologically equivalent to those obtained from discrete embeddings, and still have non-zero normal Euler classes on their compact leaves.

Let us also remark that we can obtain further examples, for which the transverse Euler class is non-zero, by taking products of the above examples. This will result in a foliated manifold with fibers a product of spheres,

$$S^{2n_1} \times \cdots \times S^{2n_p} \to V \to M \cong \Sigma_1 \times \cdots \times \Sigma_d,$$

where $n_1 + \cdots + n_p = d$, and Σ_i is a Riemann surface. The foliation $\mathcal{F}$ on V has 2^p compact leaves, each with non-zero normal Euler class. □

EXAMPLE 8.3 (Riemannian foliations) Milnor observed in his seminal paper (Theorem 3, [23]) that the Euler class for a flat bundle vanishes if the holonomy takes values in a compact subgroup of $GL^+(2n,\mathbf{R})$. This is a straightforward consequence of Chern-Weil theory. The corresponding geometric condition for foliations also forces the normal Euler class to vanish.

A foliation $\mathcal{F}$ is *Riemannian* if there exists a Riemannian metric on the normal bundle ν which is projectable. This means that on any foliation coordinate chart, $U \subset V$ with transverse coordinate function $\phi : U \to \mathbf{R}^q$ defining the foliation $\mathcal{F}|U$ as the fibers of ϕ, the metric on $\nu|U$ is the pull-back under ϕ of a metric on $T\mathbf{R}^q$.

PROPOSITION 8.4 *Let $\mathcal{F}$ be a C^2-Riemannian foliation of codimension $2n$. Then the transverse Euler class $E(\nu, \mathcal{F}) \in H^{2n}(\mathcal{F})$ is zero.*

Proof. The Bott covariant derivative operator $\nabla^{\mathcal{F}}$ preserves the metric on ν, so takes values in the orthogonal Lie algebra $\mathbf{so}(2n)$. We can therefore calculate an Euler form $e(\nu)$ on V using the Chern-Weil construction with the Bott connection. The curvature tensor of the Bott connection is projectable for a Riemannian foliation, so for each foliation chart $\phi : U \to \mathbf{R}^{2n}$, the Euler form $e(\nu)$ is locally a pull-back of a $2n$-form on $\mathbf{R}^{2n}$. In particular, the restrictions of $e(\nu)$ to leaves of $\mathcal{F}$ vanishes identically for this representative so that the leafwise class determined by $e(\nu)$ is zero. □

Note that Proposition 8.4 does not assert that the Euler class of ν is zero for a Riemannian foliation. For example, for the even dimensional sphere S^{2n} with $n > 1$, there exist a cocompact lattice Γ in a some connected Lie group G and a representation $\rho : \Gamma \to SO(2n)$ with dense image (cf. [34]). The suspension foliation of $V = \Gamma\backslash(G \times S^{2n})$ is Riemannian, with all leaves dense, and yet the Euler class of ν restricts to a nonzero class on the submanifolds $S^{2n} \subset V$. □

EXAMPLE 8.5 (Amenable group actions) When the group Γ is amenable, then every C^1-action of Γ on a compact space is amenable. We thus obtain the general result:

PROPOSITION 8.6 *Let M be a compact manifold with amenable fundamental group Γ. Then for every orientation-preserving C^1- action $\hat{\rho} : \Gamma \times X \to X$ on a compact oriented manifold X of dimension $2n$, the measurable transverse Euler class $\mathcal{E}(\nu, \mathcal{F}_{\hat{\rho}}) \in H^{2n}(\mathcal{F}_{\hat{\rho}})$ vanishes for the suspension foliation $\mathcal{F}_{\hat{\rho}}$ of V.* □

For example, if Γ is a solvable group, then it is amenable. Therefore, given a fibration $\pi : V \to M$ over a solv-manifold M, every foliation of V which is transverse to the fibers of π is amenable for the Lebesgue measure class, so its measurable transverse Euler class vanishes by Theorem 1.1.

It is also possible for a group action $\Gamma \times X \to X$ on a compact manifold X to be amenable, without Γ being amenable. The typical example of this phenomenon is for $\Gamma \subset G$ a subgroup of a Lie group, and $X = G/P$ for a closed subgroup $P \subset G$. Then a basic theorem of Zimmer ([38]; Proposition 4.3.2, [40]) states that the action of Γ on X on the left is amenable whenever P is solvable. (If the group Γ is dense in G, then there is a converse to this as well.) This gives the examples:

PROPOSITION 8.7 *Let M be a compact manifold with fundamental group Γ. Let $P \subset G$ be a closed solvable subgroup of a Lie group G with compact quotient $X = G/P$ of even dimension $2n$. Then for every homomorphism $\rho : \Gamma \to G$, the induced action of $\hat{\rho} : \Gamma \times X \to X$ is amenable, and the measurable transverse Euler class $\mathcal{E}(\nu, \mathcal{F}_{\hat{\rho}}) \in H^{2n}(\mathcal{F}_{\hat{\rho}})$ vanishes for the suspension foliation $\mathcal{F}_{\hat{\rho}}$ of V.* □

The following examples are included to illustrate the complex behavior that can be encountered with compact leaves in higher codimensions.

EXAMPLE 8.8 (Arithmetic foliations) Let $G \subset GL(N, \mathbf{R})$ be an algebraic subgroup. The intersection $G_{\mathbf{Z}} = GL(N, \mathbf{Z}) \cap G$ with the integer matrices $GL(N, \mathbf{Z})$ is said to be an *arithmetic lattice* in G (cf. Chapter 6, [40]). The group $G_{\mathbf{Z}}$ is always discrete in G, and it is known that there is a subgroup $\Gamma \subset G_{\mathbf{Z}}$ which is torsion-free. We can therefore take a maximal compact subgroup $K \subset G$, and Γ acts freely on the quotient space G/K. The quotient $M = \Gamma\backslash G/K$ is thus a manifold. We say that Γ is a *cocompact* subgroup when M is compact.

The embedding of $\Gamma \subset GL(N, \mathbf{Z})$ defines an action of Γ on $\mathbf{R}^N$ which preserves the integer lattice, and hence induces an analytic action $\hat{\rho} : \Gamma \times \mathbf{T}^N \to \mathbf{T}^N$ on the N-torus.

Note that every rational point in $\mathbf{T}^N$ is periodic for the action of Γ, so that its isotropy subgroup has finite index in Γ.

The suspension foliation $\mathcal{F}_{\hat{\rho}}$ on $V = \Gamma\backslash(G/K \times \mathbf{T}^N)$ has a compact leaf associated to every periodic point $x \in \mathbf{T}^N$, which is diffeomorphic to the covering of $\Gamma\backslash G/K$ associated to the isotropy subgroup of x. As the periodic points for this action are the rational points, and these are dense, we obtain:

PROPOSITION 8.9 *For each cocompact arithmetic subgroup $\Gamma \subset G_{\mathbf{Z}} \subset G$ of an algebraic Lie group G, the suspension foliation on V has dense compact leaves.* $\square$

The interest in the family of examples given by Proposition 8.9 is that they are in many ways very similar to the examples exhibited in Example 8.1. One notable difference is that the lattice Γ utilized in Example 8.1 is not the set of integer points in $G = SL(2, \mathbf{R})$, and we conjecture that the transverse Euler class must be zero for examples arising from the suspension of an integer lattice as in Proposition 8.9. This would be a special case of the more general conjecture, based on the results of this paper and the previous paper [16], that the transverse Euler class is zero unless there is *isolated* compact leaf for $\mathcal{F}$ which "carries" the class.

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