Exotic index theory and the Novikov conjecture

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1 Introduction

The *Novikov conjecture*, that the higher signatures of a compact oriented manifold are homotopy invariant, is well-known both for its depth and for the breadth of the research it has spawned. There are two approaches to proving the conjecture for various classes of groups – one via geometric surgery techniques [8, 16, 9, 17], and the other via index theory methods. The index method appeared first in the work of Lusztig [33], which was greatly extended by Miščenko [35, 36] and Kasparov [28, 29, 30, 32]. The two methods are directly compared in Rosenberg's article [42]. Connes introduced an intermediate approach which combines index methods with the more geometric techniques of cyclic cohomology [4, 10, 11, 12, 13, 15].

The index theory approach for a compact oriented manifold with fundamental group Γ traditionally has two steps (cf. the survey by Kasparov [31]). The first is the Kasparov-Miščenko construction of the "parametrized signature class" $\sigma(M) \in K_0(C_r^*(\Gamma))$, and the key point is that this class is a homotopy invariant [25, 27, 34, 35]. Secondly, one shows that operator assembly map $\beta\colon K_*(B\Gamma) \to K_*(C_r^*\Gamma)$ from the K-homology of $B\Gamma$ to the K-theory of the reduced group C^*-algebra is rationally injective and the image of the K-homology class determined by the signature operator coincides with $\sigma(M)$. Hence, the signature K-homology class is also homotopy invariant and its pairing with group cohomology classes will be invariant – which is the conclusion of the Novikov conjecture.

The Γ-invariance of the lifted operator $\tilde{\mathcal{D}}$ on the universal cover $\widetilde{M}$ of $M \cong B\Gamma$ is used to define a class $\beta[\mathcal{D}] \in K_*(C_r^*\Gamma)$, basically given by the differences of the homotopy classes of the representation of Γ on the kernel and cokernel of $\tilde{\mathcal{D}}$. The "index data" of an operator is localized in its spectrum around 0, while the $C_r^*(\Gamma)$-index contains information related to all the unitary representations of Γ weakly contained in the regular representation, so *a priori* has in it more structure than is necessary for the study of the Novikov conjecture. The idea of index theory in coarse geometry (cf. Roe [40]) is to localize the C^*-index in a neighborhood of infinity spacially, which

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by a non-commutative uncertainty principle corresponds to localizing the index at 0 spectrally. The applications of this fundamental idea are still being developed.

The purpose of this note is to prove the injectivity of the map β for a large class of groups using the methods of exotic index theory; that is, index theory for families of coarse metric spaces. The exotic index approach has the advantage that "coarsening the index class" results in technical simplifications which allow the Miščenko-Kasparov method to be applied more broadly. As an example, exotic index methods yield new cases of the Foliation Novikov Conjecture [26, 19].

The topological context for the "index data localized at infinity" of a Γ-invariant elliptic operator is the geometric structure given by a family of parametrized metric spaces with base a model for $B\Gamma$, and fibers the coarse metric type of Γ. The basic idea is to consider such a family as a generalized vector bundle and introduce its *coarse Bott class* which is the analogue in coarse theory of the Thom class (cf. [1]). The data given by a coarse Bott class corresponds to the "dual Dirac" KK-class of Miščenko-Kasparov, but formulated within the coarse geometry category. Our most general result using this technique is that the operator assembly map is injective on K-theory for a group Γ whose associated field of metric spaces admits a coarse Bott class. This note contains a complete proof of this assertion for such Γ when $B\Gamma$ represented by a complete Riemannian manifold.

A Riemannian n-manifold $\widetilde{M}$ is *ultraspherical* (Definition 6.1, [40]) if it admits a proper map $f\colon \widetilde{M} \to \mathbf{R}^n$ where f has non-zero degree and the gradient ∇f has uniformly bounded pointwise norm on $\widetilde{M}$. Given a complete Riemannian manifold M with fundamental group Γ and universal covering $\widetilde{M}$, we say that M is Γ-*ultraspherical* if the balanced product $\widetilde{M}\Gamma = (\widetilde{M} \times \widetilde{M})/\Gamma$ admits a map $F\colon \widetilde{M}\Gamma \to TM$ which preserves fibers, and the restriction of F to each fiber $F_x\colon \{x\} \times \widetilde{M} \to T_xM$ is ultraspherical. The coarse index approach yields the following:

THEOREM 1.1 *Let Γ be a group so that the classifying space $B\Gamma$ is represented by a spin manifold M which admits a complete Riemannian metric so that M is Γ-ultraspherical. Then the operator assembly map $\beta\colon K_*(B\Gamma) \to K_*(C^*_r\Gamma)$ is injective.*

The proof of Theorem 1.1 is especially transparent when M is compact without boundary. For this reason, we give the proof in the closed case first, in sections 2 through 6. Section 7 discusses the non-compact case.

Note that the hypotheses of Theorem 1.1 do not require that M be of the homotopy type of a CW-complex of finite type. Also note, if $B\Gamma$ is represented by a finite CW complex, then it is also represented by an open complete spin manifold M: embed $B\Gamma$ in $\mathbf{R}^\ell$ for appropriate $\ell \gg 0$ and take M to be a regular neighborhood. Modify the restriction of the Euclidean metric to M so that it rapidly vanishes near the boundary of the closure of M, and one obtains a complete Riemannian metric on TM as well. The key hypothesis which must be verified, is that the universal covering $\widetilde{M}$ of M is ultraspherical for the particular choice of complete metric on TM.

There is a version of the theorem for non-spin manifolds, where the signature operator is used in place of the Dirac operator in the application of Poincaré duality.

The coarse index approach can also be used to prove:

THEOREM 1.2 *Let Γ be a finitely presented group whose classifying space $B\Gamma$ is homotopy equivalent to a finite CW complex. Suppose the universal covering $E\Gamma$ of $B\Gamma$ admits a metrizable, contractable compactification $X\Gamma$ such that*

- *The deck translation action of Γ on $E\Gamma$ extends continuously to $X\Gamma$.*
- *For any compact subset $K \subset E\Gamma$ and sequence of group elements $\{\gamma_n\} \subset \Gamma$ which tend to infinity in the word norm, the translates $K \cdot \gamma_n$ have diameter tending to zero.*

Then the operator assembly map $\beta: K_*(B\Gamma) \to K_*(C^*_r\Gamma)$ is injective.

Theorem 1.2 follows by proving that for Γ as in the theorem, there is a complete open manifold V and a free action of Γ on V so that $(V \times V)/\Gamma \to V/\Gamma \cong B\Gamma$ admits a coarse Bott class. This follows via a geometric method similar to that used in [12] to establish the theorem for word hyperbolic groups, which will be discussed in detail in a later paper. Example 8.3 gives the proof of Theorem 1.2 for the very particular case where $\Gamma \cong \pi_1(M)$ and M is a complete open manifold with restrictions on its geometry. The most general case will be treated in a subsequent paper.

The hypotheses of Theorem 1.2 correspond with those assumed in Carlsson-Pedersen [9] and Ferry-Weinberger [17] to prove injectivity of the integral assembly map via controlled surgery theory. The results of Bestvina-Mess [6] show that the hypotheses of Theorem 1.2 are satisfied for groups which are word hyperbolic.

