Exotic Index Theory for Foliations

Steven Hurder*
Department of Mathematics
University of Illinois at Chicago

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1 Introduction

In this paper we begin the study of the index theory in coarse geometry for foliations – the *exotic index theory* for leafwise elliptic operators on foliations. This is a parametrized form of the index theory for complete open manifolds developed by John Roe [53, 56, 52, 54]. The last section of this paper applies exotic index theory to prove a general result on the Foliation Novikov Conjecture, which extends previous results of Baum and Connes [4, 5] and Takai [60]:

THEOREM 7.8 *Let $\mathcal{F}$ be an oriented ultra-spherical foliation with uniformly contractible leaves and Hausdorff holonomy groupoid. Then the Foliation Novikov Conjecture is true for $\mathcal{F}$.*

A more painstaking application of the exotic index method will prove many other cases of the Foliation Novikov Conjecture – this will be addressed in a subsequent paper. The ideas of coarse geometry for foliations have their origins in techniques from dynamical systems, and the basic framework we develop in the first part of this paper has other geometric applications (cf. [38]).

Let $\mathcal{F}$ be a smooth foliation of a compact manifold M, $\mathbf{E} \to M$ a continuous Hermitian vector bundle which is leafwise smooth, and $\mathcal{D}_{\mathcal{F}}: C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$ a leafwise first order geometric operator. Connes [8, 9, 17, 47] introduced the reduced C^*-algebra $C^*_r(\mathcal{F})$ of $\mathcal{F}$ and observed that $\mathcal{D}_{\mathcal{F}}$ is "invertible" modulo this algebra, hence has a K-theoretic index $Ind(\mathcal{D}_{\mathcal{F}}, \epsilon) \in K_*(C^*_r(\mathcal{F}))$. Here is the basic problem:

PROBLEM 1.1 *Develop methods for evaluating the foliation index $Ind(\mathcal{D}_{\mathcal{F}}, \epsilon) \in K_*(C^*_r(\mathcal{F}))$, and relate $Ind(\mathcal{D}_{\mathcal{F}}, \epsilon)$ to the geometry and topology of $\mathcal{F}$.*

For example, a holonomy invariant transverse measure μ for $\mathcal{F}$ induces a linear map $T_\mu: K_*(C^*_r(\mathcal{F})) \to \mathbf{R}$. The real number $T_\mu(Ind(\mathcal{D}_{\mathcal{F}}, \epsilon))$ is a "leafwise average index" for $\mathcal{D}_{\mathcal{F}}$ restricted to the leaves in the support of μ (cf. [9, 35]). Connes' measured foliation index theorem states that the analytically defined real number $T_\mu(Ind(\mathcal{D}_{\mathcal{F}}, \epsilon))$ is obtained by pairing the degree n component of the deRham Chern character $Ch^*(\sigma(\mathcal{D}_{\mathcal{F}}, \epsilon))$ of the symbol class of $(\mathcal{D}_{\mathcal{F}}, \epsilon)$ with the degree n, Ruelle-Sullivan leafwise homology class determined by μ.

In general, the Chern character $Ch^*(\sigma(\mathcal{D}_{\mathcal{F}}, \epsilon))$ contains much more information than simply its leafwise degree n component, and this extra information is part of what is captured by the K-theory index class $Ind(\mathcal{D}_{\mathcal{F}}, \epsilon)$. For example, if the tangential distribution $T\mathcal{F}$ admits a Riemannian metric so that the leaves of $\mathcal{F}$ are simply connected with non-positive curvature and are *Spin*, then the leafwise "dual-Dirac" operator [9, 21, 59] induces a pairing map $\rho: K_*(C^*_r(\mathcal{F})) \to K^*(M)$. The "dual-Dirac" mapping captures the topological information contained in the leafwise degree 0 component of $Ch^*(\sigma(\mathcal{D}_{\mathcal{F}}, \epsilon))$.

The applications of the foliation index theorem in the literature [3, 4, 6, 9, 12, 16, 18, 20, 36, 39, 35, 58, 60] employ either the transverse measure or dual Dirac method to evaluate the foliation index.

A main point of this paper is to generalize the "dual Dirac" construction for foliations, which yields a new approach for the study of the the analytic index $Ind(\mathcal{D}_{\mathcal{F}}, \epsilon) \in K_*(C_r^*(\mathcal{F}))$. The key point is to introduce the *corona* $\partial_h \mathcal{F}$ of a foliation, modeled on Higson's corona for a complete metric space [31, 56]. Each K-theory class $[u] \in K^*(\partial_h \mathcal{F})$ of the foliation corona yields a generalized "ρ-map", $\rho[u]: K_*(C_r^*(\mathcal{F})) \to K^*(M)$, which evaluated on the analytic index $Ind(\mathcal{D}_{\mathcal{F}}, \epsilon)$ yields an "exotic index" in the K-theory of the ambient manifold. This recovers the dual Dirac map when the leaves of $\mathcal{F}$ are simply connected with non-positive curvature. There is an index theorem to calculate the exotic indices $\rho[u](Ind(\mathcal{D}_{\mathcal{F}}, \epsilon))$ in terms of the topology of their symbols. The combination of coarse geometry and index theory provides a very useful tool for studying the index and spectrum of leafwise operators, and how they are related to the topology of the foliated manifold.

The foliation corona $\partial_h \mathcal{F}$ is a topological space defined for any topological foliation $\mathcal{F}$ of a compact manifold, and has many good properties making it useful for constructing new invariants of foliations. For example, the homotopy type of $\partial_h \mathcal{F}$ depends only on the leafwise homotopy class of the foliation (Corollary 3.9) – in fact, its homotopy type is determined by the topological groupoid induced on an open complete transversal to $\mathcal{F}$. Applying a homotopy invariant functor to $\partial_h \mathcal{F}$ yields a leafwise homotopy invariant of $\mathcal{F}$. In particular, the Čech cohomology $\check{H}^*(\partial_h \mathcal{F})$ and the K-theory groups $K^*(\partial_h \mathcal{F})$ are leafwise homotopy invariants of $\mathcal{F}$.

A second fundamental property of $\partial_h \mathcal{F}$ is that its *homeomorphism type* is determined by the "coarse geometry" of a foliated compact manifold. That is, we form a parametrized family of metric spaces over the ambient manifold, where for each $x \in M$, endow the holonomy covering $\tilde{L}_x$ of the leaf L_x through x with the induced path length metric on $\tilde{L}_x$. Then the homeomorphism class of $\partial_h \mathcal{F}$ depends only on the coarse geometry of this family of metric spaces.

Coarse geometry for foliations has its antecedents in dynamical systems. The work of Kakutani on measurable equivalences of ergodic $\mathbf{Z}$-actions introduced ideas of coarse geometry for *measurable Borel equivalence relations*, and the *Kakutani equivalence* in the dynamics and ergodic theory literature (cf. section 1.2, [40]) is just measurable coarse isometry. Sections 2, 3 and 4 of this paper further develop these ideas in the setting of continuous dynamical systems.

In ergodic theory applications, the typical data about the coarse geometry is formulated in terms of properties of measurable cocycles over the equivalence relation, then used to define equivalence classes of with respect to growth rates (cf. [33, 40]). For example, the Lyapunov exponents that arise in Pesin theory are invariants of the coarse geometry. The use of coarse geometry to study index theory invariants in $K_*(C_r^*(\mathcal{F}))$ is intuitively parallel to the ergodic theory applications. Cycles in $K^*(\partial_h \mathcal{F})$ represent "almost flat" bundles [14, 15] which are continuous versions of the ϵ-tempered cocycles of [33, 40].

In the first part of this paper, we define and study the foliation corona: Section 2 recalls some basic properties of the geometry of foliations; section 3 defines the foliation corona, and establishes its topological functoriality; examples of coronas for several types of foliations are presented in section 4. The second part of the paper studies the exotic index invariants. After some analytic preliminaries, the definition of the abstract exotic index is given in section 5. The fundamental pairing between the K-theory of the corona and K-theory for uniform operators along the leaves is discussed in section 6, which can be evaluated via the Atiyah-Singer index theorem for families. In section 7 the exotic index is used to produce K-theory fundamental classes for foliations. This is applied to prove the foliation Novikov Conjecture for ultra-spherical foliations with uniformly contractible leaves.

The methods of exotic index theory for families can also be applied to the Novikov Conjecture for compact manifolds [37]. (The resulting method is similar to that of Connes, Gromov and Moscovici for manifolds with word hyperbolic fundamental groups [13], except the dual Dirac method is replaced with the exotic index map.)

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2 Metric properties of the holonomy groupoid

A *coarse metric* on a set X is a symmetric pairing $\langle \cdot, \cdot \rangle : X \times X \to [0, \infty)$ satisfying the triangle inequality

$$\langle x, z \rangle \leq \langle x, y \rangle + \langle y, z \rangle \qquad \text{for all } \; x, y, z \in X$$

A map $f : X_1 \to X_2$ is said to be *quasi-isometric* with respect to coarse metrics $\langle \cdot, \cdot \rangle_i$ if there exists constants $d_1, d_2, d_3 > 0$ so that for all $y, y' \in X_1$

$$d_1 \cdot (\langle y, y' \rangle_1 - d_3) \leq \langle f(y), f(y') \rangle_2 \leq d_2 \cdot (\langle y, y' \rangle_1 + d_3) \tag{1}$$

A subset $\mathcal{N} \subset X$ is ϵ-*dense* for $\epsilon > 0$ if for each $x \in X$ there exists $n(x) \in \mathcal{N}$ so that $\langle x, n(x) \rangle \leq \epsilon$. An ϵ-*net* is a collection of points $\mathcal{N} = \{x_\alpha \mid \alpha \in \mathcal{A}\} \subset X$ so that $\mathcal{N}$ is ϵ-dense, and there exists $c > 1$ so that distinct points of $\mathcal{N}$ are at least distance ϵ/c apart. The net $\mathcal{N}$ inherits a coarse metric from X.

DEFINITION 2.1 *A map $f : X_1 \to X_2$ is said to be a* coarse isometry *with respect to coarse metrics $\langle \cdot, \cdot \rangle_i$ if f is quasi-isometric and the image $f(X_1)$ is ϵ-dense in X_2 for some $\epsilon > 0$.*

Coarse geometry is the study of geometric properties of a complete metric space which are invariant under coarse isometries. The *fundamental property* of coarse geometry is that the inclusion of a net, $\mathcal{N} \subset X$, is a coarse isometry, which is obvious from the definitions. This has profound implications for what geometrical/topological invariants are preserved under coarse isometry (cf. Gromov [23]). The usual example to illustrate this phenomenon is that for a connected Lie group G, a cocompact lattice $\Gamma \subset G$ with the word metric is coarsely isometric to G with the left invariant Riemannian path-length metric: the integers $\mathbf{Z}$ are coarsely isometric to the real line $\mathbf{R}$. Thus, coarse geometry detects only global metric properties of a space, and ignores local properties. For further discussions of coarse geometry for metric spaces, see Gromov [22, 23] or Roe [56].

In this section, we develop aspects of the coarse geometry for foliations. A *topological foliation* $\mathcal{F}$ of a paracompact manifold M^m is a continuous partition of M into tamely embedded submanifolds (the leaves) of constant dimension p and codimension q. We require that these leaves be locally given as the level sets (plaques) of local coordinate charts. We specify this local defining data by fixing:

1. a uniformly locally-finite covering $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ of M; that is, there exists a number $m(\mathcal{A}) > 0$ so that for any $\alpha \in \mathcal{A}$ the set $\{\beta \in \mathcal{A} \mid U_\alpha \cap U_\beta \neq \emptyset\}$ has cardinality at most $m(\mathcal{A})$
2. local coordinate charts $\phi_\alpha : U_\alpha \to (-1,1)^m$, so that each map ϕ_α admits an extension to a homeomorphism $\tilde{\phi}_\alpha : \tilde{U}_\alpha \to (-2,2)^m$ where $\tilde{U}_\alpha$ contains the closure of the open set U_α
3. for each $z \in (-2,2)^q$, the preimage $\tilde{\phi}_\alpha^{-1}((-2,2)^p \times \{z\}) \subset \tilde{U}_\alpha$ is the connected component containing $\tilde{\phi}_\alpha^{-1}(\{0\} \times \{z\})$ of the intersection of the leaf of $\mathcal{F}$ through $\phi_\alpha^{-1}(\{0\} \times \{z\})$ with the set $\tilde{U}_\alpha$.

The extensibility condition in (2) is made to guarantee that the topological structure on the leaves remains tame out to the boundary of the chart ϕ_α. The collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ is called a *regular foliation atlas* for $\mathcal{F}$.

The inverse images

$$\mathcal{P}_\alpha(z) = \phi_\alpha^{-1}((-1,1)^p \times \{z\}) \subset U_\alpha$$

are topological discs contained in the leaves of $\mathcal{F}$, called the *plaques* associated to this atlas. One thinks of the plaques as "tiling stones" which cover the leaves in a regular fashion. The plaques are indexed by the *complete transversal*

$$\mathcal{T} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha$$

associated to the given covering, where $\mathcal{T}_\alpha = (-1,1)^q$. The charts ϕ_α define tame embeddings

$$t_\alpha = \phi_\alpha^{-1}(\{0\} \times \cdot) : \mathcal{T}_\alpha \to U_\alpha \subset M$$

We will implicitly identify the set $\mathcal{T}$ with its image in M under the maps t_α, though it may be that the union of these maps is only finite-to-one.

The foliation $\mathcal{F}$ is said to be C^r if the foliation charts $\{\phi_\alpha \mid \alpha \in \mathcal{A}\}$ can be chosen to be C^r-diffeomorphisms.

A *leafwise path* γ is a continuous map $\gamma : [0,1] \to M$ whose image is contained in a single leaf of $\mathcal{F}$. Suppose that a leafwise path γ has initial point $\gamma(0) = t_\alpha(z_0)$ and final point $\gamma(1) = t_\beta(z_1)$, then γ determines a local holonomy map h_γ which is a local homeomorphism from a neighborhood of z_0 to a neighborhood of z_1. More generally, if the initial point $\gamma(0)$ lies in the plaque $\mathcal{P}_\alpha(z_0)$ and $\gamma(1)$ lies in the plaque $\mathcal{P}_\beta(z_1)$, then γ again defines a local homeomorphism h_γ. Note that the holonomy of a concatenation of two paths is the composition of their holonomy maps. We say that two leafwise paths γ_1 and γ_2 with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ have the same holonomy if h_{γ_1} and h_{γ_2} agree on a common open set about z_0.

Define an equivalence relation on pointed leafwise paths by specifying that $\gamma_1 \sim_h \gamma_2$ if γ_1 and γ_2 have the same holonomy. The *holonomy groupoid* $\mathcal{G}_\mathcal{F}$ is the set of $\sim_h$ equivalence classes of pointed leafwise paths for $\mathcal{F}$, equipped with the topology whose basic sets are generated by "neighborhoods of leafwise paths" (cf. section 2, [62]). The manifold M embeds into $\mathcal{G}_\mathcal{F}$ by associating to $x \in M$ the constant path $*x$ at x.

The *fundamental groupoid* $\Pi_\mathcal{F}$ of $\mathcal{F}$ is the set of endpoint-fixed homotopy equivalence classes of leafwise paths for $\mathcal{F}$, equipped with the topology whose basic sets are generated by "neighborhoods of leafwise paths". Two paths which are endpoint-fixed homotopy equivalent have the same holonomy, so there is a natural map of groupoids $\Pi_\mathcal{F} \to \mathcal{G}_\mathcal{F}$.

There are natural continuous maps $s, r : \mathcal{G}_\mathcal{F} \to M$ defined by $s(\gamma) = \gamma(0)$ and $r(\gamma) = \gamma(1)$. For a point $x \in M$, the pre-image $s^{-1}(x) = \tilde{L}_x$ is the *holonomy cover* of the leaf L_x of $\mathcal{F}$ through x; that is, the image of a closed curve $\gamma \subset \tilde{L}_x$ always has trivial holonomy as a curve in M. We use the source map s to view the groupoid $\mathcal{G}_\mathcal{F}$ as a parametrized family of open manifolds (the holonomy covers of leaves of $\mathcal{F}$) over the base M.

Define the *transversal groupoid* $\mathcal{T}_\mathcal{F} \subset \mathcal{G}_\mathcal{F}$ to be the preimage of $\mathcal{T} \times \mathcal{T}$ under the map

$$s \times r \colon \mathcal{G}_\mathcal{F} \to M \times M.$$

That is, $\mathcal{T}_\mathcal{F}$ consists of all the equivalence classes of paths in $\mathcal{G}_\mathcal{F}$ which start and end at points in the complete transversal $\mathcal{T}$. For each $x \in \mathcal{T}$ the fiber $(s|\mathcal{T}_\mathcal{F})^{-1}(x) \subset \tilde{L}_x$ is a net in the holonomy cover $\tilde{L}_x$, so that $\mathcal{T}_\mathcal{F}$ can be considered as a (locally) continuous selection of nets for the fibers of $s \colon \mathcal{G}_\mathcal{F} \to M$.

The topological manifold structure on $\mathcal{G}_\mathcal{F}$ may not be Hausdorff: suppose there exists a leafwise closed path γ with basepoint x which has non-trivial holonomy of infinite order, but so that there is a family $\{\gamma_s \mid 1 \leq s \geq 0\}$ of closed paths, $\gamma_0 = \gamma$, and which are the transverse "push-off" of γ so that each γ_s has trivial holonomy for $s > 0$. Then every iterate of the path γ is arbitrarily close to the push-offs γ_s for s small. That is, the path $\{\gamma_s \mid s > 0\}$ intersects every neighborhood of the iterates of γ. This property of paths that there are nearby paths for which the holonomy degenerates is typical of the non-Hausdorff aspect of $\mathcal{G}_\mathcal{F}$. This was formalized by Winkelnkemper in the following result:

PROPOSITION 2.2 (Proposition 2.1, [62]) *$\mathcal{G}_\mathcal{F}$ is Hausdorff if and only if, for all $x \in M$ and $y \in L_x$ the holonomy along two arbitrary leafwise paths γ_1 and γ_2 from x to y*

are already the same if they coincide on an open subset U of their common domain, whose closure $\overline{U}$ contains x.

For example, if the holonomy of every leaf has finite order, or is analytic, or is an isometry for some transversal metric, then $\mathcal{G}_{\mathcal{F}}$ will be Hausdorff. In contrast, one knows that the holonomy of the compact leaf in the Reeb foliation of S^3 fails this criterion, so its foliation groupoid is not Hausdorff at the compact leaf.

Let $\mathcal{G}_{\mathcal{F}}^{nh} \subset \mathcal{G}_{\mathcal{F}}$ be the union of the paths for which there exists another path which has the holonomy property of Proposition 2.2. Then $\mathcal{G}_{\mathcal{F}}^{h} = \mathcal{G}_{\mathcal{F}} \setminus \mathcal{G}_{\mathcal{F}}^{nh}$ is a Hausdorff space.

A key property of the space $\mathcal{G}_{\mathcal{F}}$ is that given a compact set $K \subset \tilde{L}_x \subset \mathcal{G}_{\mathcal{F}}^{h}$ in a leaf which is Hausdorff, there exists an open neighborhood $U \subset M$ of x and an open set $W \subset \mathcal{G}_{\mathcal{F}}$ so that $s(W) = U$, $K = W \cap \tilde{L}_x$ and there is a fiber-preserving homeomorphism $W \cong K \times U$. (This is a consequence of the previous remark that the normal foliated microbundle to the topological embedding $K \hookrightarrow M$ has trivial holonomy along the slice K, hence the image has a normal disc bundle whose pullback W to $\mathcal{G}_{\mathcal{F}}$ is foliated as a product.)

Let $\mathcal{F}_i$ be a topological foliation of M_i for $i = 1, 2$. Let $f : M_1 \to M_2$ be a continuous map which sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$. Then the assignment $\gamma \mapsto f(\gamma)$ induces a map $\mathcal{G}f : \mathcal{G}_{\mathcal{F}_1} \to \mathcal{G}_{\mathcal{F}_2}$. It is clear from the definition that $s(\mathcal{G}f(\gamma)) = f(s(\gamma))$ and similarly for the range map r. Thus, $\mathcal{G}f$ maps the fibers of s over M_1 into the fibers of s over M_2. We let $\mathcal{G}f_x : \tilde{L}_x \to \tilde{L}'_{f(x)}$ denote the restriction of $\mathcal{G}f$ from the fiber of s over $x \in X_1$ to the fiber of s over $f(x) \in X_2$.

Let $\mathcal{F}_i$ be a topological foliation of M_i for $i = 1, 2$, $f_0, f_1 : M_1 \to M_2$ be continuous maps which sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$. We say that f_0 is *leafwise homotopic* to f_1 if there exists a continuous map $F : M_1 \times [0, 1] \to M_2$ such that

- $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in M_1$
- F maps the leaves of $\mathcal{F}_1 \times [0, 1]$ into the leaves of $\mathcal{F}_2$, where $\mathcal{F}_1 \times [0, 1]$ is the foliation of $M_1 \times [0, 1]$ with typical leaf $L \times [0, 1]$ for L a leaf of $\mathcal{F}_1$.

The trace of a leafwise homotopy F is the collection of curves $t \mapsto F(x, t)$ for $x \in M_1$ and the special property of a leafwise homotopy is simply that the trace consists of leafwise curves.

A continuous map $f : M_1 \to M_2$ which sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$ is a *leafwise homotopy equivalence* if there exists a continuous map $g : M_2 \to M_1$ which sends leaves of $\mathcal{F}_2$ into leaves of $\mathcal{F}_1$ so that the compositions $g \circ f$ and $f \circ g$ are both leafwise homotopic to the respective identity maps on M_1 and M_2.

We next formulate the coarse metric properties of the foliation groupoid (cf. Plante [48]; section 1, Hurder & Katok [40].) A coarse metric on $\mathcal{G}_{\mathcal{F}}$ will be a family of coarse metrics

$$\langle \cdot, \cdot \rangle_x : \tilde{L}_x \times \tilde{L}_x \to [0, \infty)$$

parametrized by $x \in M$. It is natural to also require a type of "coarse continuity" of the family, which is satisfied by the examples presented below, but we will not formalize this notion.

Given groupoids $s\colon \mathcal{G}_i \to X_i$ equipped with coarse metrics $\langle \cdot, \cdot \rangle_x^i$ for $i = 1, 2$, a groupoid map $F\colon \mathcal{G}_1 \to \mathcal{G}_2$ is a *quasi-isometry* if there exists constants $d_1, d_2, d_3 > 0$ so that for all $x \in X_1$ and $y, y' \in s^{-1}(x)$

$$d_1 \cdot \Big(\langle y, y' \rangle_x^1 - d_3 \Big) \leq \langle F_x(y), F_x(y') \rangle_{f(x)}^2 \leq d_2 \cdot \Big(\langle y, y' \rangle_x^1 + d_3 \Big) \tag{2}$$

where $f\colon X_1 \to X_2$ is the map on objects induced by F. We say that F is a *coarse isometry* if there exists $\epsilon > 0$ so that $F_x(s^{-1}(x)) \subset s^{-1}(f(x))$ is ϵ-dense for all $x \in X_1$.

Fix a regular foliation atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ for $\mathcal{F}$. For $x \in M$ and a leafwise path $\gamma\colon [0, 1] \to \tilde{L}_x$, define the *plaque length function* $\mathcal{N}_{\mathcal{T}}(\gamma)$ to be the least number of plaques required to cover the image of γ. Define the *plaque distance function* $D_x(\cdot, \cdot)$ on the holonomy cover $\tilde{L}_x$ using the plaque length function: for $y, y' \in \tilde{L}_x$,

$$D_x(y, y') = \inf \{\mathcal{N}_{\mathcal{T}}(\gamma) \ \mid \gamma \ \text{ is a leafwise path from } \ y \text{ to } y'\}$$

In other words, $D_x(y, y')$ is the minimum number of plaques in $\tilde{L}_x$ such their union forms a connected open set in $\tilde{L}_x$ containing both y and y'. Note that $D_x(\cdot, \cdot)$ is not a distance function, for $D_x(y, y') = 1$ if and only if y and y' lie on the same plaque $\mathcal{P}_\alpha(z)$. It is immediate from the definitions that the pairings D_x satisfy the triangle inequality, so we have

LEMMA 2.3 *The family D_x is a coarse metric for the foliation groupoid $\mathcal{G}_{\mathcal{F}}$.* $\square$

LEMMA 2.4 *Suppose that $\mathcal{F}$ is a topological foliation of a compact compact M, and there are given two coverings of M by regular foliation atlases: $\{(U_\alpha^i, \phi_\alpha^i) \mid \alpha \in \{1, \ldots, k(i)\}\}$ for $i = 1, 2$, with corresponding plaque distance functions D_x^i. Then there exists constants $c_1, c_2 > 0$ so that for all $x \in M$ and $y, y' \in \tilde{L}_x$*

$$c_1 \cdot D_x^1(y, y') \leq D_x^2(y, y') \leq c_2 \cdot D_x^1(y, y') \tag{3}$$

Hence, the identity map is a coarse isometry of $\mathcal{G}_{\mathcal{F}}$ endowed with the coarse metrics D_x^1 and D_x^2.

Proof. Assume that $\{(U_\alpha^2, \phi_\alpha^2) \mid \alpha \in \{1, \ldots, k(2)\}\}$ is a refinement of $\{(U_\alpha^1, \phi_\alpha^1) \mid \alpha \in \{1, \ldots, k(1)\}\}$. Thus, for each $1 \leq \beta \leq k(2)$ there is $1 \leq \alpha(\beta) \leq k(1)$ so that $U(\beta) \subset U(\alpha(\beta))$. Let c_2 denote the maximum number of distinct open sets of the second cover contained in any fixed open set of the first cover. This is called the subdivision number for the refinement. Then a leafwise curve γ with D_x^1-plaque-length $|\gamma|_1$ has D_x^2-plaque-length at most $c_2 \cdot |\gamma|_1$ which yields the right-hand-side of (3). Conversely, if γ has D_x^2-plaque-length $|\gamma|_2$ then it clearly has D_x^1-plaque-length at most $|\gamma|_2$.

For the general case, we form a common refinement $\{(U^3_\alpha, \phi^3_\alpha) \mid \alpha \in \{1, \dots, k(3)\}\}$ of the two given covers. Then take c_2 equal to the subdivision number of the first cover, and c_1 the reciprocal of the subdivision number of the second cover, and we obtain (3). $\square$

When the foliation $\mathcal{F}$ is at least C^1, then we can give the leaves a Riemannian metric, and define a leafwise Riemannian distance function d_x on $\tilde{L}_x$ by taking the infimum over the lengths of paths in the holonomy cover between y and y'. The family d_x is a coarse metric on $\mathcal{G}_\mathcal{F}$.

LEMMA 2.5 *Suppose that $\mathcal{F}$ is a C^1-foliation, M is compact, and $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \{1, \dots, k\}\}$ is a regular foliation atlas with a finite number of open charts. Then there exists constants $c_1, c_2 > 0$ so that for all $x \in M$ and $y, y' \in \tilde{L}_x$*

$$c_1 \cdot (D_x(y, y') - 1) \leq d_x(y, y') \leq c_2 \cdot D_x(y, y') \tag{4}$$

Hence, the identity map is a coarse isometry of $\mathcal{G}_\mathcal{F}$ endowed with the metrics D_x and d_x, respectively.

Proof. The regular foliation atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \{1, \dots, k\}\}$ defines a finite number of transversals $\mathcal{T}_\alpha$ with compact closures, and each closed set $\overline{\mathcal{T}}_\alpha$ parametrizes a continuous family of compact discs $\overline{\mathcal{P}}_\alpha(z) \subset M$. By the continuity of the leafwise metrics $d_x(\cdot, \cdot)$ and the fact that each chart is regular, there is an upper bound c_2 for the diameters in the Riemannian metric of the plaques defined by $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \{1, \dots, k\}\}$. Therefore, if $\gamma\colon [0, 1] \to \tilde{L}_x$ has Riemannian length $|\gamma|$, then it cannot be covered by fewer than $|\gamma|/c_2$ plaques. Choose a leafwise path which realizes the distance $d_x(y, y')$, then this estimate yields the right-hand-side estimate in (4).

Let $c_1 = \epsilon(\{U_\alpha \mid \alpha \in \{1, \dots, k\}\}) > 0$ be the Lebesgue number for the open covering. Let $y, y' \in \tilde{L}_x$, then choose a path $\gamma : [0, 1] \to \tilde{L}_x$ with length $d_x(y, y')$. Divide the image of γ into segments $\{\mathcal{I}_i \mid 1 \leq i \leq \ell\}$ of length exactly c_1 each, except possibly for the last segment $\mathcal{I}_\ell$ which has length at most c_1. Let $\lceil |\gamma|/c_1 \rceil$ denote the least integer greater than $|\gamma|/c_1$. By the definition of the Lebesgue number, each segment $\mathcal{I}_i$ is contained in some open set U_{α_i}, and hence in some plaque $\mathcal{P}_{\alpha_i}(z_i)$ for $z_i \in \mathcal{T}$. Therefore the path γ can be covered by $\lceil |\gamma|/c_1 \rceil$ plaques, which gives the estimate

$$D_x(y, y') \leq \lceil |\gamma|/c_1 \rceil \leq (|\gamma|/c_1 + 1)$$

from which the left-hand-side of (4) follows immediately. $\square$

LEMMA 2.6 *Let M_1 be a compact manifold, and $f : M_1 \to M_2$ be a continuous function which sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$. Then there exists a constant $d_2 > 0$ so that for all $x \in M_1$ and $y, y' \in \tilde{L}_x$, the induced map $\mathcal{G}f_x : \tilde{L}_x \to \tilde{L}'_{f(x)}$ on holonomy covers satisfies the estimate*

$$D_{f(x)} \left(\mathcal{G}f_x(y), \mathcal{G}f_x(y')\right) \leq d_2 \cdot D_x \left(y, y'\right) \tag{5}$$

The induced map $\mathcal{G}f_x : \tilde{L}_x \to \tilde{L}'_{f(x)}$ is said to be eventually Lipshitz.

Proof. Let $\{(U_i, \phi_i) \mid 1 \leq i \leq k\}$ be a regular foliation atlas for $\mathcal{F}_1$. Each image $f(\overline{U}_i)$ is compact, hence is covered by a finite number N_i of foliation charts for $\mathcal{F}_2$. Let $d_2 = \max\{N_i \mid 1 \leq i \leq k\}$. Then for $y, y' \in \tilde{L}_x$ with a leafwise path γ between them with plaque-length $\mathcal{N}_{\mathcal{T}}(\gamma)$, the image curve $f(\gamma)$ has plaque-length at most $d_2 \cdot \mathcal{N}_{\mathcal{T}}(\gamma)$ from which the estimate (5) follows. $\square$

The induced map $\mathcal{G}f_x : \tilde{L}_x \to \tilde{L}'_{f(x)}$ need not be a quasi-isometry, or even proper, though both M_1 and M_2 are assumed to be compact. The first inequality in (1) fails in the following simple example. Let $M_1 = \mathbf{T}^2$ be the 2-torus with $\mathcal{F}_1$ the linear foliation by lines with irrational slope. Let $M_2 = \mathbf{T}^2$ also, with $\mathcal{F}_2$ the foliation having exactly one leaf. The identity map satisfies the estimate (5). On the other hand, the leaves of $\mathcal{F}_1$ contain paths of arbitrarily long length, which map to segments in $\mathbf{T}^2$ which are $\sim_h$ equivalent to a "shortcut" in $\mathbf{T}^2$ of length at most $2\sqrt{2}\pi$, where we assume that each circle factor in $\mathbf{T}^2$ has length 2π. Thus, for this example there is no estimate for the minimum plaque-length of a leafwise path for $\mathcal{F}_1$ in terms of the minimum plaque-length of its image in $\mathcal{F}_2$.

There is a natural condition to impose on f which forces the fiberwise maps $\mathcal{G}f_x$ to be quasi-isometries whenever M_1 is compact: f is *injective on holonomy* if, given two paths γ_1 and γ_2 contained in a leaf of $\mathcal{F}_1$ with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$, then $f(\gamma_1) \sim_h f(\gamma_2)$ implies that $\gamma_1 \sim_h \gamma_2$. This property is satisfied whenever f is a leafwise homotopy equivalence, which is the idea behind the proof of the next result.

PROPOSITION 2.7 *Let $\mathcal{F}_i$ be a topological foliation of a compact manifold M_i for $i = 1, 2$ and $f : M_1 \to M_2$ a leafwise homotopy equivalence. Then there exists constants $d_1, d_2 > 0$ so that for all $x \in M_1$ and $y, y' \in \tilde{L}_x$ with $D_x(y, y') \geq d_3$, the induced map $\mathcal{G}f_x : \tilde{L}_x \to \tilde{L}'_{f(x)}$ satisfies the estimate*

$$d_1 \cdot D_x(y, y') \leq D_{f(x)}(\mathcal{G}f_x(y), \mathcal{G}f_x(y')) \leq d_2 \cdot D_x(y, y') \tag{6}$$

Thus, $\mathcal{G}f\colon \mathcal{G}_{\mathcal{F}_1} \to \mathcal{G}_{\mathcal{F}_2}$ is a coarse isometry with respect to the coarse metrics D^1_x and D^2_x.

Proof. Choose a leaf-preserving continuous map $g : M_2 \to M_1$ and a leafwise homotopy $F\colon M_1 \times [0, 1] \to M_1$ between $g \circ f$ and the identity. Let K denote the maximum plaque-lengths of the leafwise traces $t \mapsto F(x, t)$ for $x \in M_1$. Let d'_2 denote the constant for g and d_2 the constant for f given by Lemma 2.6. Given a leafwise path γ between $z = \mathcal{G}f(y)$ and $z' = \mathcal{G}f(y')$, the images $\mathcal{G}g(z)$ and $\mathcal{G}g(z')$ are connected to y and y' by leafwise paths with plaque-lengths at most K each. (This is true for their images in M_1 so by the covering path lifting property also holds for the points in $\tilde{L}_x$.) Applying Lemma 2.6 to g we then obtain

$$D_x(y, y') \leq D_x(\mathcal{G}g_{f(x)}(z), \mathcal{G}g_{f(x)}(z')) + 2K \leq d'_2 \cdot D_{f(x)}(z, z') + 2K$$

hence

$$1/d'_2 \cdot (D_x(y, y') - 2K) \leq D_{f(x)}(z, z')$$

Take $d_3 = 4K$ and $d_1 = 1/(2d'_2)$ and the estimate (6) follows. $\square$

COROLLARY 2.8 *Let $\mathcal{F}_i$ be a topological foliation of a compact manifold M_i for $i = 1, 2$ and $f : M_1 \to M_2$ a leafwise homotopy equivalence. Then $\mathcal{G}f$ is a proper map.*

Proof. Let $K \subset \mathcal{G}_{\mathcal{F}_2}$ be a compact set. Then there is a finite collections of leafwise paths $\{\gamma_1, \ldots, \gamma_d\}$ for $\mathcal{F}_2$ and a covering of K by basic foliation charts formed from the γ_i. It follows that there is a constant C_K so that K is contained in the diagonal set

$$\Delta(\mathcal{G}_{\mathcal{F}_2}, C_K) = \{y \in \mathcal{G}_{\mathcal{F}_2} \mid D_{s(y)}(y, *s(y)) \leq C_K\}$$

where $*s(y)$ is the canonical basepoint in the fiber $\tilde{L}_{s(y)}$. The inequality (6) implies that the preimage $\mathcal{G}f^{-1}(K)$ is contained in the diagonal set $\Delta(\mathcal{G}_{\mathcal{F}_1}, C_K/d_1)$. Hence $\mathcal{G}f^{-1}(K)$ is a closed set contained in a finite union of basic foliation charts on $\mathcal{G}_{\mathcal{F}_1}$ so is compact. □

We conclude this section with the foliation version of the fundamental property of coarse geometry. Fix a regular foliation atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \{1, \ldots, k\}\}$ for $\mathcal{F}$ on the compact manifold M, which defines the transversal $\mathcal{T}$. The transversal groupoid $\mathcal{T}_\mathcal{F}$ has an intrinsic *transversal length function* $D_\mathcal{T}$, defined analogously to the word length function for groups. (The choice of the transversal $\mathcal{T}$ corresponds to the choice of a generating set for a group.) We say that two points $y \in \mathcal{T}_\alpha$ and $y' \in \mathcal{T}_\beta$ are *adjacent* if their plaques $\mathcal{P}_\alpha(y) \cap \mathcal{P}_\beta(y') \neq \emptyset$. The choice of a path $\gamma_{y,y'} \subset \mathcal{P}_\alpha(y) \cup \mathcal{P}_\beta(y')$ connecting adjacent points y, y' determines a canonical equivalence class $[\gamma_{y,y'}] \in \mathcal{T}_\mathcal{F}$. For $[\gamma_y] \neq [\gamma_{y'}] \in \mathcal{T}_\mathcal{F}$ define

$$\begin{aligned} D_\mathcal{T}([\gamma_y], [\gamma_{y'}]) \quad = \quad \min\{n > 0 \mid & \text{ there exits a chain of points } \ y = y_0, y_2, \ldots, y_n = y' \\ & \text{with } (y_i, y_{i+1}) \text{ adjacent for each } \ 0 \leq i < n \ \text{ and} \\ & [\gamma_{y'}] = [\gamma_y] * [\gamma_{y_1,y_2}] * \cdots * [\gamma_{y_{n-1},y_n}]\} \end{aligned} \qquad (7)$$

and set $D_\mathcal{T}([\gamma_y], [\gamma_{y'}]) = \infty$ if no such chain exists, and set $D_\mathcal{T}([\gamma_y], [\gamma_y]) = 0$.