Theorems 1.1, 1.2, 6.2 and 7.6 have similar conclusions to all approaches to the Novikov conjecture using the KK-approach. Kasparov and Skandalis [32] construct a Dirac and dual Dirac class for C^*-algebras associated to Bruhat-Tits buildings, and use this to prove the conjecture for many classes of arithmetic groups. A. Connes, M. Gromov, and H. Moscovici [12] developed the theory of proper Lipschitz cohomology (a work that directly inspired this note) and used it to give the first proof of the Novikov Conjecture for word hyperbolic groups. Our notion of a "coarse Bott class" for a group is clearly related to their Lipschitz-Poincaré dual of a group (cf. Epilogue to [12].) The non-finite type case of Theorem 1.1 yields a new class of examples where the conjecture has been verified, for the case where $B\Gamma$ is represented by a complete manifold not of finite type.

Here is an outline of this paper. The coarse index class is introduced in section 2. The parametrized corona $\partial_\pi \Gamma$ of the fundamental group is introduced in section 3. In section 4 we develop a pairing between the K-theory $K^1(\partial_\pi \Gamma)$ and the exotic indices. Section 5 considers the exotic index for operators on compact manifolds. The coarse Bott class hypotheses is introduced in section 6, where we prove that the μ-map is injective for a group with this condition. It is immediate that a group satisfying the hypotheses of Theorem 1.1 admits a coarse Bott class, so its proof is completed. Section 7 details the modifications needed in the open case. Finally, in section 8 we show that the groups satisfying the hypotheses of Theorem 1.2 satisfy the hypotheses of Theorem 1.1.

The results of this paper were presented to the meeting on *Novikov Conjectures, Index Theorems and Rigidity* at Oberwolfach, September 1993. There are close parallels between the operator methods of this paper and those used in the most recent controlled K-theory approaches to the Novikov Conjecture [9, 17]. This manuscript is a substantial revision of an earlier preprint (dated April 16, 1993) with these parallels accentuated.

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2 The coarsening map

We begin by recalling some basic ideas of coarse geometry. A *pseudometric* on a set X is a non-negative, symmetric pairing $d(\cdot,\cdot) : X \times X \to [0,\infty)$ satisfying the triangle inequality

$$d(x,z) \leq d(x,y) + d(y,z) \qquad \text{for all } \ x,y,z \in X$$

A map $f : X_1 \to X_2$ is said to be *coarsely quasi-isometric* with respect to pseudometrics $d_i(\cdot,\cdot)$ on the X_i if there exists constants $A, B > 0$ so that for all $y, y' \in X_1$

$$A^{-1} \cdot (d_1(y,y') - B) \leq d_2(f(y), f(y')) \leq A \cdot (d_1(y,y') + B) \tag{2.1}$$

This notion is more strict than Roe's notion of *uniform bornologous* (cf. Definition 2.1, [40]) where the estimator $A = S(R)$ need not be a linear function of $R = d(y, y')$.

A subset $\mathcal{Z} \subset X$ is called *C-dense* if for each $x \in X$, there exists $n(x) \in \mathcal{Z}$ so that $d(x, n(x)) \leq C$. A map $f : X_1 \to X_2$ is said to be a *coarse isometry* with respect to pseudometrics $d_i(\cdot, \cdot)$ if f is a coarse quasi-isometry and the image $f(X_1)$ is C-dense in X_2 for some C.

A *net*, or *quasi-lattice*, is a collection of points $\mathcal{N} = \{x_\alpha \mid \alpha \in \mathcal{A}\} \subset X$ so that there are $C,\ D > 0$ with $\mathcal{N}$ a C-dense set, and distinct points of $\mathcal{N}$ are at least distance D apart. The inclusion of a net $\mathcal{N} \subset X$ is a coarse isometry for the restricted metric on $\mathcal{N}$.

For sections 2 through 6, we will assume that M is a compact spin manifold without boundary, and we fix a Riemannian metric on M. Let $\widetilde{M} \to M$ denote the universal covering of M, with the fundamental group $\Gamma = \pi_1(M)$ acting via translations $T\colon \widetilde{M} \times \Gamma \to \widetilde{M}$ on the right. The Riemannian metric on TM lifts to a Γ-invariant complete Riemannian metric on $T\widetilde{M}$, and $\widetilde{M}$ is endowed with the Γ-invariant path length metric. Introduce the *balanced product* $\widetilde{M}\Gamma = (\widetilde{M} \times \widetilde{M})/\Gamma$, with projections $\pi_i\colon \widetilde{M}\Gamma \to M$ for $i = 1, 2$ onto the first and second factors, respectively. The Γ-invariant metric on $\widetilde{M}$ induces a Γ-invariant metric on the second factor of $\widetilde{M} \times \widetilde{M}$, which descends to a fiberwise metric on $\widetilde{M}\Gamma$: for $x \in M$ and $y, y' \in \widetilde{M}_x$ then $d_x(y, y')$ denotes the "fiberwise" distance from y to y'. Note that that $\widetilde{M}_x = \pi_1^{-1}(x)$ is naturally isometric to $\widetilde{M}$.

For $x \in \widetilde{M}$, let $\Delta(x) \in \widetilde{M}\Gamma$ denote the equivalence class of $x \times x$, with $\pi_1(\Delta(x)) = \pi_2(\Delta(x)) = x$. This factors through a map, also denoted by $\Delta\colon M \to \widetilde{M}\Gamma$, which includes M along the diagonal.

Let $\mathcal{H}(\widetilde{M}, N)$ denote the Hilbert space of L^2-sections of the trivial bundle $\widetilde{M} \times \mathbf{C}^N$, where the inner product is determined by the Riemannian volume form on $\widetilde{M}$. We will often abuse notation and write $\mathcal{H}(\widetilde{M}) = \mathcal{H}(\widetilde{M}, N)$ where $N \gg 0$ is a sufficiently large integer determined as needed. Introduce the continuous field of Hilbert spaces over M,

$$\mathcal{H}_M = \{\mathcal{H}_x = \mathcal{H}(\widetilde{M}_x) \mid x \in M\}$$

where each fiber $\mathcal{H}_x$ is isomorphic to $\mathcal{H}(\widetilde{M})$. Note the inner product of two sections of $\mathcal{H}_M$ is a continuous function on M by definition.

The *Roe algebra* $C^*(X)$ of a complete metric space X equipped with a Radon measure is the norm closure of the *-algebra $\mathcal{B}^*_\rho(X)$ of bounded propagation, locally compact, bounded operators on $\mathcal{H}(X)$ (cf. section 4.1 [40]). The compact operators on $\mathcal{H}(X)$ satisfy $\mathcal{K}(\mathcal{H}(X)) \subset C^*(X)$. The inclusion is strict if and only if X is non-compact.

We next extend the construction of the Roe algebra from the open manifold $\widetilde{M}$ to the fibers of the map $\pi: \widetilde{M}\Gamma \to M$. This technical step is one of the main ideas of the coarse approach to the Novikov Conjecture. We begin with a non-equivariant version of the extension. Let $\mathcal{B}^*_\rho(\widetilde{M}\times\widetilde{M},\pi)$ be the *-algebra generated by the continuous families of operators $A = \{A_x \in \mathcal{B}^*_\rho(\widetilde{M}_x) \mid x \in \widetilde{M}\}$ with uniformly bounded propagation. That is, we require that there exists a constant C so that for each $x \in \widetilde{M}$, the operator A_x can be represented by a kernel on the fiber $\widetilde{M}_x$ supported in a C-neighborhood of the diagonal in $\widetilde{M}_x \times \widetilde{M}_x$, and the function $x \mapsto A_x$ is continuous in x for the operator norm on $\mathcal{B}^*_\rho(\widetilde{M}_x) \cong \mathcal{B}^*_\rho(\widetilde{M}) \subset \mathcal{B}(\mathcal{H}(\widetilde{M}))$. Using this last identification we have a natural isomorphism

$$\mathcal{B}^*_\rho(\widetilde{M}\times\widetilde{M},\pi) \cong C_{up}(\widetilde{M},\mathcal{B}_\rho(\widetilde{M})) \tag{2.2}$$

where the subscript "*up*" indicates the uniform control on propagation.