PROPOSITION 2.9 *The inclusion $T\colon \mathcal{T}_\mathcal{F} \subset \mathcal{G}_\mathcal{F}$ induces a coarse isometry for the transversal length function $D_\mathcal{T}$ on $\mathcal{T}_\mathcal{F}$ and the plaque distance function on $\mathcal{G}_\mathcal{F}$.*

Proof: It follows from definitions that

$$D_x(T[\gamma_y], T[\gamma_{y'}]) = D_\mathcal{T}([\gamma_y], [\gamma_{y'}]) + 1$$

for all $[\gamma_y], [\gamma_{y'}]) \in \mathcal{T}_\mathcal{F}$ so that T is a quasi-isometry.

Each point $y \in \tilde{L}_x$ is within one plaque-distance from a point in a transversal $\mathcal{T}_\alpha$, so the image $T(\mathcal{T}_\mathcal{F}) \subset \mathcal{G}_\mathcal{F}$ is $\epsilon = 2$ dense in each fiber $\tilde{L}_x = s^{-1}(x)$. □

The main conclusion of the results of this section is that associated to a topological foliation of a compact manifold, there is a unique (up to coarse isometry) coarse metric on the foliation groupoid, which depends only on the leafwise homotopy class of the foliation, and its coarse isometry class is determined by the canonical coarse metric on the transversal groupoid to $\mathcal{F}$. It remains to extract from this coarse metric on $\mathcal{G}_\mathcal{F}$ analytical and topological information about $\mathcal{F}$.

3 The corona of a foliation

It is a fundamental problem to define a "good" compact boundary ∂X for a complete metric space X (cf. § 2, [23]), so that a coarse isometry of metric spaces induces a homeomorphism of their boundaries – in particular, ∂X should depend only on the coarse isometry class of X. Higson (section 3, [31]) and Roe (Chapter 5, [56]) introduced the *corona* $\partial_h X$ of a complete metric space X which is such a boundary, and is canonical with respect to certain pairings with operator K-theory. In this section, we extend their construction of the corona to topological foliations of compact manifolds, and show it also has good functorial properties.

We briefly recall the construction of the corona when X is a C^1-manifold with a complete Riemannian metric [31]. Let $C_h(X)$ denote the C^*-algebra closure (in the sup norm on functions) of the functions on X whose *gradients* tend to zero at infinity. The algebra of continuous functions which vanish at infinity, $C_0(X)$, is a closed C^*-subalgebra of $C_h(X)$. The Higson corona of X, denoted by $\partial_h X$, is defined to be the spectrum of the quotient C^*-algebra $C_h(X)/C_0(X)$.

There is an inclusion of closed C^*-algebras, $C_0(X) \subset C_h(X) \subset C(X)$, so that $\partial_h X$ is an intermediate boundary between the maximal Stone-Čech compactification $\check{X} = spec(C(X))$ and the one-point compactification $X \cup \infty = spec(C_0(X))$. One can show that if the coarse metric on X is not bounded, then $\partial_h X$ is non-separable.

One motivation for introducing the algebra $C_h(X)$ is that the vanishing gradient condition is exactly what is required to obtain a well-defined index pairing between the K-theory groups $K_*(C_h(X))$ and first order geometric operators on X with "bounded geometry". Roe abstracted Higson's construction to complete metric spaces, replacing the decay condition on the gradient with a decay condition on the variation function (cf. Definition 3.1 below).

The construction of the corona for a foliation groupoid $\mathcal{G}_{\mathcal{F}}$ equipped with a coarse metric uses a leafwise decay condition on the variation functions along the holonomy covers of the leaves. There is a subtlety in the groupoid case, in that the space of continuous functions on $\mathcal{G}_{\mathcal{F}}$ is closed under pointwise multiplication of functions only if $\mathcal{G}_{\mathcal{F}}$ is Hausdorff.

Let $C(\mathcal{G}_{\mathcal{F}})$ denote the topological vector space of continuous functions on the groupoid $\mathcal{G}_{\mathcal{F}}$, with the uniform norm topology obtained from the sup-norm on functions:

$$\sup |h| = \sup_{y \in \mathcal{G}_{\mathcal{F}}} |h(y)|$$

Recall that $\mathcal{G}_{\mathcal{F}}$ need not be Hausdorff, and hence $C(\mathcal{G}_{\mathcal{F}})$ is defined as the vector space spanned by locally-finite sums $\sum_i f_i$ of continuous functions supported in basic open sets $U_i \subset \mathcal{G}_{\mathcal{F}}$ (i.e., regular neighborhoods of paths $\gamma_i \in \mathcal{G}_{\mathcal{F}}$) on which the topology of $\mathcal{G}_{\mathcal{F}}$ is Hausdorff (cf. [8, 9]). Denote by $C_u(\mathcal{G}_{\mathcal{F}}) \subset C(\mathcal{G}_{\mathcal{F}})$ the closed subspace consisting of uniformly continuous functions, and $C_c(\mathcal{G}_{\mathcal{F}}) \subset C_u(\mathcal{G}_{\mathcal{F}})$ the subspace spanned by finite sums of continuous functions supported in basic open sets in $\mathcal{G}_{\mathcal{F}}$.

When $\mathcal{G}_{\mathcal{F}}$ is a Hausdorff space, $C(\mathcal{G}_{\mathcal{F}})$ is an algebra. In general, the pointwise-product of two functions $h, k \in C(\mathcal{G}_{\mathcal{F}})$ will have a set of discontinuities contained in the union of the leaves in M at which $\mathcal{G}_{\mathcal{F}}$ fails to be Hausdorff.

Define $C_u(\mathcal{F})$ to be the closed topological commutative algebra generated by $C_u(\mathcal{G}_{\mathcal{F}})$. Let $C_0(\mathcal{F}) \subset C_u(\mathcal{F})$ denote the closed topological subalgebra generated by the functions $C_c(\mathcal{G}_{\mathcal{F}})$.

DEFINITION 3.1 *For $x \in M$ and $r > 0$, define the* fiberwise variation function

$$\begin{aligned} V_s(x,r) : C(\tilde{L}_x) &\rightarrow [0,\infty) \\ V_s(x,r)(h)(y) &= \sup\{|h(y') - h(y)| \text{ such that } D_x(y,y') \leq r\} \end{aligned}$$

Here are the basic properties of the variation (cf. Chapter 5, [56]):

1. $V_s(x,r)(h) \leq 2 \sup\ |h|$
2. $V_s(x,r)(h \pm k) \leq V_s(x,r)(h) + V_s(x,r)(k)$
3. $V_s(x,r)(h \cdot k) \leq V_s(x,r)(h) \cdot \sup\ |k| + \sup\ |h| \cdot V_s(x,r)(k)$
4. If $|h(y)| \rightarrow 0$ as $D_x(y, *x) \rightarrow \infty$ then $V_s(x,r)(h)(y) \rightarrow 0$ as $D_x(y, *x) \rightarrow \infty$

We say that $f \in C(\mathcal{G}_{\mathcal{F}})$ has *uniformly vanishing variation at infinity* if there exists a function $D : (0,\infty) \rightarrow [0,\infty)$ so that if $D_x(y, *x) > D(\epsilon)$ then $V_s(x,r)(i_x^* f)(y) < \epsilon$. Let $C_h(\mathcal{F}) \subset C_u(\mathcal{F})$ denote the subspace of uniformly continuous functions which have uniformly vanishing variation at infinity. The proof of the following lemma is then exactly the same as for Lemma 5.3 of [56]:

LEMMA 3.2 *$C_h(\mathcal{F})$ is a commutative C^*-algebra. $C_0(\mathcal{F})$ is a closed C^*-subalgebra of $C_h(\mathcal{F})$.* □

DEFINITION 3.3 *Let $\mathcal{F}$ be a topological foliation of a paracompact manifold M equipped with a regular foliation atlas. The* corona*, $\partial_h\mathcal{F}$, of $\mathcal{F}$ is the spectrum of the quotient C^*-algebra $C_h(\mathcal{F})/C_0(\mathcal{F})$.*

Introduce the topological Hausdorff spaces $\mathcal{G}_{\mathcal{F}}^d = spec(C_0(\mathcal{F}))$ and $\overline{\mathcal{G}_{\mathcal{F}}^d} = spec(C_h(\mathcal{F}))$. Note that $C_h(\mathcal{F})$ contains the constant function $h = 1$ which is the multiplicative unit, so the topological space $\overline{\mathcal{G}_{\mathcal{F}}^d}$ is a compactification of $\mathcal{G}_{\mathcal{F}}^d$ obtained by adding on the corona $C_h(\mathcal{F})$ at infinity. As sets, $\mathcal{G}_{\mathcal{F}}^d = \mathcal{G}_{\mathcal{F}}$. The identity map $\mathcal{G}_{\mathcal{F}}^d \subset \mathcal{G}_{\mathcal{F}}$ is always continuous, and is a homeomorphism if $\mathcal{G}_{\mathcal{F}}$ is Hausdorff. When $\mathcal{G}_{\mathcal{F}}$ is not Hausdorff, $\mathcal{G}_{\mathcal{F}}^d$ is the topological space obtained from $\mathcal{G}_{\mathcal{F}}$ by giving it the coarsest Hausdorff topology such that the inclusion map is continuous. Note there is an exact sequence of algebras

$$0 \longrightarrow C_0(\mathcal{F}) \longrightarrow C_h(\mathcal{F}) \cong C(\overline{\mathcal{G}_{\mathcal{F}}^d}) \longrightarrow C_h(\mathcal{F})/C_0(\mathcal{F}) \cong C(\partial_h\mathcal{F}) \longrightarrow 0 \tag{8}$$

PROPOSITION 3.4 .

1. *The source projection extends to a continuous map* $\overline{s}\colon \overline{\mathcal{G}^d_{\mathcal{F}}} \longrightarrow M$.
2. *For each* $x \in M$ *there is an inclusion* $\iota_x\colon \overline{\tilde{L}_x} = spec(C_h(\tilde{L}_x)) \hookrightarrow \overline{\mathcal{G}^d_{\mathcal{F}}}$.
3. *For each* $x \in M$ *there is an inclusion* $\partial\iota_x\colon \partial_h \tilde{L}_x \hookrightarrow \partial_h \mathcal{F}$, *where* $\partial_h \tilde{L}_x$ *is the Higson corona of* $\tilde{L}_x$.

Proof: 1) The inclusion of unital algebras $s^*\colon C(M) \hookrightarrow C_h(\mathcal{F})$ induces a continuous map of spectra, $\overline{s}$, whose restriction to the interior $\mathcal{G}^d_{\mathcal{F}} \subset \overline{\mathcal{G}^d_{\mathcal{F}}}$ is clearly s.

2) For each $x \in M$ the restriction map $\iota_x^* : C(\mathcal{G}_{\mathcal{F}}) \to C(\tilde{L}_x)$ into the continuous functions on the fibers of s is a map of algebras, as the fiber $\tilde{L}_x$ is Hausdorff so the product of continuous functions restricts to a continuous function of $\tilde{L}_x$. Thus, ι_x^* restricts to an algebra map $\iota_x^*\colon C_h(\mathcal{F}) \to C_h(\tilde{L}_x)$. Each function $h \in C_h(\tilde{L}_x)$ is absolutely continuous, and an elementary extension construction shows that ι_x^* is surjective, so ι_x is injective.

3) $\iota_x^* : C_c(\mathcal{G}_{\mathcal{F}}) \subset C_c(\tilde{L}_x)$, hence the quotient map $\iota_x^* : C_h(\mathcal{F})/C_0(\mathcal{F}) \to C_h(\tilde{L}_x)/C_0(\tilde{L}_x)$ is surjective, so the induced map on boundaries $\partial\iota_x$ is injective □

We next establish three functorial properties of the foliation corona.

PROPOSITION 3.5 *Let* M_1 *be a compact manifold, and* $f : M_1 \to M_2$ *be a continuous function which sends leaves of* $\mathcal{F}_1$ *into leaves of* $\mathcal{F}_2$ *and induces a proper map of groupoids* $\mathcal{G}f\colon \mathcal{G}^d_{\mathcal{F}_1} \to \mathcal{G}^d_{\mathcal{F}_2}$. *Then there is an induced map*

$$\overline{f} : \overline{\mathcal{G}^d_{\mathcal{F}_1}} \to \overline{\mathcal{G}^d_{\mathcal{F}_2}}.$$

Proof. By Lemma 2.6, the pull-back map $f^* : C(\mathcal{G}_{\mathcal{F}_2}) \to C(\mathcal{G}_{\mathcal{F}_1})$ maps the subspace $C_h(\mathcal{F}_2)$ into the subspace $C_h(\mathcal{F}_1)$, and hence induces a map $\overline{f}$ on spectrum. □

PROPOSITION 3.6 *For* $i = 1, 2$, *let* $\mathcal{F}_i$ *be a topological foliation of a compact manifold* M_i. *Then a leafwise homotopy equivalence* $f : M_1 \to M_2$ *induces a continuous map*

$$\partial_h f : \partial_h \mathcal{F}_1 \to \partial_h \mathcal{F}_2$$

Proof. By Corollary 2.8 the pull-back $f^* : C_c(\mathcal{G}_{\mathcal{F}_2}) \to C_c(\mathcal{G}_{\mathcal{F}_1})$, so f^* induces a map on quotients

$$f^* : C_h(\mathcal{F}_2)/C_0(\mathcal{F}_2) \longrightarrow C_h(\mathcal{F}_1)/C_0(\mathcal{F}_1)$$

which yields the map $\partial_h f : \partial_h \mathcal{F}_1 \to \partial_h \mathcal{F}_2$. □

COROLLARY 3.7 *For $i = 1, 2$, let $\mathcal{F}_i$ be a topological foliation of a compact manifold M_i. Then a leaf-preserving homeomorphism $f : M_1 \to M_2$ induces a homeomorphism*

$$\partial_h f : \partial_h \mathcal{F}_1 \xrightarrow{\cong} \partial_h \mathcal{F}_2$$

PROPOSITION 3.8 *Let $\mathcal{F}$ be a topological foliation of a compact manifold M and $f : M \to M$ be a leaf-preserving continuous map which is leafwise-homotopic to the identity map. Then $\partial_h f : \partial_h \mathcal{F} \to \partial_h \mathcal{F}$ is the identity map.*

Proof. Let $F : M \times [0,1] \to M$ be a homotopy from f to the identity. As noted in the proof of Proposition 2.7 there is an upper bound K denote the maximum plaque-lengths of the leafwise traces $t \mapsto F(x,t)$ for $x \in M_1$, and therefore given $x \in M$ and $y \in \tilde{L}_x$ the lifted path $t \mapsto \mathcal{G}F(y,t)$ is covered by at most K plaques. That is, $D_{s(y)}(y, f(y)) \leq K$ for all $y \in \mathcal{G}_{\mathcal{F}}$. We then follow the proof of Proposition 5.11 of [56], observing for $h \in C_h(\mathcal{F})$ that

$$|h(y) - h(f(y))| \leq V_s(s(y), K)(h)(y)$$

which tends to 0 as $D_{s(y)}(y, *s(y)) \to \infty$. This implies the difference $h - f^*(h) \in C_0(\mathcal{F})$ and hence the induced map on $C_h(\mathcal{F})/C_0(\mathcal{F})$ equals the identity. □

COROLLARY 3.9 *For $i = 1, 2$, let $\mathcal{F}_i$ be a topological foliation of a compact manifold M_i. Then a leafwise homotopy equivalence $f : M_1 \to M_2$ induces a homeomorphism $\partial_h f : \partial_h \mathcal{F}_1 \cong \partial_h \mathcal{F}_2$.*

Proof. Let $g : M_2 \to M_1$ be a leafwise-homotopy inverse for f. Then the composition $\partial_h g \circ \partial_h f = \partial_h (g \circ f)$ is the identity map by Proposition 3.8. □

4 Topological aspects of the foliation corona

In this section we investigate some of the topological properties of foliation coronas. First note that the foliation corona $\partial_h \mathcal{F}$ of a topological foliation with non-compact leaves of a compact manifold is non-separable, and is a truly enormous space. (The problem is that the criteria for a function to be in $C_h(\mathcal{G}_{\mathcal{F}})$ imposes no restrictions on the rate of decay of the variation. No decay estimates are needed to formulate the appropriate pairings with operator K-theory in the next section, so no decay estimates are imposed!) On the other hand, we are principally interested in the K-theory groups $K^*(\partial_h \mathcal{F})$, which are determined by maps of $\partial_h \mathcal{F}$ to finite complexes. Our investigation of the topology of foliation coronas will thus be restricted to studying them via their separable quotient spaces.

A *separable corona* (X, q) for $\mathcal{F}$ is a separable compact space X equipped with a continuous surjection $q: \partial_h \mathcal{F} \to X$. A separable corona (X, q) determines a separable subalgebra

$$\mathcal{A}_X = \{f \in C_h(\mathcal{F}) \text{ such that } f|\partial_h \mathcal{F} = g \circ q \text{ for some } g \in C(X)\}$$

Conversely, given a separable C^*-subalgebra $\mathcal{A} \subset C_h(\mathcal{F})$ containing $C_0(\mathcal{F})$ there is a natural map

$$q: \partial_h \mathcal{F} \to spec(\mathcal{A}_X) \equiv X_{\mathcal{A}}$$

which defines a separable corona for $\mathcal{F}$. A natural way to obtain a separable corona for $\mathcal{F}$ is to construct such a subalgebra $\mathcal{A}$ which is generated by functions in $C_h(\mathcal{F})$ satisfying a "rate-of-decay" condition on their variations. For example, the *endset* (or *Freudenthal*) compactification of $\mathcal{G}^d_{\mathcal{F}}$ is obtained by requiring that the variation vanish outside some compact set. The *Gromov-Roe boundary* of $\mathcal{F}$ is obtained by requiring that the variation have rapid decay.

Let $C_\epsilon(\mathcal{F}) \subset C_h(\mathcal{F})$ be the closed topological subalgebra generated by the functions which are constant outside a compact set. That is, $h \in C_h(\mathcal{F})$ is in $C_\epsilon(\mathcal{F})$ if and only if there is a compact subset $K_h \subset \mathcal{G}_{\mathcal{F}}$ so that the restriction of h to $C_\epsilon(\mathcal{F}) \setminus K_h$ is constant. Note that $C_0(\mathcal{F}) \subset C_\epsilon(\mathcal{F})$.

DEFINITION 4.1 *The* endset *of a foliation $\mathcal{F}$ is the compact topological space $\epsilon(\mathcal{F})$ defined as the spectrum of the unital topological algebra $C_\epsilon(\mathcal{F})/C_0(\mathcal{F})$.*

PROPOSITION 4.2 *$\epsilon(\mathcal{F})$ is a corona for $\mathcal{F}$.*

Proof. A point in the spectrum of $C_h(\mathcal{F})/C_0(\mathcal{F})$ can be identified with an evaluation

$$\hat{\epsilon} : C_h(\mathcal{F})/C_0(\mathcal{F}) \to \mathbf{C},$$

which naturally restricts to an evaluation $\hat{\epsilon} : C_\epsilon(\mathcal{F})/C_0(\mathcal{F}) \to \mathbf{C}$. Thus, there is a natural map $\partial_h \mathcal{F} \to \epsilon(\mathcal{F})$. $C_\epsilon(\mathcal{F})/C_0(\mathcal{F})$ has a unit so $\epsilon(\mathcal{F})$ is compact. There is a countable base for the space of the functions which are constant outside a compact set, hence $\epsilon(\mathcal{F})$ is separable. Finally, let us show that $\epsilon(\mathcal{F})$ is the Freudenthal compactification for $\mathcal{G}^d_{\mathcal{F}}$. A function which is constant outside of a compact set in $\mathcal{G}_{\mathcal{F}}$ extends continuously to the Freudenthal compactification, hence $C_\epsilon(\mathcal{F})/C_0(\mathcal{F})$ is contained in the continuous functions on the Freudenthal compactification. The functions in $C_\epsilon(\mathcal{F})/C_0(\mathcal{F})$ separate the ends on $\mathcal{G}_{\mathcal{F}}$, so by the Stone-Weierstrass Theorem it must equal the standard end compactification. (We are indebted to John Roe for pointing out this last trick.) □

The endset can be effectively described for the class of suspension foliations (cf. Chapter 5, [7]). Let X denote a compact topological manifold. Let Γ be a finitely-generated group isomorphic to the fundamental group $\pi_1(B, b_0)$ of a compact manifold B, with Γ acting on the universal covering $\tilde{B} \to B$ by deck translations on the left. Given a continuous action $\varphi : \Gamma \times X \to X$, form the product of the deck action with φ to obtain an action of Γ on $\tilde{B} \times X$. Introduce the quotient compact topological manifold,

$$M_\varphi = \Gamma \setminus (\tilde{B} \times X).$$

The product foliation on $\tilde{B} \times X$, with typical leaf $\tilde{L} = \tilde{B} \times \{x\}$ for $x \in X$, descends to a topological foliation on M_φ denoted by $\mathcal{F}_\varphi$. The projection onto the first factor map,

$\tilde{B} \times X \to \tilde{B}$, descends to a map $\pi : M_\varphi \to B$, and π restricted to the leaves $\mathcal{F}_\varphi$ is a covering map. A Riemannian metric on TB lifts via π to a leafwise metric on $T\mathcal{F}_\varphi$, so that the foliation always carries a leafwise Riemannian distance function (even though $\mathcal{F}_\varphi$ need only be a topological foliation).

Let $K_\varphi \subset \Gamma$ denote the subgroup of elements which act trivially on X under φ, let $\Gamma_\varphi = \Gamma/K_\varphi$ denote the quotient group and $\tilde{B}_\varphi$ the covering of B corresponding to Γ_φ. Then Γ_φ is isomorphic to a subgroup of $\mathrm{Homeo}(X)$, called the *global holonomy group* $\mathcal{H}_{\mathcal{F}_\varphi} \subset \mathrm{Homeo}(X)$ of $\mathcal{F}_\varphi$.

The action φ is *effective* if for all open subsets $U \subset X$ and all $\gamma \in \Gamma$, if $\varphi(\gamma)$ restricts to the identity on U, then $\varphi(\gamma)$ acts as the identity on X. Winkelnkemper showed that the holonomy groupoid of the suspension of an effective action is Hausdorff, and there is a homeomorphism

$$\mathcal{G}_{\mathcal{F}_\varphi} \cong \Gamma \setminus \left(\tilde{B} \times X \times \tilde{B}_\varphi \right) \tag{9}$$

PROPOSITION 4.3 *Let $\mathcal{F}_\varphi$ be the suspension foliation associated to an effective continuous action φ. Then the foliation endset $\epsilon(\mathcal{F}_\varphi)$ fibers over M_φ with fiber homeomorphic to the endset $\epsilon(\Gamma_\varphi)$ of the global holonomy group.*

Proof: The endset $\epsilon(\Gamma_\varphi)$ is homeomorphic to the usual endset of the connected topological space $\tilde{B}_\varphi$ on which Γ_φ acts freely with cocompact quotient. The action of Γ_φ on $\tilde{B}_\varphi$ extends to a continuous action on the endset compactification $\tilde{B}_\varphi \cup \epsilon(\Gamma_\varphi)$. Thus, there is a natural map

$$\epsilon_\Gamma \colon \mathcal{G}_{\mathcal{F}_\varphi} \longrightarrow \mathcal{E}(B, \Gamma, \varphi) \equiv \Gamma \setminus \left(\tilde{B} \times X \times \left\{ \tilde{B}_\varphi \cup \epsilon(\Gamma_\varphi) \right\} \right)$$

for which the subalgebra $\epsilon_\Gamma^* C(\mathcal{E}(B, \Gamma, \varphi)) \subset C(\mathcal{F})$ is seen to agree with $C(\epsilon(\mathcal{F}))$. We thus obtain the more precise result identifying

$$\epsilon(\mathcal{F}_\varphi) \cong \Gamma \setminus \left(\tilde{B} \times X \times \epsilon_{\Gamma_\varphi} \right)$$

from which the claim of the proposition follows. $\square$

Let us next introduce a family of foliation coronas, parametrized by a real number $\tau > 0$. For $f \in C(\mathcal{G}_\mathcal{F})$, we say that the variation of f has *uniform τ-decay* if for each $r > 0$ there exists $C(f, k, r) > 0$ and a uniform estimate

$$V_\tau(x, r)(i_x^* f)(y) < C(f, k, r) \, [D_x(y, *x) + 1]^{-\tau} \text{ for each } x \in M \text{ and all } y \in \tilde{L}_x \tag{10}$$

The τ-decay condition is especially useful when $\tau > 1$ for it then implies an estimate on the change in the value of f along paths in the fibers (cf. the proof of Proposition 4.7).

Let $C_\tau(\mathcal{F}) \subset C_h(\mathcal{F})$ be the closed topological subalgebra generated by the functions whose variations have uniform τ-decay.

DEFINITION 4.4 *Let $\mathcal{F}$ be a topological foliation of a compact manifold M. For $\tau > 0$ the* τ-boundary $\partial_\tau \mathcal{F}$ *of $\mathcal{F}$ is the spectrum of the quotient C^*-algebra $C_\tau(\mathcal{F})/C_0(\mathcal{F})$.*

The variation of f has *uniformly rapid decay* if it has uniform τ-decay for all $\tau > 0$. Let $C_\infty(\mathcal{F}) \subset C_h(\mathcal{F})$ be the closed topological subalgebra generated by the functions whose variations have uniformly rapid decay. Roe proved that for a complete metric space X which is hyperbolic in the sense of Gromov, the spectrum of the algebra of functions with rapid decay is homeomorphic to the geodesic compactification of X (Proposition 2.3, [54]). This boundary is well-defined for any metric space, so we propose the

DEFINITION 4.5 *Let $\mathcal{F}$ be a topological foliation of a compact manifold M. The* Gromov-Roe boundary $\partial_\infty \mathcal{F}$ *of $\mathcal{F}$ is the spectrum of the quotient C^*-algebra $C_\infty(\mathcal{F})/C_0(\mathcal{F})$.*

Let us consider an important class of examples of foliations for which these boundaries can be effectively described. Assume there is given:

- a compact CW-complex Z and a fibration $\Pi: Z \to M$,
- a fiberwise metric $\Re_x: Z_x \times Z_x \to [0,1]$ which varies continuously with x,
- a continuous "weight" function $\Phi: M \times [0,\infty) \to [0,\infty)$ with $\Phi(M \times \{0\}) = 0$ and each restriction $\Phi_x: [0,\infty) \to [0,\infty)$ is monotone-increasing and unbounded.

The *parametrized cone* determined by the map Π is the fibration $C\Pi: C(Z,\Pi) \to M$, where for each $x \in M$ the fiber $CZ_x \equiv C\Pi^{-1}(x)$ over x is the cone with vertex x and base $Z_x = \Pi^{-1}(x)$. The additional data $\Re$ and Φ determines a fiberwise metric $C_\Phi \Re$ on $C(Z,\Pi)$, where the fiber CZ_x has the cone metric determined by Φ_x and $\Re_x$ (cf. section (3.46) of [56]). The data $\{C\Pi: C(Z,\Pi) \to M, C\Re\}$ is called the *parametrized metric cone* on $\{\Pi: Z \to M, \Re, \Phi\}$.

DEFINITION 4.6 *A foliation $\mathcal{F}$ is* cone-like with base $\Pi: Z \to M$ *if there exists*

- *a parametrized metric cone $\{C\Pi: C(Z,\Pi) \to M, C\Re\}$*
- *a fiber-preserving map $C\mathcal{F}: C(Z,\Pi) \to \mathcal{G}_{\mathcal{F}}$ which covers the identity on M,*
- *constants d_1, d_2, d_3, ϵ so that for each $x \in M$ the restriction $C\mathcal{F}_x: CZ_x \to \tilde{L}_x$ is a coarse isometry with respect to these constants (cf. Definition 2.1).*

PROPOSITION 4.7 *Let $\mathcal{F}$ be a cone-like foliation with base $\Pi: Z \to M$. Then there are fiber-preserving continuous surjections*

$$\partial_h \mathcal{F} \stackrel{\partial C\mathcal{F}}{\longrightarrow} Z \stackrel{\partial_\tau C\mathcal{F}}{\longrightarrow} \partial_\tau \mathcal{F}$$

for $1 < \tau \leq \infty$ such that the composition is the canonical map $\partial_h \mathcal{F} \to \partial_\tau \mathcal{F}$. In particular, $\partial_\tau \mathcal{F}$ is a separable corona for $1 < \tau \leq \infty$.

Proof. Let $C_\rho(Z) \subset C_u(C(Z,\Pi))$ be the C^*-subalgebra of functions which are uniformly asymptotic along rays. That is, for each $z \in Z$ parametrize the ray $z \times [0,\infty) \subset CZ_x$ by the arclength parameter t. Then for each $f \in C_\rho(Z)$ and $\epsilon > 0$ there exists $T(f,\epsilon)$ so that for all $z \in Z$,

$$|f(z \times t) - f(z \times K(f,\epsilon))| < \epsilon \text{ for all } t > K(f,\epsilon)$$

Clearly, the quotient algebra $C_\rho(Z)/C_0(C(Z,\Pi)) \cong C(Z)$ so $spec(C_\rho(Z))$ compactifies $C(Z,\Pi)$ by adding Z at infinity. The induced map $C\mathcal{F}_*: C_\rho(Z) \to C(\mathcal{F})$ is injective with image in $C_h(\mathcal{F})$, and maps $C_0(C(Z,\Pi))$ into $C_0(\mathcal{F})$. Thus, there is an induced surjective map on spectra $\partial C\mathcal{F}: \partial_h\mathcal{F} \to Z$.

Let $f \in C_\tau(\mathcal{F})$, the pull-back $C\mathcal{F}_x^* f$ has uniform τ-decay along each ray $z \times [0,\infty) \subset CZ_x$. Let t denote the arclength parameter along the ray. Since $\tau > 1$, for any $\epsilon > 0$ there exists T_ϵ so that for for $t_0, t_1 > T_\epsilon$ the difference

$$|C\mathcal{F}_x^* f(z \times t_0) - C\mathcal{F}_x^* f(z \times t_1)| < \int_{t_0}^{t_1} V_\tau(x,1)(i_x^* f)(z \times t)dt < C(f,k,1)\int_{t_0}^{t_1} t^{-\tau}dt \quad (11)$$

Hence, there is a well-defined asymptotic value along each ray $z \times [0,\infty)$. Asymptotic evaluation defines a map of algebras $A_\tau C\mathcal{F}_*: C_\tau(\mathcal{F}) \to C(Z)$ which vanishes when restricted to $C_0(\mathcal{F})$, and restricts to the identity on the subalgebra $\pi_1^* C(M)$. Hence, $A_\tau C\mathcal{F}_*$ determines a continuous map $\partial_\tau C\mathcal{F}: Z \to \partial_\tau \mathcal{F}$ which covers the identity on M. The kernel of the asymptotic evaluation map $A_\tau C\mathcal{F}_*$ consist of functions which uniformly tend to zero at infinity, hence the factor map $A_\tau C\mathcal{F}_*: C_\tau(\mathcal{F})/C_0(\mathcal{F}) \to C(Z)$ is injective which implies that $\partial_\tau C\mathcal{F}$ is surjective. $\square$

REMARK 4.8 The surjection $\partial_\tau C\mathcal{F}: Z \longrightarrow \partial_\tau\mathcal{F}$ need not be a homeomorphism. For example, when all leaves of $\mathcal{F}$ are metrically Euclidean of dimension greater than 1, then it is a nice exercise to show that each fiber of $\partial_\tau\mathcal{F} \to M$ is a point for $\tau > 1$. At the other extreme, when the leaves of $\mathcal{F}$ admit metrics of uniformly negative curvature, the arguments of Roe (cf. the proof of Proposition 2.3, [54]) show that $\partial_\infty\mathcal{F}$ is a fibration over M with fibers S^{p-1}.

Our last set of examples and remarks concern foliations where the corona can be shown to fiber over the base M. We first establish a general result, then consider geometric special cases to illustrate it.

DEFINITION 4.9 *A foliation $\mathcal{F}$ is said to be* coarsely geodesic *if*

- *$\mathcal{G}_\mathcal{F}$ is a Hausdorff space, with $s: \mathcal{G}_\mathcal{F} \to M$ a fibration.*
- *For each $x \in M$ there exists an open neighborhood $x \in U \subset M$ and a trivialization $T_U: s^{-1}U \to \tilde{L}_x \times U$, so that for each $y \in U$ the restriction $T_{U,y}: s^{-1}(y) \to \tilde{L}_x \times \{y\}$ is a coarse isometry, with uniform constants independent of $y \in U$.*

A coarsely taut foliation $\mathcal{F}$ has a "typical leaf" $\tilde{L}$ which is a complete metric space, so that for all $x \in M$ the holonomy cover $\tilde{L}_x$ is diffeomorphic and coarsely isometric to $\tilde{L}$.

PROPOSITION 4.10 *Let $\mathcal{F}$ be a coarsely geodesic foliation. Then the corona of $\mathcal{F}$ fibers*

$$\partial_h \tilde{L} \longrightarrow \partial_h \mathcal{F} \xrightarrow{\partial s} M$$

Proof: Proposition 3.4 implies that the source map extends to a continuous map on the boundary, $\partial s \colon \partial_h \mathcal{F} \to M$. We need to show that for a sufficiently small open disk $U \subset M$ and $x \in U$, the fibrations

$$\begin{array}{rcl} s\colon (\partial s)^{-1}(U) & \longrightarrow & U \\ \pi_1 \colon \partial_h \tilde{L}_x \times U & \longrightarrow & U \end{array}$$

are homeomorphic.

Choose a trivialization $T_U \colon s^{-1}U \to \tilde{L}_x \times U$ which restricts to a uniform coarse isometry on fibers, $T_{U,y} \colon s^{-1}(y) \to \tilde{L}_x \times \{y\}$. There is an induced map of algebras

$$(\pi_1 \times T_U)^* \colon C_h(\tilde{L}_x) \otimes C_0(U) \longrightarrow C_h((\partial s)^{-1}(U))$$

where the left-hand-side is the algebraic tensor product. The image is dense in the uniform topology of functions, because $C_h((\partial s)^{-1}(U)) \subset C_h(\mathcal{F}) \subset C_u(\mathcal{F})$ – so $(\pi_1 \times T_U)^*$ induces a homeomorphism of their spectra,

$$\overline{\pi_1 \times T_U} \; : \; \overline{(\partial s)^{-1}(U)} \quad \to \quad \overline{\tilde{L}_x \times U} \;\; \cong \;\; \overline{\tilde{L}_x} \times U$$

whose restriction to the boundary gives the desired trivialization. $\Box$

There are many constructions which yield coarsely geodesic foliations.

Recall the construction of the suspension foliation $M_\varphi = \Gamma \setminus (\tilde{B} \times X)$ associated to a continuous action $\varphi : \Gamma \times X \to X$ on a compact topological manifold X the fundamental group $\Gamma \cong \pi_1(B, b_0)$ of a compact manifold B. Let $\Gamma_\varphi = \Gamma / K_\varphi$ denote the quotient by the maximal subgroup K_φ which acts trivially on X under φ, and $\tilde{B}_\varphi$ the covering of B corresponding to Γ_φ. The deck translations act via isometries on $\tilde{B}_\varphi$ so induce a continuous action on the compactification $\overline{\tilde{B}_\varphi} = \tilde{B}_\varphi \cup \partial_h \tilde{B}_\varphi$. There is a Γ-equivariant homeomorphism of boundaries $\partial_h \tilde{B}_\varphi \cong \partial_h \Gamma$, so by the identification (9) and an application of the Proposition 4.10 we obtain:

PROPOSITION 4.11 *Let $\varphi : \Gamma \times X \to X$ be an effective on a compact topological manifold X. Then the foliation corona is homeomorphic to the suspension fibration obtained from the induced action of Γ on the Higson corona of the global holonomy group Γ_φ*

$$\partial_h \mathcal{F} \cong \Gamma \setminus \left(\tilde{B} \times X \times \partial_h \Gamma_\varphi \right)$$

Another class of examples are provided by locally free Lie group actions. Let G be a connected Lie group. A topological action $\varphi : G \times M \to M$ is *locally-free* if for all $x \in M$ the isotropy subgroup $G_x \subset G$ is a finite subgroup. The action is *effective* if g must be the identity element whenever there is an open set $U \subset M$ so that $\varphi(g)$ restricts to the identity on U.