The right translation action of Γ induces a right action on operators $\mathcal{B}_\rho(\widetilde{M})$. Given $A = \{A_x\}$ and $\gamma \in \Gamma$ we obtain an action

$$\gamma^* A = (T\gamma)^* A_{T\gamma^{-1}(x)}.$$

Let $\mathcal{B}^*_\rho(\widetilde{M}\times\widetilde{M},\pi)^\Gamma$ denote the *-subalgebra of Γ-invariant elements of $C_{up}(\widetilde{M}, \mathcal{B}_\rho(\widetilde{M}))$, and $C^*(\widetilde{M}\Gamma,\pi) \subset \mathcal{B}(\mathcal{H}_M)$ its C^*-closure for the *-representation on $\mathcal{H}(\widetilde{M})$. An element of $C^*(\widetilde{M}\Gamma,\pi)$ is a continuous field of operators $\{A_x \mid x \in M\}$ where for each $x \in M$, $A_x \in C^*(\widetilde{M}_x)$. The C^*-completion commutes with the Γ-action, hence:

PROPOSITION 2.1 *There is a natural isomorphism*

$$C^*(\widetilde{M}\Gamma,\pi) \cong C(\widetilde{M}, C^*(\widetilde{M}))^\Gamma.$$

Let $C^*_r(\Gamma)$ denote the reduced C^*-algebra of Γ, and $\mathcal{B}(\mathcal{H}(\widetilde{M}))^\Gamma \subset \mathcal{B}(\mathcal{H}(\widetilde{M}))$ the C^*-subalgebra of Γ-invariant bounded operators.

PROPOSITION 2.2 *There is an injective map of C^*-algebras, $\mathcal{C}: C^*_r(\Gamma) \to C^*(\widetilde{M}\Gamma,\pi)$. The induced map on K-theory, called the* coarsening map *for Γ,*

$$\chi: K_*(C^*_r(\Gamma)) \to K_*(C^*(\widetilde{M}\Gamma,\pi)), \tag{2.3}$$

is independent of the choices made in defining $\mathcal{C}$.

Proof: Let $\mathcal{K} \subset \mathcal{B}(\mathcal{H}(\widetilde{M}))$ denote the C^*-subalgebra of compact operators. Let $P_\xi \in \mathcal{K}$ denote the projection onto a unit vector $\xi \in \mathcal{H}(\widetilde{M})$. Choose ξ with compact support, so that $P_\xi \in \mathcal{B}_\rho(\mathcal{H}(\widetilde{M}))$. Define the inclusion of C^*-algebras, $P^*_\xi: C^*_r(\Gamma) \to C^*_r(\Gamma)\hat{\otimes}\mathcal{K}$ mapping the unit element $\mathbf{1}_e \mapsto \mathbf{1}_e \otimes P_\xi$.

PROPOSITION 2.3 [13] *There is an isomorphism of C^*-algebras*

$$C^*_r(\Gamma)\hat{\otimes}\mathcal{K} \cong C^*(\widetilde{M})^\Gamma \subset \mathcal{B}(\mathcal{H}(\widetilde{M}))^\Gamma$$

where the image of the projection $\mathbf{1}_e \otimes P_\xi$ is contained in $\mathcal{B}^*_\rho(\widetilde{M}\Gamma, \pi)^\Gamma$.

The natural inclusion of $C^*(\widetilde{M})^\Gamma$ into $C^*(\widetilde{M}\Gamma, \pi)$ is induced from the inclusion of the constant functions in terms of the isomorphism (2.2). The composition

$$\mathcal{C}: C^*_r(\Gamma) \to C^*_r(\Gamma)\hat{\otimes}\mathcal{K} \cong C^*(\widetilde{M})^\Gamma \subset C^*(\widetilde{M}\Gamma, \pi)$$

is the desired map. Note that the choice of ξ affects $\mathcal{C}$ but not χ. $\square$

Here is a simple but typical example of the above construction, for the group $\Gamma = \mathbf{Z}^n$ with model $M = \mathbf{T}^n$ the n-torus, and $\widetilde{M} = \mathbf{R}^n$. Fourier transform gives an isomorphism $C^*_r(\mathbf{Z}^n) \cong C_0(\mathbf{T}^n)$, so that the K-theory $K_*(C^*_r(\mathbf{Z}^n)) \cong K^*(\mathbf{T}^n)$. Higson and Roe [24, 23] have calculated the K-theory $K_*(C^*(X))$ for a wide range of complete metric spaces X, which in the case of $\mathbf{R}^n$ yields $K_*(C^*(\mathbf{R}^n)) \cong K^*(\mathbf{R}^n)$. There is a natural *unparametrized* coarsening map $K_*(C^*_r(\mathbf{Z}^n)) \to K_*(C^*(\mathbf{R}^n))$, which assigns to a $\mathbf{Z}^n$-invariant elliptic operator on $\mathbf{R}^n$ its index in the coarse group $K_*(C^*_\rho(\mathbf{Z}^n)) \cong K_*(C^*(\mathbf{R}^n)) \cong K^*(\mathbf{R}^n)$. This coarsening process retains only the information contained in the top degree in K-homology.

The Higson-Roe results and standard algebraic topology yield

$$K_*(C^*(\mathbf{R}^n\mathbf{Z}^n, \pi)) \cong K^*(\mathbf{T}^n \times \mathbf{R}^n) \cong K^{*+n}(\mathbf{T}^n)$$

where the last isomorphism is cup product with the Thom class for the bundle $\widetilde{M}\Gamma = \mathbf{R}^n\mathbf{Z}^n = (\mathbf{R}^n \times \mathbf{R}^n)/\mathbf{Z}^n \to \mathbf{T}^n$. In this case the parametrized coarsening map

$$\chi: K_*(C(\mathbf{T}^n)) \cong K_*(C^*_r(\mathbf{Z}^n)) \to K_*(C^*(\mathbf{R}^n\mathbf{Z}^n, \pi)) \cong K^*(\mathbf{T}^n \times \mathbf{R}^n)$$

is an isomorphism.

3 Parametrized coronas

In this section, the parametrized (Higson) corona $\partial_\pi\widetilde{M}\Gamma$ is defined, which encapsulates the topological data relevant to the coarse indices of Γ-invariant operators on $\widetilde{M}$. Begin by introducing subalgebras of the bounded continuous functions on $\widetilde{M}\Gamma$:

• $C(\widetilde{M}\Gamma)$ denotes the algebra of bounded continuous functions on $\widetilde{M}\Gamma$, with the usual *sup*-norm

$$|h| = \sup_{y\in\widetilde{M}\Gamma} |h(y)|.$$

• $C_u(\widetilde{M}\Gamma)$ is the unital algebra of uniformly continuous bounded functions on $\widetilde{M}\Gamma$.

• $C_0(\widetilde{M}\Gamma)$ is the uniform closure of the compactly supported functions $C_c(\widetilde{M}\Gamma)$.