LEMMA 4.12 *Let $\varphi : G \times M \to M$ be a locally-free effective C^1-action. Then the orbits of the action φ define a C^1-foliation $\mathcal{F}_\varphi$ of M, and there is a natural homeomorphism*

$$\mathcal{G}_{\mathcal{F}_\varphi} \cong G \times M \tag{12}$$

Choose an orthonormal framing of the Lie algebra of G, which determines a right-invariant Riemannian metric on TG. At each $x \in M$ the left action of G on M induces a framing of the orbit of G through x. The action of G is locally free, so the resulting continuous vector fields on M are linearly independent at each point, hence yields a global framing of the leaves of $\mathcal{F}_\varphi$. Declare this to be an orthonormal framing to obtain a Riemannian metric on the leaves. Note that the identification (12) maps $G \times \{x\}$ to the holonomy cover of the orbit of G through x, which by the essentially free hypotheses is exactly the orbit Gx. The induced Riemannian metrics on G and Gx are identical, hence hence they are trivially quasi-isometric. By the identification (12) and an application of the Proposition 4.10 we obtain:

PROPOSITION 4.13 *The foliation corona of $\mathcal{F}_\varphi$ is homeomorphic to a product, $\partial_h \mathcal{G}_{\mathcal{F}_\varphi} \cong \partial_h G \times M$.* $\square$

The coronas of Riemannian foliations on compact manifolds can be explicitly determined. As the leafwise geometric operators for Riemannian foliations are a generalization of the study of almost-periodic operators, the study of their analysis and index theory is a natural extension of more classical topics, and the corona construction gives an additional topological tool for their investigation. Recall that a C^1-foliation $\mathcal{F}$ is *Riemannian* [46, 49] if there exists a Riemannian metric on the normal bundle to $\mathcal{F}$ which is invariant under the linear holonomy transport. This has many consequences for the topology of M and the structure of the foliation [46]. For example, for a compact manifold M there is an open dense set of leaves in a Riemannian foliation which have no holonomy, and the holonomy covers of all of the leaves of $\mathcal{F}$ are homeomorphic. The homeomorphisms are induced by first forming the principal $O(q)$-bundle $P \to M$ of orthogonal frames to the foliation $\mathcal{F}$, where q is the codimension. The foliation lifts to a foliation $\hat{\mathcal{F}}$ without holonomy, and the leaves of $\hat{\mathcal{F}}$ cover those of $\mathcal{F}$. The compact manifold P carries a collection of linearly independent vector fields which span the normal bundle to $\hat{\mathcal{F}}$, whose flows induce leaf preserving homeomorphisms of P and which are transitive on the leaf space of $\hat{\mathcal{F}}$. Thus, given any two leaves of $\mathcal{F}$, there is a homeomorphism of their holonomy covers which is realized by a sequence of homeomorphisms, each the flow associated to a vector field on P. As noted by Winkelnkemper (section3, Corollary [62]), this implies that the foliation groupoid is a fibration over the base M,

$$L \longrightarrow \mathcal{G}_{\mathcal{F}} \xrightarrow{s} M \tag{13}$$

where L is called the "typical" leaf of $\mathcal{F}$ – as almost every leaf of $\mathcal{F}$ is diffeomorphic to L. The explicit construction of the homeomorphisms between the fibers of (13) as the composition of flows on the compact manifold P implies that the fibration transition functions are coarse isometries on fibers, so the typical leaf also has a well-defined coarse isometry type. By the identification (13) and an application of the Proposition 4.10 we obtain:

PROPOSITION 4.14 *Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold M, with typical leaf L. Then the foliation corona of $\mathcal{F}$ fibers*

$$\partial_h L \longrightarrow \partial_h \mathcal{F} \longrightarrow M \qquad \square$$

The other coronas $\partial_\tau \mathcal{F}$ for $\tau > 0$ constructed above also fiber in this way over the base M.

We conclude this discussion of examples with a class of foliations for which there is a canonically associated separable corona (X, q) for $\mathcal{F}$ where X is again a manifold of dimension $2p + q - 1$

PROPOSITION 4.15 *Let $\mathcal{F}$ be a C^2-foliation of a compact manifold M such that the holonomy cover of each leaf is simply connected. Assume there is a Riemannian metric on the tangential distribution to $\mathcal{F}$ so that each leaf has* non-positive *sectional curvatures. Then there exists a corona $\partial\mathcal{F}$ which fibers $\pi\colon \partial\mathcal{F} \to M$, where the fiber $\pi^{-1}(x) \cong S^{p-1}$ is identified with the "sphere at infinity" on the holonomy cover $\tilde{L}_x$.*

Proof. Let $T\mathcal{F} \to M$ be the tangential distribution to the leaves of $\mathcal{F}$. The metric assumption implies that the *leaf exponential map* $\exp_{\mathcal{F}}\colon T\mathcal{F} \to M \times M$ is a covering map onto each leaf. (The leaf exponential is defined by considering M with a new topology in which each leaf is an open connected component, hence the exponential spray stays inside each leaf. cf. [34, 61].) Thus, we obtain a diffeomorphism $\exp_{\mathcal{F}}\colon T\mathcal{F} \cong \mathcal{G}_{\mathcal{F}}$. Let $\overline{T\mathcal{F}}^g = T\mathcal{F} \cup \partial\mathcal{F}$ be the compactification of $T\mathcal{F}$ obtained by adding on the sphere at infinity in each fiber. Then $\exp_{\mathcal{F}}^{-1}$ extends to a continuous map of the compactifications (by the same reasoning as for Proposition 3.5)

$$\exp_{\mathcal{F}}^{-1}\colon \overline{\mathcal{G}_{\mathcal{F}}} \longrightarrow \overline{T\mathcal{F}}^g$$

which restricts to a fiber-preserving surjective map $\partial_h \mathcal{F} \to \partial\mathcal{F}$. $\square$

The compactification in Proposition 4.15 is called the geodesic compactification, as the ideal points added at infinity correspond exactly to the equivalence classes of positive geodesic rays which converge at infinity.

The following example is a generalization and combination of the two previous examples. A foliation $\mathcal{F}$ is said to be *locally symmetric* [65, 66] if there exists a connected semi-simple Lie group G and a continuous action $\varphi: G \times M \to M$ such that the isotropy subgroup $K_x \subset G$ at $x \in M$ is maximal compact and the subgroups are continuously parametrized by $x \in M$. The *Furstenberg boundary* $\partial_f G$ of the Lie group G is the quotient G-space G/H where H is a minimal parabolic (Borel) subgroup. If G has split real rank k, then $\partial_f G$ is also described as the the equivalence classes of k-flats in G/K, where K is a some maximal compact subgroup [2]. For real rank one, this recovers the sphere boundary compactification of G/K above. Define the the *Furstenberg boundary* $\partial_f \mathcal{F}$ of a locally symmetric foliation $\mathcal{F}$ to be the space of equivalence classes of k-flats in the leaves. Alternately, this boundary is described as a field over M of quotient spaces G/P_x where $x \mapsto P_x$ is a continuous family of minimal parabolic subgroups of G (cf. Zimmer [66].)

PROPOSITION 4.16 *Let $\mathcal{F}$ be a locally symmetric foliation of a compact manifold M. Then the* Furstenberg boundary *$\partial_f \mathcal{F}$ is a corona for $\mathcal{F}$.* $\square$

5 Exotic index of leafwise geometric operators

In this section we begin the study of the exotic index of a leafwise geometric operator. The exotic index is defined as a homomorphic image of the Connes index class in $K(C_r^*(\mathcal{F}))$, and represents a "coarsening" of the Connes index. For example, if all the leaves of $\mathcal{F}$ are compact, the exotic index is always zero – coarsening destroys all of the index data. On the other hand, if all leaves are contractible then conjecturally no information is lost under coarsening. There are two advantages to the exotic foliation index which justify its introduction: one, it has additional naturality, which is used to pair it with classes from the K-theory of the corona in the next section. Secondly, the exotic foliation index vanishes if there exists a uniform gap about 0 in the spectrum of the leafwise operators, reflecting its "coarse" nature. This "gap" property is an important source of relations between geometry and index for leafwise operators, via the Lichnerowicz formalism [24, 25, 56, 57, 39].

The construction of the exotic foliation index requires that $\mathcal{G}_{\mathcal{F}}$ be Hausdorff, which implies the map $s: \mathcal{G}_{\mathcal{F}} \to M$ is a local fibration and ensures that the field of fiberwise index classes constructed from the leafwise geometric operator is "locally continuous" over the base M. When $\mathcal{G}_{\mathcal{F}}$ is non-Hausdorff, the exotic foliation index can be defined over closed subsets $Z \subset M^h = M \setminus s(\mathcal{G}_{\mathcal{F}}^{nh})$ contained in the union of the Hausdorff leaves M^h. The modifications necessary for this case will be discussed at the end of this section.

We assume that $\mathcal{F}$ has a leafwise Haar system $dv_{\mathcal{F}}$ and a quasi-invariant transverse measure (cf. Renault [50], Connes [8, 9]) which combine to yield a measure dv_M on M. The fibers of $s : \mathcal{G}_\mathcal{F} \to M$ are canonically locally homeomorphic to the leaves of $\mathcal{F}$ so the leafwise Haar system $dv_\mathcal{F}$ defines a fiberwise Haar system for $\mathcal{G}_\mathcal{F}$. Let $dv_\mathcal{G} = dv_\mathcal{F} \times s^* dv_M$ denote the product measure on the groupoid $\mathcal{G}_\mathcal{F}$. When $\mathcal{F}$ a C^2-foliation of a compact manifold M without boundary, this construction can also be done using smooth volume forms: Fix a Riemannian metric on TM, which induces a Riemannian metric on $T\mathcal{G}_\mathcal{F}$. (The metric defines an orthogonal complement $Q \subset TM$ to $T\mathcal{F}$ and thus isometrically decomposes $T\mathcal{G}_\mathcal{F} \cong Ts \oplus TM$ where Ts is the bundle of tangents to the fibers of s. There is a local isomorphism between Ts and $T\mathcal{F}$ which we use to copy the metric from the latter to the former.) Let dv_M denote the smooth Riemannian volume form on M and $dv_\mathcal{G}$ the smooth Riemannian volume form on $\mathcal{G}_\mathcal{F}$.

Let $\{\mathcal{H}_x = \mathrm{L}^2(\tilde{L}_x, dv_{L_x}) \mid x \in M\}$ denote the Borel field of fiberwise Hilbert spaces over M. A section σ of this field can be identified with a Borel map $\sigma\colon \mathcal{G}_\mathcal{F} \to \mathbf{C}$ whose restriction to a fiber $s^{-1}(x) = \tilde{L}_x$ is L^2 with respect to the fiber measure. Note that $\sigma(x) \in \mathrm{L}^2(\tilde{L}_x, dv_{L_x})$ is well-defined for all $x \in M$, but we do not require that σ have finite L^2-norm on $\mathcal{G}_\mathcal{F}$. Contrast this with the construction of the foliation von Neumann algebra for $\mathcal{F}$, which is represented on $L^2(\mathcal{G}_\mathcal{F})$.

Continuity for sections can be defined on the set of Hausdorff points $\mathcal{G}^h_\mathcal{F}$: given $[\gamma] \in \mathcal{G}^h_\mathcal{F}$ and a compact set $K \subset \tilde{L}_x$ with $[\gamma]$ contained in its interior, let $W \subset \mathcal{G}_\mathcal{F}$ be a local fibered product (cf. section 2) over an open set $U \subset M$. An element $\psi \in \mathrm{L}^2(W)$ is *continuous* if it decomposes into a family $\{\psi_x \mid x \in U\}$ so that the assignment $x \mapsto \psi_x$ is continuous from U to $L^2(W_x)$, and its image in $\{\mathcal{H}_x \mid x \in M\}$ is called a basic continuous section ψ. In general, σ is continuous on $\mathcal{G}^h_\mathcal{F}$ if the restriction $\sigma|\mathcal{G}^h_\mathcal{F}$ can be written as a locally-finite sum of basic continuous sections.

A bounded operator A on $\mathcal{H}_\mathcal{F}$ is *fiberwise* if there is a direct integral decomposition

$$A \quad \cong \quad \int_M^{\oplus} \quad A_x \, dv_M(x)$$

where each A_x is a bounded linear operator on $\mathcal{H}_x$. Let $\mathcal{B}(\mathcal{H}_\mathcal{F})$ denote the algebra of operators on $\mathcal{H}_\mathcal{F}$ which are fiberwise, have uniformly bounded propagation and are continuous on $\mathcal{G}_\mathcal{F}$. That is, we require:

1. For each $x \in M$ the operator A_x has bounded propagation (cf. Roe, Chapter 4 [56]), by a constant $\rho(A) > 0$ independent of x.

2. The assignment $x \mapsto A_x$ is continuous on M^h for the weak operator topology on the fibers. That is, for $x \in M$ and a basic continuous function ψ defined on an open set W which projects to an open neighborhood $x \in U \subset M^h$, then $x \mapsto A_x\psi_x$ is a basic continuous function in a smaller neighborhood $V \subset U$ of x. (Note that this makes sense because of uniform bounded propagation, hence each $A_y\psi_y$ for $y \in U$ has support in a compact subset of $\tilde{L}_y$.)

We define special subalgebras $\mathcal{K}_\mathcal{F} \subset \mathcal{R}_\mathcal{F} \subset \mathcal{B}(\mathcal{H}_\mathcal{F})$:

The *uniform Roe algebra* $\mathcal{R}_{\mathcal{F}}$ for $\mathcal{F}$ consists of the operators $A \in \mathcal{B}(\mathcal{H}_{\mathcal{F}})$ which are uniformly leafwise locally-traceable:

3. For each $x \in M$ the operator A_x is locally-traceable (cf. [8]; Chapter 4, [56]), and there exists a uniform upper bound $T(A) > 0$ for the trace-norms of the compressions $A_x|\mathcal{P}_\alpha(z)$ to the plaques of a regular foliation atlas.

$\mathcal{K}_{\mathcal{F}}$ consists of the operators $A \in \mathcal{B}(\mathcal{H}_{\mathcal{F}})$ which are uniformly leafwise compact:

4. For each $x \in M$ the bounded linear operator A_x on $\mathrm{L}^2(\tilde{L}_x)$ is compact. (Continuity of the family of operators $\{A_x \mid x \in M\} \subset \mathcal{B}(\mathcal{H}_{\mathcal{F}})$ implies it is uniformly compact.)

The *Roe algebra* $\mathcal{R}^*_{\mathcal{F}}$ for $\mathcal{F}$ is the C^*-closure of $\mathcal{R}_{\mathcal{F}}$ in the operator norm, and the corresponding closures of the operator algebras $\mathcal{K}_{\mathcal{F}}$ and $\mathcal{B}(\mathcal{H}_{\mathcal{F}})$ are denoted respectively by $\mathcal{K}^*_{\mathcal{F}}$ and $\mathcal{B}^*(\mathcal{H}_{\mathcal{F}})$. Analogous to the complete open manifold case [56], we introduce the terminology

DEFINITION 5.1 *The operator K-groups $K_*(\mathcal{R}^*_{\mathcal{F}})$ are called the* coarse K-theory groups *of $\mathcal{F}$.*

Let $C^*_r(\mathcal{F})$ denote the reduced C^*-algebra associated to the foliation $\mathcal{F}$ with its given leafwise Haar system $dv_{\mathcal{F}}$ (cf. [8, 9, 50].) The next lemma establishes the "coarsening" map for foliation K-theory.

LEMMA 5.2 *There is a natural inclusion $C^*_r(\mathcal{F}) \subset \mathcal{R}^*_{\mathcal{F}}$ of C^*-algebras, which induces*

$$C\colon K_*(C^*_r(\mathcal{F})) \to K_*(\mathcal{R}^*_{\mathcal{F}}) \tag{14}$$

Proof. The choice of the leafwise Haar system for $\mathcal{F}$ endows the space of continuous functions with compact support $C_c(\mathcal{G}_{\mathcal{F}})$ with a convolution product. Fix a leaf $L_x \subset M$ with holonomy cover $\tilde{L}_x$. Each function $f\colon \mathcal{G}_{\mathcal{F}} \to \mathbf{C}$ defines a convolution operator on $L^2(\tilde{L}_x)$ and the correspondence between functions and kernel operators defines a *-representation $\rho_x\colon C_c(\mathcal{G}_{\mathcal{F}}) \to \mathcal{B}(L^2(\tilde{L}_x))$. The direct integral over M of all these representations yields a *-representation $\rho\colon C_c(\mathcal{G}_{\mathcal{F}}) \to \mathcal{R}_{\mathcal{F}}$. The reduced C^*-norm $||\cdot||^*$ on $C_c(\mathcal{G}_{\mathcal{F}})$ is defined to be the supremum over $x \in M$ of the semi-norms induced from the representations $\{\rho_x \mid x \in M\}$, so it is tautological that ρ induces a map of the C^*-closures. $\square$

DEFINITION 5.3 *Let $\mathcal{D}_{\mathcal{F}}$ be a leafwise-elliptic, pseudo-differential operator for $\mathcal{F}$ with self-adjoint symbol.*

- *If ϵ is a self-adjoint grading for $\mathcal{D}_{\mathcal{F}}$, and $Ind(\mathcal{D}_{\mathcal{F}}, \epsilon) \in K_0(C^*(\mathcal{F}))$ is the Connes foliation index class [9, 17], then the* even exotic index *is the image under the coarsening map (14)*
$$Ind_e(\mathcal{D}_{\mathcal{F}}, \epsilon) = C\,(Ind(\mathcal{D}_{\mathcal{F}}, \epsilon)) \in K_0(\mathcal{R}^*_{\mathcal{F}})$$

- *For the ungraded case, with $Ind(\mathcal{D}_\mathcal{F}) \in K_1(C^*(\mathcal{F}))$ the Toeplitz foliation index class [17, 19], then the* odd exotic index *is the image under the coarsening map (14)*

$$Ind_e(\mathcal{D}_\mathcal{F}) = C\left(Ind(\mathcal{D}_\mathcal{F})\right) \in K_1(\mathcal{R}^*_\mathcal{F})$$

The exotic foliation index has an intuitive formulation for even-dimensional leaves when 0 is uniformly isolated in the spectrum of each fiberwise operator $\mathcal{D}_x$. Let $\mathcal{D}_\mathcal{F}$ be a leafwise-elliptic, pseudo-differential operator $\mathcal{D}_\mathcal{F}$ of degree 1, with coefficients in an Hermitian bundle $\mathbf{E} \to M$, so that for each $x \in M$ the leafwise operator $\mathcal{D}_x$ on the compactly supported smooth sections $C^\infty_c(\mathbf{E}_{\tilde{L}_x} \to \tilde{L}_x)$ is an essentially self-adjoint densely-defined unbounded operator. Form the projection operator $\Pi_x \colon L^2(\mathbf{E}_{\tilde{L}_x}) \to L^2(\mathbf{E}_{\tilde{L}_x})$ onto the kernel of $\mathcal{D}_x$. The spectral assumption implies that the family of projections $\Pi^\pm_x$ are continuous as a function of x (cf. [51].) The grading ϵ anti-commutes with $\mathcal{D}_x$ so leaves its kernel invariant. Introduce the associated projections $\Pi^\pm_x$ onto the $\pm$ eigenspaces of $\epsilon|\ker(\mathcal{D}_\mathrm{x})$. The abstract exotic index $Ind_e(\mathcal{D}_\mathcal{F}, \epsilon) \in K_0(\mathcal{R}^*_\mathcal{F})$ is the formal difference of the projections, $[\Pi^+_x] - [\Pi^-_x]$, as an element of a Grothendieck group of "locally-finite" Hermitian subbundles of the field of Hilbert spaces bundles $x \mapsto \mathcal{H}_x$ over M. In this way, the exotic index is a natural generalization of the Gromov-Lawson index of a family of "locally Fredholm" operators on an open manifold [24, 25].