For $x \in M$ and $r > 0$, introduce the *fiberwise variation function*

$$\begin{array}{rcl} \mathbf{Var}_{(x,r)} : C_u(\widetilde{M}_x) & \to & C(\widetilde{M}_x,\ [0,\infty)) \\ \mathbf{Var}_{(x,r)}(h)(y) & = & \sup\{|h(y') - h(y)| \text{ such that } d_x(y,y') \leq r\} \end{array}$$

and also set

$$\begin{array}{rcl} \mathbf{Var}_r : C_u(\widetilde{M}\Gamma) & \to & C(\widetilde{M}\Gamma,\ [0,\infty)) \\ \mathbf{Var}_r(h)(y) & = & \mathbf{Var}_{(\pi(y),r)}(h)(y) \end{array}$$

We say $f \in C_u(\widetilde{M}\Gamma)$ has *uniformly vanishing variation at infinity* if for each $r > 0$ there exists a function $D(f,r) : [0,\infty) \to [0,\infty)$ so that $d_x(y, \Delta(x)) \geq D(f,r)(\epsilon) \Longrightarrow \mathbf{Var}(x,r)(f)(y) \leq \epsilon$. When M is compact, this condition is equivalent to saying that $\mathbf{Var}_r(f) \in C_0(\widetilde{M}\Gamma)$ for all $r > 0$. Let $C_h(\widetilde{M}\Gamma, \pi) \subset C_u(\widetilde{M}\Gamma)$ denote the subspace of functions which have uniformly vanishing variation at infinity.

LEMMA 3.1 (cf. 5.3, [40]) *$C_h(\widetilde{M}\Gamma, \pi)$ is a commutative C^*-algebra.* □

The spectrum of the C^*-algebra $C_h(\widetilde{M}\Gamma, \pi)$, denoted by $\overline{\widetilde{M}\Gamma}$, is a compactification of $\widetilde{M}\Gamma$, where the inclusion of the ideal $C_0(\widetilde{M}\Gamma) \hookrightarrow C_h(\widetilde{M}\Gamma, \pi)$ induces a topological inclusion $\widetilde{M}\Gamma \subset \overline{\widetilde{M}\Gamma}$ as an open dense subset. Define the *parametrized* (Higson) *corona* of $\widetilde{M}\Gamma$ as the boundary

$$\partial_\pi \widetilde{M}\Gamma = \overline{\widetilde{M}\Gamma} - \widetilde{M}\Gamma,$$

which is homeomorphic to the spectrum of the quotient C^*-algebra

$$C_h(\widetilde{M}\Gamma, \pi)/C_0(\widetilde{M}\Gamma).$$

The functions in $C(M)$ act as multipliers on $C_h(\widetilde{M}\Gamma, \pi)/C_0(\widetilde{M}\Gamma)$, hence the projection $\pi = \pi_1$ extends to continuous maps $\widehat{\pi}: \overline{\widetilde{M}\Gamma} \to M$ and $\partial\pi: \partial_\pi \widetilde{M}\Gamma \to M$. One can show, using that $C_h(\widetilde{M}\Gamma, \pi)$ is a subalgebra of the uniformly continuous functions on $\widetilde{M}\Gamma$, that both maps $\widehat{\pi}$ and $\partial\pi$ are fibrations. The typical fiber of $\partial\pi: \partial_\pi \widetilde{M}\Gamma \to M$ is not metrizable, even in the simplest case where $M = \mathbf{S^1}$ (cf. page 504, [38]).

4 The boundary pairing

The K-theory of the parametrized corona $\partial_\pi \widetilde{M}\Gamma$ naturally pairs with the coarse indices of Γ-invariant operators on $\widetilde{M}$. This pairing is a parametrized extension of N. Higson's observation [20, 21] that the vanishing condition on gradients for C^1-functions on a open complete Riemannian manifold X is the exact analytic condition required to form a pairing between the K-theory of its corona $\partial_h X$ and the K-homology of X. Roe extended this idea to complete metric spaces [40], and a parametric form of this construction was introduced in [26]. We require the following result:

THEOREM 4.1 *There is a natural pairing, for $p, q = 0, 1$,*

$$B_e \colon K_q(C_r^*(\Gamma)) \otimes K^p(\partial_\pi \widetilde{M}\Gamma) \longrightarrow K^{p+q+1}(M) \tag{4.1}$$

For each $[u] \in K^p(\partial_\pi \widetilde{M}\Gamma)$ evaluation of the pairing on $[u]$ yields the exotic index map

$$B_e[u] \colon K_q(C_r^*(\Gamma)) \longrightarrow K^{q+p+1}(M) \tag{4.2}$$

Proof: The idea of the proof is to construct a natural map

$$\partial_e \colon K^p(\partial_\pi \widetilde{M}\Gamma) \to KK_{p+1}(C^*(\widetilde{M}\Gamma, \pi), C(M)) \tag{4.3}$$

The map (4.1) is obtained as the Kasparov product between the images of the maps (2.3) and (4.3):

$$\begin{array}{rcl} \chi \otimes \partial_e & : & K_q(C_r^*(\Gamma)) \otimes K^p(\partial_\pi \widetilde{M}\Gamma) \\ & \longrightarrow & KK_q(\mathbf{C}, C^*(\widetilde{M}\Gamma, \pi)) \otimes KK_{p+1}(C^*(\widetilde{M}\Gamma, \pi), C(M)) \\ & \longrightarrow & KK_{p+q+1}(\mathbf{C}, C(M)) \cong K^{p+q+1}(M) \end{array}$$

The map (4.3) is exactly the coarse analogue of the dual Dirac construction. We give the essential points of the construction below – further details can be found in section 6, [26]. For example, one must use care because both the full Roe algebra $C^*(\widetilde{M}\Gamma, \pi)$ and the Higson corona $\partial_\pi \widetilde{M}\Gamma$ are not separable, so it is actually necessary to work with separable subalgebras and direct limits to define the pairing (4.3). This poses no problem, as every class in $K^p(\partial_\pi \widetilde{M}\Gamma)$ factors thru a separable quotient of the corona $\partial_\pi \widetilde{M}\Gamma$, and the image of a projection in $C_r^*(\Gamma)$ lies in a separable subalgebra of $C^*(\widetilde{M}\Gamma, \pi)$.

Suppose $p = 1$. Let $[u] \in K^1(\partial_\pi \widetilde{M}\Gamma)$; then we will define $\partial_e[u]$. Represent $[u]$ by a continuous map $u : \partial_\pi \widetilde{M}\Gamma \to U(N)$ for some $N > 0$. Let $j : U(N) \subset GL(N, \mathbf{C}) \subset \mathbf{C}^{N^2}$ be the embedding obtained by the standard coordinates on matrices. Apply the Tietze Extension Theorem to the map from the boundary of $\widetilde{M}\Gamma$ into a regular neighborhood retract of $U(N)$, $u \colon \partial_\pi \widetilde{M}\Gamma \to$

$U(N) \subset N(U(N)) \subset \mathbf{R}^{N^2}$, to obtain a continuous extension $\overline{\widetilde{M}\Gamma} \to \mathrm{End}(\mathbf{C}^{\mathrm{N}})$, and then use the retraction from $N(U(N))$ back to $U(N)$ to obtain a map $\hat{u}\colon \overline{\widetilde{M}\Gamma} \to \mathrm{End}(\mathbf{C}^{\mathrm{N}})$ such that $\hat{u}(x)$ is a unitary matrix for x in an open neighborhood of $\partial_\pi \widetilde{M}\Gamma$ in $\overline{\widetilde{M}\Gamma}$.

The Kasparov $(C^*(\widetilde{M}\Gamma,\pi), C(M))$-bimodule that represents $\partial_e[u]$ is constructed from $\hat{u}$ following the method of G. Yu [44], as adapted to the bivariant context. (Yu's method implements a K-theory duality in the index theory of coarse spaces, in the sense of Higson's original paper introducing these ideas [20].) Introduce the KK-cycle:

$$(\mathcal{E}_0 \oplus \mathcal{E}_1, \Phi = \begin{bmatrix} 0 & F^* \\ F & 0 \end{bmatrix}, \phi_0 \oplus \phi_1, \psi) \tag{4.4}$$

whose components are defined as follows:

• Let the integer N be determined by the representative $u : \partial_\pi \widetilde{M}\Gamma \to U(N)$. Recall that $\mathcal{H}_\Gamma$ consists of fiberwise sections of the Hermitian vector bundle $\mathbf{E} = \overline{\widetilde{M}\Gamma} \times \mathbf{C}^N$, so there is a natural module action ϕ_i of $C(M)$ on $\mathcal{H}_\Gamma$ via the induced map $\pi_1^*\colon C(M) \to C(\widetilde{M}\Gamma)$. Set $\mathcal{E}_i = \mathcal{H}_\Gamma$ for $i = 0, 1$.

• The matrix-valued function $\hat{u}$ induces a map of bundles $F\colon \mathcal{E}_0 \to \mathcal{E}_1$ which is an Hermitian isomorphism outside of a compact set in $\widetilde{M}\Gamma$.

• Let ψ be the diagonal representation of $C^*(\widetilde{M}\Gamma,\pi)$ on $\mathcal{E}_0 \oplus \mathcal{E}_1$. Note that ψ is a $C(M)$-representation, as the operators in $C^*(\widetilde{M}\Gamma,\pi)$ are fiberwise, and the module action of $C(M)$ is via fiberwise constant multipliers.

It is now routine to check

PROPOSITION 4.2 (cf. Lemma 3 [44]) $(\mathcal{E}_0 \oplus \mathcal{E}_1, \Phi, \phi_0 \oplus \phi_1, \psi)$ *defines a Kasparov* $(C^*(\widetilde{M}\Gamma,\pi), C(M))$*-bimodule. Its* KK*-class*

$$\partial_e[u] \in KK(C^*(\widetilde{M}\Gamma,\pi), C(M))$$

depends only on the class $[u] \in K^1(\partial_\pi \widetilde{M}\Gamma)$.

Proof: To establish that this yields a KK-bimodule, it only remains to check that for all $a \in C^*(\widetilde{M}\Gamma,\pi)$, the graded commutator $[\Phi, \psi(a)]$ is a uniformly fiberwise compact operator. This follows from the compact support of Φ and the bounded propagation property of a; the calculation follows exactly as that for a single complete open manifold by Higson and Roe (cf. Proposition 5.18, [40]). The homotopy invariance of the KK-class follows by standard methods (cf. Chapter VIII, [7]).

A similar construction applies in the even case, where we represent a K-theory class $[\mathbf{E}] \in K^0(\partial_\pi \widetilde{M}\Gamma)$ by a map to a Grassmannian embedded in Euclidean space, then follow the above outline to obtain

$$\partial_e[\mathbf{E}] \in KK_1(C^*(\widetilde{M}\Gamma,\pi), C(M)).$$

5 Coarse geometry and the exotic index

The exotic indices for an elliptic differential operator $\mathcal{D}$ on M are obtained from its Γ-index class by the pairing (4.1) with boundary classes.

Let $\mathcal{D}$ be a first order differential operator defined on the smooth sections $C^\infty(M, \mathbf{E}_0)$ of an Hermitian bundle $\mathbf{E}_0 \to M$, determining a K-homology class $[\mathcal{D}] \in K_*(M)$. The cap product $\cap\colon K_*(M) \otimes K^*(M) \to K_*(M)$ is realized by an operator pairing: given $[\mathbf{E}] \in K(M)$ the choice of an Hermitian metric on a representative $\mathbf{E} = \mathbf{E}_+ - \mathbf{C}^\ell$ determines an extension of $\mathcal{D}$ to an elliptic first order operator $\mathcal{D} \otimes \mathbf{E}$ acting on $C_c^\infty(M, \mathbf{E}_0 \otimes \mathbf{E})$. The K-homology class $[\mathcal{D} \otimes \mathbf{E}] \in K_*(M)$ represents $[\mathcal{D}] \cap \mathbf{E}$. We recall a fundamental result:

THEOREM 5.1 ([5]; Corollary 4.11 [30]) *Let M be a closed spin-manifold and $\mathcal{D} = \partial$ the Dirac operator on spinors, then $[\mathcal{D}] \cap : K^*(M) \to K_*(M)$ is the Poincaré duality isomorphism on K-theory.*

We define the Baum-Connes map [3] $\mu[\mathcal{D}]\colon K^*(M) \to K_*(C_r^*(\Gamma))$ in terms of the lifts of operators to coverings: For $[\mathbf{E}] \in K^*(M)$, the lift of $\mathcal{D} \otimes \mathbf{E}$ to a Γ-invariant differential operator $\widetilde{\mathcal{D} \otimes \mathbf{E}}$ on the compactly supported sections $C_c^\infty(\widetilde{M}, \widetilde{\mathbf{E}_0 \otimes \mathbf{E}})$ has a Γ-index

$$\mu[\mathcal{D}]\,[\mathbf{E}] = [\widetilde{\mathcal{D} \otimes \mathbf{E}}] \in K_*(C_r^*(\Gamma))$$

When $M = B\Gamma$ the Baum-Connes map can be written simply as $\mu[\mathcal{D}]([\mathbf{E}]) = \beta \circ [\mathcal{D}] \cap [\mathbf{E}]$.

DEFINITION 5.2 *For each class $[u] \in K^p(\partial_\pi \widetilde{M}\Gamma)$ and K-homology class $[\mathcal{D}] \in K_q(M)$, define the exotic index map*

$$\mathrm{Ind}_e([u], [\mathcal{D}]) = B_e[u] \circ \mu[\mathcal{D}] : K^*(M) \longrightarrow K^{*+p+q+1}(M) \tag{5.1}$$

The exotic index map (5.1) can be evaluated in terms of ordinary indices of a family [14]. Let $\tilde{\mathcal{D}}_\pi$ denote the differential operator along the fibers of $\pi\colon \widetilde{M}\Gamma \to M$, obtained from the suspension of the Γ-invariant operator $\tilde{\mathcal{D}}$. It determines a Connes-Skandalis index class $\mathrm{Ind}_\pi(\tilde{\mathcal{D}}_\pi) \in KK_*(C_0(\widetilde{M}\Gamma), C(M))$ where in the case of graded operators we have suppressed notation of the grading. Use the boundary map δ in K-theory for $(\widetilde{M}\Gamma, \partial_\pi \widetilde{M}\Gamma)$ and the the KK-external product

$$KK(\mathbf{C}, C_0(\widetilde{M}\Gamma)) \otimes KK_*(C_0(\widetilde{M}\Gamma), C(M)) \longrightarrow KK(\mathbf{C}, C(M)) \cong K^*(M)$$

to pair $\delta[u] \in KK(\mathbf{C}, C_0(\widetilde{M}\Gamma))$ with $\mathrm{Ind}_\pi(\tilde{\mathcal{D}}_\pi)$ to obtain a map

$$\mathrm{Ind}_\pi\Big(\delta[u] \otimes \tilde{\mathcal{D}}_\pi\Big) \in K^*(M).$$

The Kasparov pairing $\delta[u] \otimes (\tilde{\mathcal{D}}_\pi)$ is operator homotopic to a family of fiberwise operators over M which are invertible off a compact set – exactly the class of operators considered by Gromov and Lawson [18, 39]. An elegant homotopy argument of G. Yu for the indices of special vector bundles on an open complete manifold (Theorem 2, [44]) adapts to the parametrized case to relate these two indices in $K^*(M)$:

PROPOSITION 5.3 (Exotic families index theorem)

$$\mathrm{Ind}_e([u],[\mathcal{D}])[\mathbf{E}] = \mathrm{Ind}_\pi\Big(\delta[u] \otimes (\widetilde{\mathcal{D} \otimes \mathbf{E}})_\pi\Big) \in K^*(M) \tag{5.2}$$

REMARK 5.4 Formula (5.2) gives an expression for the exotic indices as the indices of a family of operators — a decidedly non-exotic index. The notation that $\mathrm{Ind}_e([u],[\mathcal{D}])[\mathbf{E}]$ is an "exotic" index is retained because it results from pairing with coefficients $\delta[u]$ that are the transgression of a K-theory class "at infinity". The description of the index invariants as "exotic" thus parallels exactly the usage in characteristic class theory, where classes arising from boundary constructions are usually called exotic or secondary. It should be noted that almost all other authors now refer to such constructions as *coarse* invariants, due to the role of coarse geometry in their construction.