The technical difficulty with the above intuitive approach is that there is no reason why 0 should be isolated in the spectrum of an elliptic operator on an open manifold. For odd dimensional leaves, the exotic index vanishes if there is a gap in the spectrum so the above approach is useless. This forces one to define the foliation index using "almost-projection" operators, interpreted as an index class via the K-theory formalism (cf. [9, 17, 29, 30, 31, 42]).

Fix a smooth function $\chi \colon \mathbf{R} \to [-1, 1]$ so that $\chi(-x) = -x$ and $\chi(x) = 1$ for $x \geq 1$ (called a "chopping function" by Roe [56].)

When the leaves have even dimension, and $\mathcal{D}_\mathcal{F}$ anti-commutes with the grading operator ϵ, the exotic K_0-index is defined as follows (cf. section 4, [56]). Choose a constant $\delta > 0$, and use the functional calculus to define the fiberwise bounded operator $F \colon \mathrm{L}^2(\mathcal{G}_\mathcal{F}, dv_\mathcal{G}) \otimes \mathbf{E} \to \mathrm{L}^2(\mathcal{G}_\mathcal{F}, dv_\mathcal{G}) \otimes \mathbf{E}$, which on the fiber over $x \in M$ is given by

$$F_x = \chi(\mathcal{D}_x/\delta)$$

Estimates using the wave operator and finite propagation speed technique imply that $F \in \mathcal{B}(\mathcal{H}_\mathcal{F})$, and $F^2 - \mathrm{Id} \in \mathcal{R}^*_\mathcal{F}$. The "$2 \times 2$" trick yields an exact involution $G \in \mathcal{B}(\mathcal{H}_\mathcal{F}) \otimes \mathbf{C}^2$ so that $G - F \in \mathcal{B}(\mathcal{H}_\mathcal{F})$. We then define $Ind_e(\mathcal{D}_\mathcal{F}, \epsilon)$ to be the equivalence class of the formal difference

$$\left[G\left(\frac{1+\epsilon}{2}\right)G\right] - \left[\frac{1-\epsilon}{2}\right] \in K_0(\mathcal{R}^*_\mathcal{F}) = \ker\left\{K_0(\widetilde{\mathcal{R}^*_\mathcal{F}}) \to K_0(\mathbf{C}) = \mathbf{Z}\right\} \tag{15}$$

which is independent of the choice $\delta > 0$.

When the leaves have odd dimension, the exotic K_1-index is defined as follows. Choose a continuous function $\lambda: M \to \mathbf{R}$, and a constant $\delta > 0$. Define the fiberwise unitary operator $U: \mathrm{L}^2(\mathcal{G}_{\mathcal{F}}, dv_{\mathcal{G}}) \otimes \mathbf{E} \to \mathrm{L}^2(\mathcal{G}_{\mathcal{F}}, dv_{\mathcal{G}}) \otimes \mathbf{E}$, which on the fiber over $x \in M$ is given by

$$U_x = \exp\left\{\pi\sqrt{-1}\ \chi((\mathcal{D}_x - \lambda(x))/\delta)\right\}$$

Estimates using the wave operator and finite propagation speed technique imply that the $U + \mathrm{Id} \in \mathcal{R}^*_{\mathcal{F}}$ and $Ind_e(\mathcal{D}_{\mathcal{F}})$ is defined to be the equivalence class

$$[U] \in K_1(\mathcal{R}^*_{\mathcal{F}}) = \ker\left\{K_1(\widetilde{\mathcal{R}^*_{\mathcal{F}}}) \to K_1(\mathbf{C}) = 0\right\} \tag{16}$$

which is independent of the choice of functions χ and λ, and of $\delta > 0$. Note that $U_x\psi_x = -\psi_x$ whenever $\psi_x \in \mathrm{L}^2(\mathcal{G}_{\mathcal{F}}, dv_{\mathcal{G}}) \otimes \mathbf{E}$ lies in the range of either of the spectral projections $\chi_{[\lambda(x)+\delta,\infty)}(\mathcal{D}_x)$ or $\chi_{(-\infty,\lambda(x)-\delta]}(\mathcal{D}_x)$, hence the operator U_x depends on $\mathcal{D}_x$ only for functions in the range of the spectral projection $\chi_{[\lambda(x)-\delta,\lambda(x)+\delta]}(\mathcal{D}_x)$.

Recall that the spectrum $\sigma(\mathcal{D}_x) \subset \mathbf{R}$ can *a priori* be any closed subset of $\mathbf{R}$. A point $\lambda \in \sigma(\mathcal{D}_x)$ which is isolated must correspond to an eigensection in $L^2(\mathbf{E}_{\tilde{L}_x}) \cap C^\infty(\mathbf{E}_{\tilde{L}_x})$. In general, though, a cluster point $\lambda \in \sigma(\mathcal{D}_x)$ need not coincide with an eigensection, but rather to a sequence of "approximate eigensections" which eventually vanish on compact sets in $\tilde{L}_x$. This suggests there should be a relation between the topology of $\sigma(\mathcal{D}_x)$, especially its derived set, and the coarse geometry of the holonomy covers $\tilde{L}_x$. Roe observed (Proposition 5.21 [56]) that the existence of a gap in the spectrum implies the vanishing of the exotic index for open complete manifolds. Roe's proofs carry over verbatim to the case of foliations:

DEFINITION 5.4 *We say that the spectrum of $\mathcal{D}_{\mathcal{F}}$ has a* uniform gap *about $\lambda \in \mathbf{R}$ if there exists $\delta > 0$ such that, for each $x \in M$, the intersection $\sigma(\mathcal{D}_x) \cap (\lambda - \delta, \lambda + \delta)$ is empty for all $x \in M$.*

PROPOSITION 5.5 *Let $\mathcal{D}_{\mathcal{F}}$ be a leafwise geometric operator for $\mathcal{F}$ with coefficients in an Hermitian bundle $\mathbf{E} \to M$.*

1. *Suppose that $\mathcal{D}_{\mathcal{F}}$ has uniform gap about 0. Then for any self-adjoint grading ϵ for $\mathcal{D}_{\mathcal{F}}$, the even degree exotic index $Ind_e(\mathcal{D}_{\mathcal{F}}, \epsilon) \in K_0(\mathcal{R}^*_{\mathcal{F}})$ vanishes.*
2. *If there exists $\lambda \in \mathbf{R}$ such that $\mathcal{D}_{\mathcal{F}}$ has a* uniform gap *about λ, then the odd degree exotic index $Ind_e(\mathcal{D}_{\mathcal{F}}) \in K_1(\mathcal{R}^*_{\mathcal{F}})$ vanishes.*

Proof: Assume that $\mathcal{D}_{\mathcal{F}}$ anti-commutes with a grading ϵ. Let F be the fiberwise operator constructed from $\mathcal{D}_{\mathcal{F}}$ as above, with $\delta > 0$ chosen so that $(-\delta, \delta) \cap \sigma(\mathcal{D}_x)$ is empty for each

$x \in M$. Then we can choose $G = F$ for the involution used to define the index class, hence $G\epsilon + \epsilon G = 0$ and the left-hand-side of (15) vanishes.

For the ungraded case, the operator $\chi((\mathcal{D}_x - \lambda(x))/\delta)$ has spectrum contained in the set $\{+1, -1\}$ for each $x \in M$, hence $U = -\mathrm{Id}$ and its class $[U] = 0$. $\square$

We mention an open problem connected related to extending Proposition 5.5.1:

PROBLEM 5.6 *Determine the image of the natural map* $K_*(\mathcal{K}^*_{\mathcal{F}}) \to K_*(\mathcal{R}^*_{\mathcal{F}})$.

The inclusion of the compact operators into the Roe algebra on a non-compact complete manifold space induces the trivial map in K-theory (cf. comments at the end of section 4, [56].) It is possible that a similar conclusion holds for the fiberwise inclusion of the compact operators into the uniform operators on the holonomy covers $\tilde{L}_x$ over M, when every fiber $\tilde{L}_x$ is non-compact. *A priori*, it is necessary to deal with the possibility of spectral flow with respect to the parameter $x \in M$, which could result in non-zero classes in the image of $K_*(\mathcal{K}^*_{\mathcal{F}}) \to K_*(\mathcal{R}^*_{\mathcal{F}})$. The point is to show these exotic spectral flow invariants are zero.

Proposition 5.5 has the customary application to the existence of metrics of positive scalar curvature (cf. Rosenberg [57]; Zimmer [64]; section 6C of Roe [56]):

COROLLARY 5.7 *Let $\mathcal{F}$ be a C^∞-foliation with even dimensional leaves of a compact manifold M, and assume the tangential distribution $T\mathcal{F}$ admits a spin structure. If there exists a Riemannian metric on $T\mathcal{F}$ so that each leaf of $\mathcal{F}$ has positive scalar curvature, then the exotic index class $Ind_e(\not{D}, \epsilon) \in K_0(\mathcal{R}_{\mathcal{F}})$ of the leafwise Dirac operator vanishes. For a foliation with odd dimensional leaves, the corresponding statement holds for the odd exotic index class.* $\square$

One of the principal applications of exotic index theory is to prove the homotopy invariance of characteristic classes associated to a foliated manifold (cf. [4].) Underlying these applications, discussed in the next sections, is Theorem 5.2 of Hilsum and Skandalis [32], which extends to foliations the results of Miscenko [45] on the homotopy invariance of the C^*-signature of a compact manifold: for a C^2-foliation of a compact manifold M the index of the leafwise signature operator, $Ind(d_{\mathcal{F}} * - * d_{\mathcal{F}}) \in K_{(m)}(C^*(\mathcal{F}))$ is a homotopy invariant of the foliation, where (m) denotes the parity of the leaf dimension. Actually, they show much more, that the higher signatures associated to almost flat bundles are homotopy invariants, which combined with Lemma 5.2 yields:

THEOREM 5.8 *Let $\mathcal{F}$ be a C^2-foliation of a compact manifold M with leaves of dimension m, and $\mathbf{E} \to M$ an almost flat bundle for $\mathcal{F}$. Then the exotic index of the signature operator with $\mathbf{E}$-coefficients,*

$$Ind_e\left((d_{\mathcal{F}} * - * d_{\mathcal{F}}) \otimes \mathbf{E}\right) \in K_{(m)}(\mathcal{R}^*_{\mathcal{F}})$$

is a homotopy invariant. $\square$

The above constructions have versions for the case when $\mathcal{G}_{\mathcal{F}}$ is non-Hausdorff, which are useful for studying the index of foliations with respect to their saturated Borel structure (cf. [27]). Let $Z \subset M^h = M \setminus s(\mathcal{G}_{\mathcal{F}}^{nh})$ be a closed subset of the Hausdorff leaves of $\mathcal{F}$. For each $x \in Z$ the holonomy cover $\tilde{L}_x$ has the local product neighborhood property for each compact subset $K \subset \tilde{L}_x$; that is, there exists a locally open subset $U \cap Z \subset Z$ and a relatively open neighborhood $K \subset W \subset \mathcal{G}_{\mathcal{F}}^{nh}$ which is a fibered over $U \cap Z$. The Borel field $\{\mathcal{H}_x = \mathrm{L}^2(\tilde{L}_x, dv_{L_x}) \mid x \in M\}$ restricts to a Borel field of Hilbert spaces $\{\mathcal{H}_x \mid x \in Z\}$. Then exactly as before, we introduce basic continuous sections in $\{\mathcal{H}_x \mid x \in Z\}$, and continuous sections over Z. The restricted Roe algebra $\mathcal{R}^*_{\mathcal{F}|Z}$ can be defined as operators on the field $\{\mathcal{H}_x \mid x \in Z\}$. The C^*-algebra $C^*_r(\mathcal{F})$ represents on the restricted field, so defines a restricted "Z-coarsening" map for foliation K-theory:

$$C_Z \colon K_*(C^*_r(\mathcal{F})) \to K_*(\mathcal{R}^*_{\mathcal{F}|Z}) \tag{17}$$

DEFINITION 5.9 *Let $\mathcal{D}_{\mathcal{F}}$ be a leafwise-elliptic, pseudo-differential operator for $\mathcal{F}$ with self-adjoint symbol. Let $Z \subset M^h = M \setminus s(\mathcal{G}_{\mathcal{F}}^{nh})$ be a closed subset of the Hausdorff leaves of $\mathcal{F}$.*

- *For the graded case, the Z-*even exotic index* is $Ind_e(\mathcal{D}_{\mathcal{F}}|Z, \epsilon) = C_Z\left(Ind(\mathcal{D}_{\mathcal{F}}, \epsilon)\right) \in K_0(\mathcal{R}^*_{\mathcal{F}|Z})$*
- *For the ungraded case, the Z-*odd exotic index* is $Ind_e(\mathcal{D}_{\mathcal{F}}|Z) = C_Z\left(Ind(\mathcal{D}_{\mathcal{F}})\right) \in K_1(\mathcal{R}^*_{\mathcal{F}|Z})$*

6 Exotic foliation index theorems

The corona of a metric space X is the boundary for a compactification $\overline{X}$, hence for each cohomology theory there is a transgression map from boundary classes to classes compactly supported in the interior of X. Higson and Roe showed that the transgressed K-theory classes from the corona play the rôle of generalized "dual-Dirac" classes on X, and pair naturally with the K-theoretic exotic index of a geometric operator [31, 53, 56]. For the foliation corona, this boundary is "fibered" over the parameter space M, which leads to the construction of generalized dual Dirac classes in Kasparov bivariant-KK-theory. In this section, we will establish the properties of these generalized dual Dirac classes, then prove an index theorem for calculating their pairing with the exotic index.

THEOREM 6.1 *Let $\mathcal{F}$ be a C^2-foliation of a compact manifold M. Then there is a natural map*

$$\rho \colon K^*(\partial_h \mathcal{F}) \longrightarrow KK_{*+1}(C^*_r(\mathcal{F}), C(M)) \tag{18}$$

whose image consists of generalized dual-Dirac classes.

The construction of the map ρ in (18) will be based on a modification, due to Guoliang Yu [63], of Higson's construction in [31] of a pairing between the K-theory of the corona and that of the Roe algebra for open complete manifolds. Before proving the theorem, we point out the main application, which follows from composing the boundary map (18) with the coarsening map (14), and using the natural pairing of KK-groups:

COROLLARY 6.2 *Let $k, \ell = 0, 1$ be fixed. Then for each $[u] \in K^{\ell}(\partial_h \mathcal{F})$ there is an exotic index map*

$$\rho[u]: K_k\left(C^*(\mathcal{F})\right) \longrightarrow K^{k+\ell+1}(M) \tag{19}$$

The exotic index $\rho[u](Ind(\mathcal{D}_{\mathcal{F}}, \epsilon)) \in K^*(M)$ is an "integral" invariant of $\mathcal{D}_{\mathcal{F}}$. This contrasts with the real-valued measured index of a leafwise operator for a foliation admitting a holonomy-invariant transverse measure, which is typically a renormalized index with values in $\mathbf{R}$.

Proof of (6.1). Consider first the case $\ell = 1$. Given a class $[u] \in K^1(\partial_h \mathcal{F})$, it is represented by a continuous map $u : \partial_h \mathcal{F} \to U(N)$ for some $N > 0$. The key idea is to mimic the construction of the boundary map in the exact sequence

$$\cdots \longrightarrow K^1(\overline{\mathcal{G}_{\mathcal{F}}}, \partial_h \mathcal{F}) \longrightarrow K^1(\partial_h \mathcal{F}) \xrightarrow{\delta} K^0(\mathcal{G}(\mathcal{F})) \longrightarrow \cdots \tag{20}$$

The unitary u is extended to $\overline{\mathcal{G}_{\mathcal{F}}}$ then its K-theory boundary is a *special Hermitian bundle* on $\mathcal{G}_{\mathcal{F}}$ in the sense of Yu (section 4, [63]). A special Hermitian bundle determines a class in $KK_0(\mathcal{R}^*_{\mathcal{F}}, C(M))$ which then restricts to a class $\rho[u] \in KK_0(C^*_r(\mathcal{F}), C(M))$.

The first step is to extend u to $\hat{u}: \overline{\mathcal{G}_{\mathcal{F}}} \to \mathrm{End}(\mathbf{C}^{\mathrm{N}})$. Let $j : U(N) \subset GL(N, \mathbf{R}) \subset \mathbf{R}^{N^2}$ be the embedding obtained by the standard coordinates on matrices. By the Tietze extension theorem, $j \circ u$ extends to a continuous map $\tilde{u} : \overline{\mathcal{G}_{\mathcal{F}}} \to \mathbf{R}^{N^2}$. Let $\nu \to U(N)$ denote the normal bundle to j with Riemannian metric pulled-back from $\mathbf{R}^{N^2}$. For $\epsilon > 0$ sufficiently small, the exponential map $\exp: \nu \to \mathbf{R}^{N^2}$ defined over j is an embedding when restricted to the ϵ-disc neighborhood of zero $D_\epsilon(j) \subset \nu$. The image $N_\epsilon(j) = \exp(D_\epsilon(j))$ is an open neighborhood retract of $j(U(N)) \subset \mathbf{R}^{N^2}$ equipped with a smooth fibration map $\pi : N_\epsilon(j) \to j(U(N))$. Choose a smooth function $s : [0, \epsilon) \to [0, 1]$ so that $s(t) = 1$ for $0 \leq t \leq \epsilon/3$ and $s(t) = 0$ for $t \geq 2\epsilon/3$. Compose $s(r)$ with the (normal) radial distance function on $N_\epsilon(j)$, and extend as the zero function, to obtain a compactly supported smooth function $\hat{s} : \mathbf{R}^{N^2} \to [0, 1]$ with

$$\begin{array}{lll} \hat{s}(x) = 1 & \text{for} & x \in \ \exp(D_{\epsilon/3}(j)) \\ \hat{s}(x) = 0 & \text{for} & x \notin \exp(D_{2\epsilon/3}(j)). \end{array}$$

Extend the inclusion $U(N) \subset \mathrm{End}(\mathbf{C}^{\mathrm{N}})$ to a compactly supported map $\hat{i} : \mathbf{R}^{N^2} \to \mathrm{End}(\mathrm{N}, \mathbf{C})$. Pull the inclusion back via the fibration π and multiply with the function $\hat{s}$ to obtain an extension $\hat{i}$ on the open subset $N_\epsilon(j) \subset \mathbf{R}^{N^2}$ which is the zero endomorphism on the boundary. Extend $\hat{i}$ as the zero endomorphism on the complement of $N_\epsilon(j)$.

Let $\hat{u} = \hat{\imath} \circ \tilde{u} : \overline{\mathcal{G}_{\mathcal{F}}} \to \mathrm{End}(\mathbf{C}^{\mathrm{N}})$ be the continuous function which is the composition of the Tietze extension function $\tilde{u}$ with the Whitney extension function $\hat{\imath}$. Note that $\hat{u}(x)$ is a unitary matrix for x in the open neighborhood $\tilde{u}^{-1}(\exp(D_{\epsilon/3}(j))) \subset \overline{\mathcal{G}_{\mathcal{F}}}$ of the foliation corona $\partial_h \mathcal{F}$.