6 Coarse Bott classes

The notion of a *coarse Bott class* in K-theory for a family of pseudometric spaces is introduced, and we prove the map $\mu\colon K_*(B\Gamma) \to K_*(C^*_r\Gamma)$ is injective for a group Γ with a coarse Bott class coming from the corona. There is a close analogy between the coarse techniques of this section and the constructions of "proper Lipschitz cocycles" introduced by Connes-Gromov-Moscovici [12].

The definition of coarse Bott classes is motivated by the Bott class for vector bundles:

DEFINITION 6.1 *Let $M = \widetilde{M}/\Gamma$ be a connected, compact spin-manifold, and ∂ the Dirac operator on spinors for M. We say that $\Theta \in K^*(\widetilde{M}\Gamma)$ is a* coarse Bott class *if there exists $[u_\Theta] \in K^*(\partial_\pi\widetilde{M}\Gamma)$ so that $\Theta = \delta[u_\Theta]$, and for some $x \in M$, hence for all x, the index of the operator $\partial_x \otimes (\Theta|\widetilde{M}_x)$ on the fiber $\widetilde{M}_x$ satisfies $\mathrm{Ind}(\partial_x \otimes (\Theta|\widetilde{M}_x)) = \pm 1$.*

Here is the main result of this note for the case where $B\Gamma \cong M$ is compact:

THEOREM 6.2 *Let Γ be a group so that the classifying space $B\Gamma$ is represented by a complete, orientable Riemannian spin manifold $M = \widetilde{M}/\Gamma$ such that there is a coarse Bott class $\Theta \in K^*(\widetilde{M}\Gamma)$. Then $\beta\colon K_*(B\Gamma) \to K_*(C^*_r\Gamma)$ is injective.*

Proof: Let $u_\Theta \in K^*(\partial_\pi \widetilde{M}\Gamma)$ with $\Theta = \delta u_\Theta$ be a coarse Bott class, and let ∂ be the Dirac operator on M. By Theorem 5.1 it will suffice to show that the exotic index map

$$\mathrm{Ind}_e([\mathcal{D}], [u_\Theta]) = B_e[u_\Theta] \circ \mu[\mathcal{D}] : K^*(M) \longrightarrow K^*(M)$$

is injective. This in turn follows from a simple topological observation and a calculation. The observation is contained in the following lemma, which is a simple reformulation of Kasparov's "homotopy lemma" (cf. the proof of **Theorem**, section 6.5, page 193 [30]).

LEMMA 6.3 *For each* $[\mathbf{E}] \in K^*(M)$, $\Theta \otimes \pi_1^*[\mathbf{E}] = \Theta \otimes \pi_2^*[\mathbf{E}] \in K^*(\widetilde{M}\Gamma)$

Proof: The projections $\pi_i \colon \widetilde{M}\Gamma \to M$ for $i = 1, 2$ are homotopic, so the induced module actions π_1^* and π_2^* of $K^*(M)$ on $K(\widetilde{M}\Gamma)$ coincide. $\square$

Theorem 6.2 now follows from a calculation using Proposition 5.3 (the exotic families index theorem) applied to the the fibration $\pi = \pi_1 \colon \widetilde{M}\Gamma \to M$.

LEMMA 6.4 *For each* $[\mathbf{E}] \in K^*(M)$, *the exotic index* $\mathrm{Ind}_e([u_\Theta], [\partial])[\mathbf{E}] \in K^*(M)$ *is non-zero.*

Proof:

$$\begin{aligned}
\mathrm{Ind}_e([u_\Theta], [\partial])[\mathbf{E}] &= \mathrm{Ind}_\pi\Big(\Theta \otimes (\partial \widetilde{\otimes \mathbf{E}})_\pi\Big) \\
&= \mathrm{Ind}_\pi\Big(\Theta \otimes \pi_2^* \mathbf{E} \otimes \widetilde{\partial}_\pi\Big) \\
&= \mathrm{Ind}_\pi\Big(\Theta \otimes \pi_1^* \mathbf{E} \otimes \widetilde{\partial}_\pi\Big) \\
&= \mathrm{Ind}_\pi\Big(\Theta \otimes \widetilde{\partial}_\pi\Big) \otimes [\mathbf{E}] \\
&\neq 0
\end{aligned}$$

The last conclusion follows as $\mathrm{Ind}(\partial \otimes (\Theta|\widetilde{M}_x)) = \pm 1$ for all $x \in M$ implies $\mathrm{Ind}_\pi\Big(\Theta \otimes \widetilde{\partial}_\pi\Big) \in K^*(M)$ is invertible in $K^*(M)$.

Note that one can weakened the notion of a coarse Bott class, requiring only that the indices of the operators $\partial_x \otimes (\Theta|\widetilde{M}_x)$ on the fibers $\widetilde{M}_x$ satisfy $\mathrm{Ind}(\partial_x \otimes (\Theta|\widetilde{M}_x)) \neq 0$. Then in the above proof, $\mathrm{Ind}_\pi\Big(\Theta \otimes \widetilde{\partial}_\pi\Big) \in K^*(M)$ will be invertible in $K^*(M) \otimes \mathbf{Q}$ which implies the rational injectivity of the operator assembly map β in this case. We expect this more general case will prove useful for the study of further classes of groups.

7 The relative case

We extend Theorem 6.2 to the case where $B\Gamma \cong M$ is a complete open manifold. We do not assume that M has finite type, so the main result of this section, Theorem 7.6, implies the Novikov Conjecture for certain classes groups $\Gamma = \pi_1(M)$ which need not be of finite type.

Fix a complete Riemannian metric on TM. As in the compact case, the metric lifts to a Γ-invariant Riemannian metric on $T\widetilde{M}$, and the induced path-length metric on $\widetilde{M}$ is Γ-invariant and complete. Endow the fibers of $\widetilde{M}\Gamma \to M$ with the quotient metric obtained from that on $\widetilde{M}$ – for each $x \in M$ the fiber $\widetilde{M}_x = \pi_1^{-1}(x)$ is isometric to $\widetilde{M}$. Recall that $\Delta\colon M \to \widetilde{M}\Gamma$ is the quotient of the diagonal mapping $\Delta\colon \widetilde{M} \to \widetilde{M} \times \widetilde{M}$.

$\mathcal{B}^*_\rho(\widetilde{M}\Gamma, \pi)$ is the C^*-algebra with typical element a family of operators $A = \{A_x \in \mathcal{B}^*_\rho(\widetilde{M}_x) \mid x \in M\}$ with uniformly bounded propagation, exactly as in § 2 for the compact case, with a natural isomorphism

$$\mathcal{B}^*_\rho(\widetilde{M}\Gamma, \pi) \cong C_{up}(\widetilde{M}, \mathcal{B}_\rho(\widetilde{M}))^\Gamma \tag{7.1}$$

A uniformly continuous function $\psi\colon \widetilde{M}\Gamma \to \mathbf{C}$ has Δ-*compact support* if there exists a constant $R_\psi > 0$ so that the support of ψ is contained in the fiberwise R_ψ-tube $N(\Delta, R_\psi)$ around the diagonal $\Delta(M) \subset \widetilde{M}\Gamma$:

$$N(\Delta, R_\psi) = \{y \in \widetilde{M}\Gamma \mid d_x(y, \Delta(\pi_1(y))) \leq R_\psi\}$$

The construction of the parametrized corona for $\widetilde{M}\Gamma$ remains unchanged in the case where M is non-compact: Γ acts naturally on the Higson corona $\partial_h \widetilde{M}$, and we set

$$\partial_\pi \widetilde{M}\Gamma \equiv (\widetilde{M} \times \partial_h \widetilde{M})/\Gamma$$

Let $\pi\colon \partial_\pi \widetilde{M}\Gamma \to M$ be induced by projection onto the first factor.

A key difference in the non-compact case is that we must introduce the "locally finite" K-theory $K^*_{lf}(\partial_\pi \widetilde{M}\Gamma)$ of $\partial_\pi \widetilde{M}\Gamma$. An *exhaustion for M* is a nested increasing sequence $\{M_n\}$ of closed submanifolds $M_0 \subset M_1 \subset \cdots M$ whose union is all of M. For each $n \geq 1$ define $\widetilde{M}_n\Gamma = \pi^{-1}(M_n)$ and define its corona to be $\partial_\pi \widetilde{M}_n\Gamma = \partial\pi^{-1}(M_n)$.

We will take for an operating definition

$$\begin{aligned} K^*_{lf}(\partial_\pi \widetilde{M}\Gamma) = \{u \;\; &= \;\; \lim_{\leftarrow}[u_n] \in \lim_{\leftarrow} K^*(\partial_\pi \widetilde{M}_n\Gamma) \mid \exists R > 0 \;\; \forall n > 0 \\ &\qquad \delta[u_n] \text{ has support in } N(\Delta, n, R)\} \end{aligned}$$

where $N(\Delta, n, R) = \{y \in \widetilde{M}_n\Gamma \mid d_{\pi(y)}(y, \Delta\pi(y)) \leq R\}$

The support condition on an element $u = \lim_{\leftarrow}[u_n]$ has a subtle aspect that we comment on. The sequence of representatives $\{[u_n]\}$ are supported in a uniform tube $N(\Delta, R) \subset \widetilde{M}\Gamma$ around $\Delta(M)$. If we suppose that M admits a compactification to a manifold with boundary, $\overline{M}$, then $\widetilde{M}\Gamma$ can be trivialized on a collar neighborhood of $\overline{M} - M$ hence $\widetilde{M}\Gamma \to M$ extends to a fibration over $\overline{M}$, with a section

$$\tau: \overline{M} \to (\widetilde{\overline{M}} \times \widetilde{M})/\Gamma$$

With respect to the "intuitively natural" section τ, the tube $N(\Delta, R)$ intersects the fiber $\widetilde{M}_x$ in a section which becomes infinitely far from $\tau(x)$ as x tends to the boundary $\overline{M} - M$. In particular, given any compact set $K \subset \widetilde{M}\Gamma$, the distance between $\{y \in \widetilde{M}_x \mid d_x(y, \Delta(x)) \leq R\}$ and K tends to infinity as x tends to the boundary $\overline{M} - M$. This latter condition makes sense whether M admits a manifold compactification or not, and suggests that the proper way to consider the fiberwise supports of the sections $\{[u_n]\}$ is that they lie in supports tending to infinity with respect to a geometric "cross-section".

Another unique aspect of the open manifold case, is the need to introduce the algebra of fiberwise operators $A = \{A_x \mid x \in M\}$ on $\widetilde{M}\Gamma$ whose support in a fiber $\widetilde{M}_x$ tends to infinity with respect to the basepoint $\Delta(x)$ as x tends to infinity in M. To be precise, introduce the subalgebra $\mathcal{B}^*_{c\rho}(\widetilde{M}\Gamma, \pi) \subset \mathcal{B}^*_{\rho}(\widetilde{M}\Gamma, \pi)$ generated by the continuous fields of operators $A = \{A_x \mid x \in M\}$ such that for each Δ-compactly supported function $\psi: \widetilde{M}\Gamma \to \mathbf{C}$, the field $\psi \cdot A = \{\psi \cdot A_x \mid x \in M\}$ has compact support in M. Let $C^*_{c\rho}(\widetilde{M}\Gamma, \pi) \subset C^*_{\rho}(\widetilde{M}\Gamma, \pi)$ denote the C^*-closure of $\mathcal{B}^*_{c\rho}(\widetilde{M}\Gamma, \pi)$.

PROPOSITION 7.1 *There is a natural injective map of unital C^*-algebras, $\mathcal{C}: C^*_r(\Gamma) \to C^*_{c\rho}(\widetilde{M}\Gamma, \pi)$, with the coarsening map for Γ given by the induced map on K-theory,*

$$\chi: K_*(C^*_r(\Gamma)) \to K_*(C^*_{c\rho}(\widetilde{M}\Gamma, \pi)).$$

Proof: Choose a compactly supported function $\xi \in C_0(M) \subset L^2(M)$ with L^2-norm one, and compact support $K_\xi \subset M$. The function ξ defines a projection $P_\xi \in \mathcal{K} \cong \mathcal{K}(M)$ which lifts to a Γ-invariant operator $\widetilde{P_\xi}$ on $\widetilde{M}$. Let P_ξ also denote the fiberwise operator in $\mathcal{B}^*_{\rho}(\widetilde{M}\Gamma, \pi)$. The product $\psi \cdot P_\xi$ then has compact support in the R-tube over the support K_ξ of ξ.

As in the proof of Proposition 2.2, $\widetilde{P_\xi}$ determines a map

$$P^*_\xi: C^*_r(\Gamma) \to C^*_r(\Gamma) \hat{\otimes} \mathcal{K} \subset C^*(\widetilde{M}\Gamma, \pi)$$

whose image similarly consists of classes represented by operators with supports contained in the fibers over the support of ξ, hence lie in $C^*_{c\rho}(\widetilde{M}\Gamma, \pi)$. $\square$

The construction of the boundary pairing B_e proceeds exactly as before, where we note the important detail that by Proposition 7.1 its range is the K-theory with compact supports of M:

THEOREM 7.2 *There is a natural pairing, for $p, q = 0, 1$,*

$$B_e \colon K_q(C_r^*(\Gamma)) \otimes K_{lf}^p(\partial_\pi \widetilde{M}\Gamma) \longrightarrow K^{p+q+1}(M) \tag{7.2}$$

The definition of coarse Bott classes is modified as follows:

DEFINITION 7.3 *Let $M = \widetilde{M}/\Gamma$ be a connected, complete spin-manifold, and ∂ the Dirac operator on spinors for M. We say that $\Theta \in \lim\limits_{\leftarrow} K^*(\widetilde{M}_n\Gamma)$ is a* coarse Bott class *if there exists $[u_\Theta] \in K_{lf}^*(\partial_\pi \widetilde{M}\Gamma)$ so that $\Theta = \delta[u_\Theta]$, and for some $x \in M$ hence for all x, the fiber index* $\mathrm{Ind}(\partial \otimes (\Theta|V_x)) = \pm 1$.