Let $\mathcal{A}, \mathcal{B}$ denote C^*-algebras. Recall that a cycle $(\mathcal{E}, \Phi, \phi)$ for the Kasparov group $KK(\mathcal{A}, \mathcal{B})$ consists of a separable Hilbert space $\mathcal{E}$ equipped with a C^*-module action of $\mathcal{B}$, a representation ϕ of $\mathcal{A}$ on the $\mathcal{B}$-module $\mathcal{E}$ and an adjoinable operator $\Phi \colon \mathcal{E} \to \mathcal{E}$ such that for every $a \in \mathcal{A}$ the expressions $(\phi(a)\Phi - \Phi\phi(a))$, $\phi(a)(\Phi - \Phi^*)$ and $\phi(a)(\Phi^2 - 1)$ are compact Hilbert $\mathcal{B}$-module operators. The equivalence relation between cycles is generated by operator homotopy and addition of degenerate bimodules (cf. [42]; section 1 [31]; Definition 3.29 [30].)

Recall the construction of a Kasparov $(\mathbf{C}, C_0(\mathcal{F}))$-bimodule from $\hat{u}$. Form Hermitian vector bundles $\mathbf{E}_i = \overline{\mathcal{G}_{\mathcal{F}}} \times \mathbf{C}^N$ for $i = 0, 1$, and let $\mathcal{E}_i$ be the Hilbert-space closure of the space of compactly supported continuous sections of $\mathbf{E}_i$. Let $\mathbf{C}$ act on each space $\mathcal{E}_i$ via the natural extension of the identity map. There is a natural module action ϕ_i of $C_0(\mathcal{G}_{\mathcal{F}}) = C_0(\mathcal{F})$ on $\mathcal{E}_i$. The matrix-valued function $\hat{u}$ induces a map of bundles $F \colon \mathcal{E}_0 \to \mathcal{E}_1$ which is an Hermitian isomorphism outside of a compact set in $\mathcal{G}_{\mathcal{F}}$. The K-theory boundary $\delta[u] \in KK(\mathbf{C}, C_0(\mathcal{F}))$ of the sequence (20) is the class of the cycle $(\mathcal{E}_0 \oplus \mathcal{E}_1, \begin{bmatrix} 0 & F^* \\ F & 0 \end{bmatrix}, \phi_0 \oplus \phi_1)$. However, in passing to this K-theory group, we lose the information about the explicit representative for $\delta[u]$ obtained from the corona, in particular that it is represented by a map which has vanishing gradient near infinity. We show next that the data $(\mathcal{E}_0 \oplus \mathcal{E}_1, \begin{bmatrix} 0 & F^* \\ F & 0 \end{bmatrix}, \phi_0 \oplus \phi_1)$ also determines a $(\mathcal{R}^*_{\mathcal{F}}, C(M))$-bimodule, whose KK-class captures this extra information.

The Hilbert space $\mathcal{H}_{\mathcal{F}} \otimes \mathbf{C}^{2N}$ is a $C(M)$-module via the map $s \colon \mathcal{G}_{\mathcal{F}} \to M$. Let ϕ be the representation of $\mathcal{R}^*_{\mathcal{F}}$ on $\mathcal{H}_{\mathcal{F}} \otimes \mathbf{C}^{2N}$ acting as operators extended as the identity on the factor $\mathbf{C}^{2N}$. Note that ϕ is a $C(M)$-representation by the assumption that the operators in $\mathcal{R}^*_{\mathcal{F}}$ are fiberwise, hence commute with the action of $C(M)$ as multipliers on the Hilbert field $\{\mathcal{H}_x \mid x \in M\}$. Let $\Phi = 1 \otimes \begin{bmatrix} 0 & F^* \\ F & 0 \end{bmatrix}$ be the extension of $\hat{u}$ to an operator on

$$\mathcal{H}_{\mathcal{F}} \otimes \mathbf{C}^{2N} \cong \mathcal{H}_{\mathcal{F}} \otimes_{C_h(\mathcal{F})} (\mathcal{E}_0 \oplus \mathcal{E}_1)$$

LEMMA 6.3 (cf. Lemma 3 [63]) *$(\mathcal{H}_{\mathcal{F}} \otimes \mathbf{C}^{2N}, \Phi, \phi)$ is a Kasparov $(\mathcal{R}^*_{\mathcal{F}}, C(M))$-bimodule.*

Proof. We check the three conditions on a Kasparov bi-module for $(\mathcal{R}_{\mathcal{F}}, C(M))$, and then note that the argument is stable under closure in the operator norm, so that the conditions also hold for $(\mathcal{R}^*_{\mathcal{F}}, C(M))$. Note that $B \in \mathcal{B}^*(\mathcal{H}_{\mathcal{F}})$ is a compact Hilbert $C(M)$-module operator precisely when $B \in \mathcal{K}^*_{\mathcal{F}}$. Let $A \in \mathcal{R}_{\mathcal{F}}$. $\Phi^* = \Phi$ by construction, so

$0 = \phi(A)(\Phi - \Phi^*) \in \mathcal{K}^*_{\mathcal{F}}$. Also, the expression

$$(\Phi^2 - 1) = \begin{bmatrix} F^*F - 1 & 0 \\ 0 & FF^* - 1 \end{bmatrix}$$

is compactly supported in $\mathcal{G}_{\mathcal{F}}$ by the choice of $\hat{u}$, so that the composition $\phi(A)(\Phi^2 - 1) \in \mathcal{K}^*_{\mathcal{F}}$.

Finally, the key point is to show that the commutator $(\phi(A)\Phi - \Phi\phi(A)) \in \mathcal{K}_{\mathcal{F}}$. The operator $\phi(A)$ restricts to each leaf fiber $\tilde{L}_x$ of $s: \mathcal{G}_{\mathcal{F}} \to M$ to give a locally compact operator, whose kernel is supported in a uniform tube in $\tilde{L}_x \times \tilde{L}_x$ of radius $\rho(A)$ about the diagonal. The entries of the matrix operator Φ have vanishing gradient, so given $\epsilon > 0$ there is a compact set $K(\epsilon, A) \subset \mathcal{G}_{\mathcal{F}}$ so that the variation of Φ is at most ϵ on any disc of radius $\rho(A)$ contained in a fiber $\tilde{L}_x$ and outside of $K(\epsilon, A)$. This implies that the commutator $(\phi(A)\Phi - \Phi\phi(A)) \in \mathcal{K}_{\mathcal{F}}$ vanishes at infinity, hence is a fiberwise compact operator. The details of this argument are exactly those of Proposition 5.18 of Roe [56], applied to each fiber of $s: \mathcal{G}_{\mathcal{F}} \to M$, so are omitted. We need to note that the fiberwise operators $(\phi(A)\Phi - \Phi\phi(A))_x \in \mathcal{K}(L^2(\tilde{L}_x))$ are continuous in x, because of the uniform estimate on their supports and the local fibration properties of the map $s: \mathcal{G}_{\mathcal{F}} \to M$. □

Define $\rho[u] = [(\mathcal{H}_{\mathcal{F}} \otimes \mathbf{C}^{2N}, \Phi, \phi)] \in KK(\mathcal{R}^*_{\mathcal{F}}, C(M))$. It is straightforward to check that $\rho[u]$ is independent of the choice of N, the representative $u: \partial_h \mathcal{F} \to U(N)$ for $[u]$, and of the extension $\hat{u}$. This completes the construction of the map (18) for the case of $[u] \in K^1(\partial_h^{\mathcal{F}})$.

The case for $[p] \in K^0(\partial_h \mathcal{F})$ proceeds similarly to the odd case, so we just indicate the modifications. Given a representative $p: \partial_h \mathcal{F} \to G(N, n)$ of $[p]$, we choose an embedding of $G(N, n)$ into $\mathbf{R}^\ell$ for ℓ sufficiently large. Each point in $G(N, n)$ corresponds to a self-adjoint projection operator in $\mathbf{C}^{N+n}$ with complex rank n, so that the map p determines a field of self-adjoint projections in the trivial bundle $\partial_h \mathcal{F} \times \mathbf{C}^{N+n}$. By the same extension methods above, we can lift these projections back to a self-adjoint vector bundle endomorphisms $\hat{p}$ on the trivial bundle $\mathcal{G}_{\mathcal{F}} \times \mathbf{C}^{N+n}$ which are projections outside of a compact set in $\mathcal{G}_{\mathcal{F}}$. The exponential $\exp(2\pi\sqrt{-1}\,\hat{p})$ defines a unitary automorphism of $\mathcal{G}_{\mathcal{F}} \times \mathbf{C}^{N+n}$ which is trivial outside of a compact set in $\mathcal{G}_{\mathcal{F}}$. We then set $U = 1 \otimes \exp(2\pi\sqrt{-1}\,\hat{p})$ acting on $\mathcal{H}_{\mathcal{F}} \otimes \mathbf{C}^{N+n}$, and obtain an odd Kasparov $(D^*(C_h(\mathcal{F}), C_0(\mathcal{F})), C(M))$-bimodule $(\mathcal{H}_{\mathcal{F}} \otimes \mathbf{C}^{N+n}, U, \phi)$. Define $\rho[p]$ to be the KK-class of this bimodule. □

The use of the Tietze extension theorem above is comparable with the method of Roe in section 5.3, [56] used to define the pairing between coarse cohomology and the K-theory of uniform algebras.

The construction of abstract boundary maps in K-homology by Higson (cf. Lemma 1.3 [31]) requires the choice of a completely positive section of a restriction map of C^*-algebras. In the above proof, the choice of a map $u : \partial_h \mathcal{F} \to U(N)$ defines a foliation corona $\mathcal{A}_u$ for which the positive extension $\hat{s} \circ \tilde{u}$ has the rôle of a completely positive section.

The constants $0 < \epsilon/3 < 2\epsilon/3 < \epsilon$ used to define the cut-off function $s(r)$ are completely arbitrary. In fact, one could introduce parameters $0 < \lambda < \mu < \epsilon$ and let $s_{\lambda\mu}(r)$ be the corresponding cut-off function. Then we obtain Kasparov $(D^*(C_h(\mathcal{F}), C_0(\mathcal{F})), C(M))$-bimodules whose classes in $KK(\mathcal{R}^*_{\mathcal{F}}, C(M))$ are independent of λ and μ. Letting $\lambda, \mu \to 0$ these bimodules have "compact support" contained in any arbitrary open neighborhood of the corona $\partial_h \mathcal{F}$.

The ordinary K-theory transgression class obtained from u

$$\delta[u] = [(\mathcal{E}_0 \oplus \mathcal{E}_1, \begin{bmatrix} 0 & F^* \\ F & 0 \end{bmatrix}, \phi_0 \oplus \phi_1)] \in KK(\mathbf{C}, C_0(\mathcal{F})) \cong K_0(C_0(\mathcal{F}))$$

is similarly represented by a cycle with compact support, which can be chosen arbitrarily close to infinity. Classes of this type are called *mobile*, following a suggestion of John Roe, as they are not localized to any one region of the manifold.

There is an alternate interpretation of the pairing (19) for the indices of leafwise Dirac operators, in terms of an index theorem for families. Let $\mathcal{D}_{\mathcal{F}}$ be a leafwise Dirac operator on an even dimensional foliation, and $[u] \in K^1(\partial_h \mathcal{F})$ an odd K-theory class from the corona, with transgression $\delta[u] \in K_0(C_0(\mathcal{F}))$. The relative index theorem for open manifolds (cf. [55]) defines a fiberwise pairing between the leafwise operator $\mathcal{D}_{\mathcal{F}}$ and the compactly supported K-theory class $\delta[u]$: For each $x \in M$, the restriction $\mathcal{D}_x$ on the leaf L_x through x is a geometric operator, and the data $\delta[u]$ restricts to the fiber $L_x \subset \mathcal{G}_{\mathcal{F}}$ to yield a compactly supported finite-dimensional bundle $\mathbf{E}_x \to L_x$. The relative index pairing of $\mathcal{D}_x$ with $\mathbf{E}_x$ yields a finite-dimensional vector space for each $x \in M$. The local continuity of the index bundles for leafwise operators implies that this family of vector spaces over M determines a K-theory class

$$Ind(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}} \otimes s^! \mathbf{E}) \in KK(\mathbf{C}, C(M)) \cong K^0(M) \tag{21}$$

The rôle of the boundary K-theory in the construction of $Ind(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}} \otimes s^! \mathbf{E})$ is to choose a mobile, compactly supported K-theory class on the leaves which is then paired with the leafwise operators to yield a total K-theory class on the ambient manifold.

The intuitive discussion above can be made rigorous using the foliation index theorem of Connes and Skandalis, which leads to a topological expression for the exotic foliation index (19). Note that the groupoid $\mathcal{G}_{\mathcal{F}}$ is foliated by the fibers $\tilde{L}_x = s^{-1}(x)$ of $s\colon \mathcal{G}_{\mathcal{F}} \to M$, and a leafwise operator $\mathcal{D}_{\mathcal{F}}$ induces a fiberwise operator denoted $\mathcal{D}_{\mathcal{G}_{\mathcal{F}}}$. The C^*-algebra of this foliation is Morita equivalent to the algebra of functions on the quotient space, $C(M)$. In this situation, the Connes-Skandalis construction [17] yields a KK-index class $Ind(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}}) \in KK_*(C_0(\mathcal{F}), C(M))$: choose a smooth chopping function $f\colon \mathbf{R} \to [-1, 1]$ which is odd, and $f(\xi) = 1$ for $\xi > 1$. Then for each $x \in M$,

$$f(\mathcal{D}_x)\colon C_c^\infty(\mathbf{E}_{\tilde{L}_x}) \to C_c^\infty(\mathbf{E}_{\tilde{L}_x})$$

is a bounded symmetric operator which extends to $L^2(\mathbf{E}_{\tilde{L}_x})$, and so yields a family of operators denoted by $f(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}})$ on $\mathcal{H}_{\mathcal{F}} \otimes s^! \mathbf{E}$. If $\mathcal{D}_{\mathcal{F}}$ anti-commutes with a grading operator ϵ then the same will hold for $f(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}})$. Take the action ϕ of $C_0(\mathcal{F})$ on $\mathcal{H}_{\mathcal{F}} \otimes s^! \mathbf{E}$ to be the standard one. Then $Ind(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}})$ is the KK-class of the cycle $(\mathcal{H}_{\mathcal{F}} \otimes s^! \mathbf{E}, f(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}}), \phi)$.

The KK-product

$$KK(\mathbf{C}, C_0(\mathcal{F})) \otimes KK_*(C_0(\mathcal{F}), C(M)) \longrightarrow KK(\mathbf{C}, C(M))$$

pairs a K-theory boundary class $\delta[u] \in KK(\mathbf{C}, C_0(\mathcal{F}))$ with $Ind(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}})$ to yield a class in $K^0(M)$. This is the KK-formulation of the index class in (21).

Proposition 5.29 of Roe [56] implies that, for the open manifold case, the $KK(\mathbf{C}, \mathbf{C})$-valued pairing between the exotic index of a geometric operator and the compactly supported K-theory class transgressed from the corona, equals the relative index pairing constructed as in (21). This was reproved G. Yu (Theorem 2, [63]) in the framework of special vector bundles, by showing that explicit cycles representing the KK-pairings are homotopic as $(\mathbf{C}, \mathbf{C})$-cycles. Yu's method directly adapts to the foliation groupoid case to yield:

THEOREM 6.4 *Let $\mathcal{D}_{\mathcal{F}}$ be a leafwise geometric operator for the foliation $\mathcal{F}$. For each $[u] \in K^*(\partial_h \mathcal{F})$ the exotic index pairing is calculated by*

$$\rho[u](Ind_e(\mathcal{D}_{\mathcal{F}}, \epsilon)) = Ind\,(\delta[u] \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}}, \epsilon) \tag{22}$$

The Chern character $\mathrm{Ch}\colon K^*(M) \to H^*(M; \mathbf{R})$ applied to the right-hand-side of (22) can be calculated by an index formula for families (cf. [25, 28]), so that explicit topological formula for the exotic index follow from Theorem 6.4.

7 The Foliation Novikov Conjecture

The index class of a leafwise elliptic differential operator is a K-theory class in $K_*(C^*(\mathcal{F}))$, which intuitively can be considered as a cohomology class on the leaf space $M/\mathcal{F}$. A K-theory *fundamental class* for $\mathcal{F}$ is defined to be homomorphism $Z_* = \langle \cdot, Z \rangle \colon K_*(C^*(\mathcal{F})) \to \mathbf{Z}$ which depends only on the leafwise homotopy class of $\mathcal{F}$. Connes proved that an invariant transverse elliptic operator to $\mathcal{F}$ yields a fundamental class [11]. He later showed that a cyclic cocycle on the smooth convolution algebra $C_c^\infty(\mathcal{G}_{\mathcal{F}})$ which satisfies appropriate growth estimates yields a fundamental class [10]. By the work of Hilsum and Skandalis (cf . Theorem 5.8 above), the index class of the leafwise signature operator with coefficients in a leafwise almost flat bundle $\mathbf{E} \to M$ is a leafwise homotopy invariant, so every fundamental class Z_* yields a numerical invariant $\langle Ind\,((d_{\mathcal{F}} * - * d_{\mathcal{F}}) \otimes \mathbf{E})\,, Z \rangle$ of the leafwise homotopy class of $\mathcal{F}$. In this section, we observe that each boundary K-theory class in $K^{\ell+1}(\partial_h \mathcal{F})$ provides a family of fundamental classes for $\mathcal{F}$, thus greatly extending the list of leafwise homotopy numerical invariants. The Novikov conjecture for special contractible foliations is deduced by applying a particular case of this construction.

THEOREM 7.1 *For each* $[u] \in K^\ell(\partial_h \mathcal{F})$ *and* $[\mathcal{D}_M] \in KK(C_0(M), \mathbf{C})$*, there is a K-theory fundamental class*

$$Z([u], [\mathcal{D}_M])_* \colon K_*(C^*(\mathcal{F})) \to \mathbf{Z}$$

Proof: For $[\mathbf{e}] \in K_*(C^*(\mathcal{F}))$ define

$$\langle [\mathbf{e}], Z([u], [\mathcal{D}_M]) \rangle = \langle \rho[u]([\mathbf{e}]), [\mathcal{D}_M] \rangle = \rho[u]([\mathbf{e}]) \otimes [\mathcal{D}_M] \in KK(\mathbf{C}, \mathbf{C}) \cong \mathbf{Z} \tag{23}$$

$Z([u], [\mathcal{D}_M])_*$ is well-defined for any foliation that is leafwise homotopic to $\mathcal{F}$ by Corollary 3.9, so (23) yields a K-theory fundamental class. □

Let $\mathcal{D}_{\mathcal{F}}$ be a leafwise-elliptic, pseudo-differential operator for $\mathcal{F}$. The Connes-Skandalis construction [17] yields a KK-index class $Ind(\mathcal{D}_{\mathcal{F}}) \in KK_*(C_0(M), C^*(\mathcal{F}))$, which via the external KK-product yields a map:

$$\mu(\mathcal{D}_{\mathcal{F}}) \colon K^*(M) \cong KK(\mathbf{C}, C_0(M)) \longrightarrow KK(\mathbf{C}, C^*(\mathcal{F})) \cong K_*(C^*(\mathcal{F})) \tag{24}$$

The map (24) is a special case of the Baum-Connes "μ-map" whose domain is the K-theory of all leafwise symbols for $\mathcal{F}$ [3, 4].

We say that $\mathcal{F}$ is a *contractible foliation* if the identity map of $\mathcal{G}_{\mathcal{F}}$ is homotopic to the fiberwise projection onto the diagonal, $*s \colon \mathcal{G}_{\mathcal{F}} \to M \hookrightarrow *M \subset \mathcal{G}_{\mathcal{F}}$. If the homotopy can be chosen to preserve the fibers of s, then we say that $\mathcal{F}$ has *uniformly contractible leaves.*

THEOREM 7.2 *Let* $\mathcal{F}$ *be a contractible foliation of leaf dimension* p *with Hausdorff holonomy groupoid* $\mathcal{G}_{\mathcal{F}}$*. For each boundary K-theory class* $[u] \in K^{\ell+1}(\partial_h \mathcal{F})$ *the composition*

$$\rho[u] \circ \mu(\mathcal{D}_{\mathcal{F}}) \colon K^k(M) \longrightarrow K^{k+\ell+p}(M) \tag{25}$$

is multiplication by the exotic index class $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u]) = \rho[u](Ind_e(\mathcal{D}_{\mathcal{F}}, \epsilon)) \in K^{\ell+p}(M)$ *for* p *even and* $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u]) = \rho[u](Ind_e(\mathcal{D}_{\mathcal{F}})) \in K^{\ell+p}(M)$ *for* p *odd.*

We denote the composition (25) by $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u])$.

Proof: We give the proof for k, ℓ and p even. The other seven cases follow in exactly the same way. Let $[\mathbf{e}] \in K^0(M)$ represented by an Hermitian vector bundle $\mathbf{E} \to M$. The external product

$$\mu(\mathcal{D}_{\mathcal{F}})[\mathbf{e}] = \mathbf{E} \otimes Ind(\mathcal{D}_{\mathcal{F}}, \epsilon) \in K_0(C^*(\mathcal{F}))$$

is equal to the index of the leafwise operator obtained extending the domain of $\mathcal{D}_{\mathcal{F}}$ by tensoring with the sections of the bundle $\mathbf{E}$. Now use Theorem 6.4 to obtain

$$\begin{aligned} \rho[u] \circ \mu(\mathcal{D}_{\mathcal{F}})[\mathbf{e}] &= \rho[u]\left(Ind(\mathbf{E} \otimes \mathcal{D}_{\mathcal{F}}, \epsilon)\right) \\ &= Ind\left(\delta[u] \otimes (\mathbf{E}_{\mathcal{G}_{\mathcal{F}}} \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}})\right) \end{aligned} \tag{26}$$

The next lemma identifies the fiberwise coefficients $\mathbf{E}_{\mathcal{G}_{\mathcal{F}}}$ appearing in the parentheses on the right side of (26). Recall that $r \colon \mathcal{G}_{\mathcal{F}} \to M$ is the "range map" evaluating a leafwise-path at its endpoint.

LEMMA 7.3 $Ind\,(\delta[u] \otimes (\mathbf{E}_{\mathcal{G}_{\mathcal{F}}} \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}})) = Ind\left(\delta[u] \otimes (r^! \mathbf{E} \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}})\right)$

Proof: Recall that $r\colon \mathcal{G}_{\mathcal{F}} \to M$ restricted to the fiber of $\tilde{L}_x = s^{-1}(x) \subset \mathcal{G}_{\mathcal{F}}$ is the canonical covering map onto L_x. From the definition of the fiberwise operator $\mathbf{E}_{\mathcal{G}_{\mathcal{F}}} \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}}$ on $\mathcal{G}_{\mathcal{F}}$ it is represented fiberwise by the lift of the leafwise operator $\mathbf{E} \otimes \mathcal{D}_{\mathcal{F}}$ for $\mathcal{F}$. Calculate from the definitions

$$\begin{aligned} \left\{\mathbf{E}_{\mathcal{G}_{\mathcal{F}}} \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}} | \tilde{L}_x\right\} &= r^* \left\{\mathbf{E} \otimes \mathcal{D}_{\mathcal{F}}\right\} | \tilde{L}_x \\ &= r^* \left\{\mathbf{E}|L_x \otimes \mathcal{D}|L_x\right\} \\ &= r^! \mathbf{E} | \tilde{L}_x \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}} | \tilde{L}_x \end{aligned} \tag{27}$$

where the identification on line (27) means that we choose the fiberwise Hermitian connection on $r^! \mathbf{E}$ to be the lift under r^* of the leafwise Hermitian connection on $\mathbf{E} \to M$. The claim of the lemma follows immediately from this calculation. □

Combine Lemma 7.3 and associativity of the external KK-product to obtain

$$\begin{aligned} \rho[u] \circ \mu(\mathcal{D}_{\mathcal{F}})[\mathbf{e}] &= Ind\left(\delta[u] \otimes (r^! \mathbf{E} \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}})\right) \\ &= Ind\left((\delta[u] \otimes r^! \mathbf{E}) \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}})\right) \end{aligned} \tag{28}$$

Formula (28) for the exotic index pairing holds for any boundary class of a Hausdorff groupoid. In the case of a contractible foliation, a vector bundle over $\mathcal{G}_{\mathcal{F}}$ is determined by its restriction to the section $*M$, so there is the additional information

LEMMA 7.4 *If $\mathcal{F}$ is a contractible Hausdorff foliation,* $r^! \mathbf{E} \cong s^! \mathbf{E}$. □

The index class on the right-hand-side of (28) depends only on the compactly supported isomorphism class of $\delta[u] \otimes r^! \mathbf{E}$, so that for a contractible foliation we have

$$\begin{aligned} \rho[u] \circ \mu(\mathcal{D}_{\mathcal{F}})[\mathbf{e}] &= Ind\left((\delta[u] \otimes s^! \mathbf{E}) \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}})\right) \\ &= Ind\left(s^! \mathbf{E} \otimes (\delta[u] \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}})\right) \\ &= \mu(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}})\left(\delta[u] \otimes s^! \mathbf{E}\right) \end{aligned} \tag{29}$$

Finally, we observe that $s\colon \mathcal{G}_{\mathcal{F}} \to M$ endows $K^*(\mathcal{G}_{\mathcal{F}})$ with a $K^*(M)$-module action which commutes with the fiberwise index map

$$\mu(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}})\colon KK(\mathbf{C}, C_0(\mathcal{G}_{\mathcal{F}})) \longrightarrow KK(\mathbf{C}, C(M)) \cong K^*(M).$$

Apply these remarks to (29) to obtain

$$\rho[u] \circ \mu(\mathcal{D}_{\mathcal{F}})[\mathbf{e}] = [\mathbf{e}] \otimes Ind\,(\delta[u] \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}}, \epsilon) = \mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u]) \otimes [\mathbf{e}] \tag{30}$$

which proves Theorem 7.2. □

COROLLARY 7.5 *Let $\mathcal{F}$ be a contractible foliation of leaf dimension p with Hausdorff holonomy groupoid $\mathcal{G}_\mathcal{F}$ and $\mathcal{D}_\mathcal{F}$ be a leafwise-elliptic, pseudo-differential operator. Suppose there exists a boundary K-theory class $[u] \in K^*(\partial_h \mathcal{F})$ so that $\mathcal{I}(\mathcal{D}_\mathcal{F}, [u])$ is invertible in $K^*(M) \otimes \mathbf{Q}$. Then the leafwise index map*

$$\mu(\mathcal{D}_\mathcal{F}) \colon K^*(M) \otimes \mathbf{Q} \longrightarrow K_*(C^*(\mathcal{F})) \otimes \mathbf{Q}$$

is injective. □

A class $\mathcal{I} \in K^0(M) \otimes \mathbf{R}$ for a connected manifold M is invertible if and only if its virtual dimension is non-zero. That is, the restriction of $\mathcal{I}$ to a point $x \in M$ yields a non-trivial class in $K^0(x) \cong \mathbf{Z}$. In the above context, this implies that if $\mathcal{I}(\mathcal{D}_\mathcal{F}, [u])$ has even degree and its restriction to a fiber over each connected component of M is non-trivial, then $\mu(\mathcal{D}_\mathcal{F})$ is injective.

In the Atiyah formalism of [1], given an hermitian vector bundle $p_\mathbf{E} \colon \mathbf{E} \to M$ and an elliptic operator $\mathcal{D}_\mathbf{E}$ along the fibers of $p_\mathbf{E}$, there is a map $\alpha(\mathcal{D}_\mathbf{E}) \colon K(\mathbf{E}) \to K(M)$ given by integration along the fibers in K-theory. A key property of this map is that it commutes with the natural $p_\mathbf{E}^*$-module action of $K(M)$ on $K(\mathbf{E})$. Tensor product with the Bott class $\beta[\mathbf{E}] \in K(\mathbf{E})$ of the bundle $\mathbf{E}$ defines a map $\beta \colon K(M) \to K(\mathbf{E})$. The $K(M)$-module properties of α and β imply that $\alpha(\mathcal{D}_\mathbf{E}) \circ \beta \colon K(M) \to K(M)$ is multiplication by $\mathcal{I}(\beta[\mathbf{E}] \otimes \mathcal{D}_\mathbf{E}) \in K(M)$, which is calculated from the index theorem for families.

The constructions of the exotic index bear a strong similarity with the Atiyah approach. In the foliation context, the groupoid "fibration" $s \colon \mathcal{G}_\mathcal{F} \to M$ replaces the vector bundle $\mathbf{E} \to M$, and the fiberwise operator $\mathcal{D}_{\mathcal{G}_\mathcal{F}}$ replaces $\mathcal{D}_\mathbf{E}$. The transgression $\delta[u] \in K^*(\mathcal{G}_\mathcal{F})$ of a boundary class $[u] \in K^*(\partial_h \mathcal{F})$ replaces the Bott class $\beta[\mathbf{E}]$. There are generalized α and β maps as well:

$$\alpha(\mathcal{D}_{\mathcal{G}_\mathcal{F}}) \colon K(\mathcal{G}_\mathcal{F}) \to K(M) \tag{31}$$

$$\beta[u] \colon K(M) \to K(\mathcal{G}_\mathcal{F}) \tag{32}$$

where $\beta[u]([\mathbf{e}]) = \delta[u] \otimes [s^! \mathbf{e}]$ and $\alpha(\mathcal{D}_{\mathcal{G}_\mathcal{F}})[\mathbf{e}] = \mathit{Ind}\,([\mathbf{e}] \otimes \mathcal{D}_{\mathcal{G}_\mathcal{F}})$. The composition $\alpha(\mathcal{D}_{\mathcal{G}_\mathcal{F}}) \circ \beta[u] = \mathcal{I}(\mathcal{D}_\mathcal{F}, [u])$, so that injectivity of $\rho[u] \circ \mu(\mathcal{D}_\mathcal{F})$ is equivalent to injectivity of $\alpha(\mathcal{D}_{\mathcal{G}_\mathcal{F}}) \circ \beta[u]$.

The corona of Euclidean space $\mathbf{R}^N$ has the same K-theory as S^{N-1}, so for a vector bundle $\mathbf{E} \to M$, there is a unique boundary K-theory class which transgresses to a fiberwise fundamental class for the fibration (just as there is a unique Bott class.) For the more general situation of $s \colon \mathcal{G}_\mathcal{F} \to M$, each class $\delta[u] \in K^*(\mathcal{G}_\mathcal{F})$ can be used as a "Bott class" and the topological problem is to calculate the range of the index pairings $\mathcal{I}(\mathcal{D}_\mathcal{F}, [u])$ for the various classes $[u] \in K^*(\partial_h \mathcal{F})$.

The composition of groupoids $M \cong *M \subset \Pi_\mathcal{F} \subset \mathcal{G}_\mathcal{F}$ induces a sequence of classifying maps

$$M \simeq B(*M) \longrightarrow B\Pi_\mathcal{F} \longrightarrow B\mathcal{G}_\mathcal{F}$$

Haefliger (Corollaire 3.2.4, [26]) proved that for a foliation with uniformly contractible leaves, the composition $M \to B\mathcal{G}_{\mathcal{F}}$ is a homotopy equivalence. As a corollary, we note that the image of the induced map $H^*(\mathcal{G}_{\mathcal{F}}) \to H^*(M)$ equals the image of $H^*(B\Pi_{\mathcal{F}}) \to H^*(M)$.

The Novikov conjecture for compact manifolds was the source of inspiration for most of the index theory used in this paper. This has formulation for foliations which properly extends the conjecture for compact manifolds:

CONJECTURE 7.6 (Foliation Novikov Conjecture, [4]) *Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be oriented C^∞ foliations with M, M' compact. Let $f\colon M \to M'$ be an orientation-preserving leafwise homotopy equivalence. Then for any class $\omega \in H^*(B\Pi_{\mathcal{F}}; \mathbf{Q})$*

$$(B\pi')^*\omega \cup L(TM') = f^*\left((B\pi)^*\omega \cup L(TM)\right) \tag{33}$$

where $L(TM)$ denotes the Hirzebruch L-polynomial in the Pontrjagin classes of TM.

The Foliation Novikov conjecture is said to hold for $\mathcal{F}$ if the conclusion (33) is true for all leafwise homotopy equivalences $f\colon M \to M'$ as above. For a foliation $\mathcal{F}$ with uniformly contractible leaves, Haefliger's theorem implies it suffices to check (33) holds for all $\omega \in H^*(B\mathcal{G}_{\mathcal{F}}; \mathbf{Q}) \cong H^*(M; \mathbf{Q})$.

Baum and Connes proved this conjecture for foliations whose leaves admit a metric with non-positive sectional curvatures, using the "dual Dirac" method [4]. We next show how the exotic index applies to extend their result. First, we need the foliation formulation of an idea introduced by Roe (section 6.2, [56].) Let $T\mathcal{F} \to M$ be the tangent bundle to the leaves of $\mathcal{F}$ and $S\mathcal{F}$ the sphere bundle for $T\mathcal{F}$ considered as a corona for $T\mathcal{F}$. There is a unique class $\Theta \in H^{p-1}(S\mathcal{F})$ whose boundary $\delta\Theta = \mathbf{Th}[T\mathcal{F}] \in H^p_c(T\mathcal{F})$ is the Thom class.

DEFINITION 7.7 *A foliation $\mathcal{F}$ on a connected manifold M is said to be* ultra-spherical *if there exists a map of coronas $\sigma\colon \partial_h\mathcal{F} \to S\mathcal{F}$ which commutes with the projections onto M, and so that $\sigma^*\Theta \in H^*(\partial_h\mathcal{F})$ is non-zero.*

THEOREM 7.8 *Let $\mathcal{F}$ be an oriented ultra-spherical foliation with uniformly contractible leaves and Hausdorff holonomy groupoid. Then the Foliation Novikov Conjecture is true for $\mathcal{F}$.*

Proof: By the standard reduction of the problem (cf. [4]), it suffices to show that the map $\mu(\mathcal{D}_{\mathcal{F}})$ is injective for the leafwise Dirac operator. By Corollary 7.5, this will follow from proving there exists a boundary K-theory class $[u] \in K^*(\partial_h\mathcal{F})$ so that $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u]) \in K^*(M) \otimes \mathbf{Q}$ is invertible.

Let $\eta \in K(S\mathcal{F})$ with K-theory boundary $\beta[T\mathcal{F}] \in K(T\mathcal{F})$, and set $[u] = \sigma^*\eta$.

LEMMA 7.9 $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u])$ *is is invertible in $K^*(M) \otimes \mathbf{Q}$.*

Proof: There is a continuous extension of σ to a map of pairs (cf. proof of Lemma 6.3, [56])

$$\overline{\sigma}\colon (\overline{\mathcal{G}_{\mathcal{F}}}, \partial_h \mathcal{F}) \longrightarrow (\overline{T\mathcal{F}}, S\mathcal{F})$$

which commutes with the projection onto M. By naturality of the boundary map, $\partial[u] = \overline{\sigma}^* \beta[T\mathcal{F}]$, so that

$$\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u]) = Ind\,(\overline{\sigma}^* \beta[T\mathcal{F}] \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}}) \tag{34}$$

The index class $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u])$ has even dimension, so it suffices to show that $Ind\,(\overline{\sigma}^* \beta[T\mathcal{F}] \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}})$ is non-zero when restricted to any fiber over M. But this follows from the original calculation of Roe, Theorem 6.9 [56]. $\square$

REMARK 7.10 The sequence of hypotheses above have progressed from the least restrictive,
"$\mathcal{F}$ is contractible" to the more restrictive, "$\mathcal{F}$ is ultra-spherical" with each assumption yielding further progress towards establishing the foliation Novikov Conjecture for that class of foliations. This is precisely parallel to the development of the proof of the Novikov Conjecture for compact manifolds, where the all current methods of proof seem to require a version of the "ultra-spherical hypotheses" and speculate that the techniques extend to the uniformly contractible case. It is natural to conjecture that the above techniques will show that the map $\mu(\mathcal{D}_{\mathcal{F}})$ is injective for contractible foliations. That is, the problem is to show that all contractible foliations admit a boundary K-theory class $[u] \in K^*(\partial_h \mathcal{F})$ so that $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u])$ is a multiplicative unit in $K^*(M)$ for the leafwise signature operator $\mathcal{D}_{\mathcal{F}}$.

The hypotheses of Theorem 7.8 are readily established for many classes of foliations:

EXAMPLE 7.11 Proposition 4.13 implies that a contractible foliation $\mathcal{F}$ defined by a locally free action of a simply connected Lie group is ultraspherical.

EXAMPLE 7.12 A uniformly contractible foliation $\mathcal{F}$ whose leaves have a metric so that their holonomy covers have no conjugate points is ultraspherical.

EXAMPLE 7.13 Let $\mathcal{F}$ be a Riemannian foliation $\mathcal{F}$ whose universal leaf L is ultra-spherical. Then by the proof of Proposition 4.14, $\mathcal{F}$ satisfies the hypotheses of Theorem 7.8.

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Steven Hurder
Department of Mathematics (m/c 249)
University of Illinois at Chicago
851 S. Morgan St.
CHICAGO, IL 60607-7045 USA
email: hurder@boss.math.uic.edu