With the previous set-up, we now have the following extension of Lemma 6.3 within the framework of the fiberwise foliation index theorem applied to $\pi \colon \widetilde{M}\Gamma \to M$:

LEMMA 7.4 *Let $\Theta \in \lim\limits_{\leftarrow} K^*(\widetilde{M}_n\Gamma)$ be a coarse Bott class. For each $[\mathbf{E}] \in K^*(M)$,*

$$\mathrm{Ind}_\pi\Big(\Theta \otimes \widetilde{\partial}_\pi \otimes \pi_2^*\mathbf{E}\Big) = \mathrm{Ind}_\pi\Big(\Theta \otimes \widetilde{\partial}_\pi \otimes \pi_1^*\mathbf{E}\Big) \tag{7.3}$$

Proof: By Theorem 7.2 the index class of the operator $\Theta \otimes \widetilde{\partial}_\pi$ is supported in a compact subset of $\widetilde{M}\Gamma$, so we can apply the homotopy argument from the proof of Lemma 6.3. $\square$

Poincaré duality for a manifold with boundary extends to exhaustion sequences:

THEOREM 7.5 (Corollary 4.11 [30]) *Let M be a complete spin-manifold and ∂ the Dirac operator on spinors. Then $[\partial] \cap : K^*(M) \to K_*(M)$ is the Poincaré duality isomorphism on K-theory.*

Finally, we have established the preliminaries needed to prove the main result of this note, the version of Theorem 6.2 applicable for M an open complete manifold:

THEOREM 7.6 *Let Γ be a group so that the classifying space $B\Gamma$ is represented by a complete, orientable Riemannian spin manifold $M = \widetilde{M}/\Gamma$ such that there is a coarse Bott class $\Theta \in K^*(\widetilde{M}\Gamma)$. Then $\beta \colon K_*(B\Gamma) \to K_*(C_r^*\Gamma)$ is injective.*

Proof: This follows exactly the outline of the proof of Theorem 6.2, where we use the natural pairing

$$\varprojlim K^*(M_n) \otimes K^*(M) \longrightarrow K^*(M),$$

Poincaré duality as in Theorem 7.5, and the fact that the class

$$\mathrm{Ind}_\pi\left(\Theta \otimes \widetilde{\partial}_\pi\right) \in \varprojlim K^*(M_n)$$

is invertible. □

8 Applications

In this final section, we give three illustrations of the use of Theorem 6.2.

EXAMPLE 8.1 (cocompact lattices) Let $\Gamma \subset G$ be a torsion-free uniform lattice in a connected semi-simple Lie group G with finite-center. Choose a maximal compact subgroup $K \subset G$, then the double quotient $M = K\backslash G/\Gamma$ is compact manifold, as Γ is torsion-free and discrete. The inverse of the geodesic exponential map is a degree-one proper Lipschitz map $\log\colon V \to \mathbf{R}^n$. The geodesic ray compactification of V, corresponding the the spherical compactification of $\mathbf{R}^n$ via the map log, is a Γ-equivariant quotient of the corona $\partial_h V$ (this is the usual Miščenko calculation), so there is a fiberwise degree-one map $\widetilde{M}\Gamma \to TM$. Thus the usual Bott class in $TM \to M$ pulls back to a Bott class for $\widetilde{M}\Gamma$.

For M a spin manifold, we have satisfied all of the hypotheses of Theorem 6.2 so the operator assembly map $\beta\colon K_*(B\Gamma) \to K_*(C_r^*\Gamma)$ is injective.

The case where G is simply connected nilpotent Lie group follows similarly, except that the construction of the Γ-equivariant spherical-quotient of the corona $\partial_h V$ uses the *special vector field* technique of Rees [37].

This example reproduces the Miščenko method [35, 36] in the "coarse language"; the cases discussed below are seen to be just successive embellishments of it.

EXAMPLE 8.2 (non-uniform lattices) Consider a discrete, non-uniform torsion-free subgroup $\Gamma \subset G$ of a connected semi-simple Lie group G with finite-center. Choose a maximal compact subgroup $K \subset G$, then the quotient $\widetilde{M} = K\backslash G$ is ultra-spherical for a K-G bi-invariant Riemannian metric on TG. The double quotient $M = K\backslash G/\Gamma$ is a complete open manifold, to which we apply the methods of section 7.

This calculation reproduces the Kasparov's method [28, 29, 30] for handling non-uniform lattices. Kasparov's construction of the "realizable K-functor for C^*-algebras" has been replaced with the usual inverse limit construction on the K-theory of the corona.

EXAMPLE 8.3 (word hyperbolic groups) We give a sketch of the proof of Theorem 1.2 for the case where Γ satisfies the following hypotheses: there is an open manifold M representing $B\Gamma$ such that $\widetilde{M}\Gamma$ has a coarse Bott class. Assume

- the classifying space $B\Gamma$ has finite type.
- The universal covering $\widetilde{M}$ of M admits a metrizable, contractable compactification $\overline{\widetilde{M}}$.
- The right deck translation action of Γ on V extends continuously to $\overline{\widetilde{M}}$.
- For any compact subset $K \subset E\Gamma$ and sequence of group elements $\{\gamma_n\} \subset \Gamma$ which tend to infinity in the word norm, the translates $K \cdot \gamma_n$ have diameter tending to zero.

These assumptions imply that $\overline{\widetilde{M}}$ is an equivariant quotient of the Higson compactification of $\widetilde{M}$, so it will suffice to construct a class $\Theta \in K^*(\widetilde{M}\Gamma)$ which is the boundary of a class $\theta \in K^*((\widetilde{M} \times \overline{\widetilde{M}})/\Gamma)$.

Use the exponential map to define a local diffeomorphism from an open δ-neighborhood $N(TM, \delta) \subset TM$ of the zero section in TM to an open neighborhood of $*M \subset \widetilde{M}\Gamma$. Choose a Bott class in $H^*(TM)$ supported near the diagonal. It will pull back to a compactly supported class around the diagonal of $\widetilde{M}\Gamma$. This is induced from a relative class in $H^*((\widetilde{M} \times \overline{\widetilde{M}})/\Gamma, (\widetilde{M} \times \delta\overline{\widetilde{M}})/\Gamma)$. Replacing M with $M \times S^1$ we can assume that the restriction of the Bott class to the diagonal is trivial in cohomology, hence the image of the pull-back to $H^*((\widetilde{M} \times \overline{\widetilde{M}})/\Gamma, (\widetilde{M} \times \delta\overline{\widetilde{M}})/\Gamma)$ maps to the trivial class in $H^*((\widetilde{M} \times \overline{\widetilde{M}}))$. (This uses that $(\widetilde{M} \times \overline{\widetilde{M}})/\Gamma$ retracts to the diagonal, or that the added boundary to $\widetilde{M}\Gamma$ is a Z-set.) Choose $\omega \in H^{*-1}((\widetilde{M} \times \delta\overline{\widetilde{M}})/\Gamma)$ which maps to the pull-back class. Now use the isomorphism between rational K-theory and rational cohomology to pull these cohomology classes on the pair $((\widetilde{M} \times \overline{\widetilde{M}})/\Gamma, (\widetilde{M} \times \delta\overline{\widetilde{M}})/\Gamma)$ back to K-theory classes. This yields the θ and Θ required to have a coarse Bott class.

The more general case, where $B\Gamma$ is simply a finite CW complex, requires embedding $B\Gamma$ into Euclidean space, taking a regular neighborhood M of $B\Gamma$, then repeating this argument for the open manifold M. However, one must show that $\widetilde{M}$ admits a Z-set compactification, given that $\widetilde{M}\Gamma$ admits one. The proof of this uses an engulfing technique similar to that in [12].

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