Tangential LS category and cohomology for foliations

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Abstract. The purpose of this paper is to give lower-bound estimates on the tangential category $\mathrm{cat}_{\mathcal{F}}(M)$ of a foliated manifold in terms of cup-length in the associated foliation spectral sequence. We first show that the nilpotency index of the reduced filtered cohomology, for $r > 0$, provides a lower bound $\mathrm{nil}\, E_r^{*,+}(M, \mathcal{F}) \leq \mathrm{cat}_{\mathcal{F}}(M)$. This generalizes the estimate by the first author $\mathrm{nil}\, H_{\mathcal{F}}^{+}(M) \leq \mathrm{cat}_{\mathcal{F}}(M)$ in terms of the nilpotency of the foliated cohomology.

The second theme of this paper is to develop tools for showing that a particular cup-product in $H^{+}(M, \mathcal{F})$ or $E_1^{*,+}(M, \mathcal{F})$ is non-zero. We develop three approaches to this problem: pairing with the foliation current associated to a transverse invariant measure; pairing with foliation k-currents associated with elements of Haefliger's transverse cohomology; and evaluation of the Godbillon-Vey class of the foliation.

Singhof and Vogt [**44**] proved that for a foliation with leaf dimension m, $\mathrm{cat}_{\mathcal{F}}(M) \leq m + 1$. We give several classes of foliations for which the lower bound estimates using cup-length is $\mathrm{cat}_{\mathcal{F}}(M) \geq m+1$, hence $\mathrm{cat}_{\mathcal{F}}(M) = m+1$. For example, we prove that the category of the foliation defined by a locally free $\mathbb{R}^m$ action on a compact manifold is $m+1$. We also show that the category is $m + 1$ for the weak-stable foliation of dimension m associated to a contact Anosov flow on a compact manifold. These calculations extend to products of the foliations considered.

Contents

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1. Introduction

The manifolds, maps and foliations considered in this paper are assumed to be smooth unless otherwise noted. Let M be a manifold without boundary and $\mathcal{F}$ a foliation. Let m denote the leaf dimension and n the codimension of $\mathcal{F}$. Key definitions and background results are given in §§2,3.

The purpose of this paper is to give lower-bound estimates on the tangential category $\mathrm{cat}_{\mathcal{F}}(M)$ of the foliated manifold using cup-length in the associated foliation spectral sequence. When the estimates of this paper are combined with the upper bound estimate $\mathrm{cat}_{\mathcal{F}}(M) \leq m+1$ of Singhof and Vogt [**44**], this yields in many cases an exact calculation of the tangential category.

The foliation spectral sequence $E_r^{p,q}(M,\mathcal{F}) \Longrightarrow H^*(M)$ is a natural tool for the study of the geometry of $\mathcal{F}$. For example, the foliated cohomology $H^q_{\mathcal{F}}(M)$ studied by many authors is the first derived cohomology group $E_1^{0,q}(M,\mathcal{F})$. In the thesis of the first author, it was shown that the nilpotency index of the reduced foliated cohomology $H^+_{\mathcal{F}}(M)$ is a lower bound for $\mathrm{cat}_t(M,\mathcal{F})$. The basic idea of this paper is to extend this lower bound to the derived cohomology algebras in the foliation spectral sequence, and that there are many classes of foliations for which this extension yields an exact calculation of the tangential category.

The following is proved in §3.

Theorem 1.1.

$$\text{nil } E_\infty^{*,+} \leq \text{nil } E_2^{*,+} \leq \text{nil } E_1^{*,+} \leq \mathrm{cat}_{\mathcal{F}}(M) \tag{1.1}$$

$$\text{nil } H^+_{\mathcal{F}}(M) \leq \text{nil } E_1^{*,+} \leq \mathrm{cat}_{\mathcal{F}}(M) \tag{1.2}$$

One of the difficulties with applying the estimates (1.1) or (1.2) is that the groups $E_r^{*,+}(M,\mathcal{F})$ are often intractable to calculate. The second theme of this paper is to develop tools for showing that a particular cup-product in $H^+(M,\mathcal{F})$ or $E_1^{*,+}(M,\mathcal{F})$ is non-zero, which is then useful to obtain a lower bound on the nilpotency index.

Recall that a transverse invariant measure μ for a foliation $\mathcal{F}$ is a Borel measure defined on transversals, so that $\mu(T) = \mu(h(T))$ if h is an element of holonomy of $\mathcal{F}$ and T is a transverse manifold in the domain of μ. This concept was introduced by J. Plante [**34**]. Not all foliations admit a non-trivial transverse invariant measure, and in codimension one this is an especially strong hypothesis to make. In any case, they do frequently arise naturally. Ruelle and Sullivan observed that a transverse invariant measure yields a "homology fundamental class" $[\mu] \in H_m(M)$ for a foliation [**34, 36, 45**]. If the measure μ is defined by a integration of a closed n-form ω along transversals to $\mathcal{F}$, then $[\mu]$ is just the Poincaré dual to $[\omega]$.

The following is proved in §4.

Theorem 1.2. *Let M be a compact manifold, and μ is non-trivial transverse invariant measure for a foliation $\mathcal{F}$ with leaf dimension m. Then there exists a natural map*

$$\int_\mu : E_1^{0,m}(M,\mathcal{F}) \to \mathbb{R}$$

which is non-vanishing on the leafwise volume form of $\mathcal{F}$.

We prove more generally in §4 that

THEOREM 1.3. *There is a natural map to the transverse cohomology* $H^p(Tr\ \mathcal{F})$*,*

$$\chi_* \colon E_2^{p,m}(M, \mathcal{F}) \to H^p(Tr\ \mathcal{F})$$

Here is an application of Theorem 1.2, proved in §6.1.

THEOREM 1.4. *Let* M *be a compact manifold, and assume* $\mathcal{F}$ *is defined by a locally free action* $\mathbb{R}^m \times M \to M$*. Then* $\mathrm{cat}_{\mathcal{F}}(M) = m + 1$*.*

The following result, proved in §5 as an application of Theorem 1.1, generalizes a result of H. Shulman [**40**].

THEOREM 1.5. *Suppose that* $GV(\mathcal{F}) \neq 0$*, then* $\mathrm{cat}_{\mathcal{F}}(M) \geq n + 2$*.*

COROLLARY 1.6. *Suppose* M *is a compact manifold of dimension* $2n + 1$ *with a smooth foliation of codimension* n*, with* $GV(\mathcal{F}) \neq 0$*. Then* $\mathrm{cat}_{\mathcal{F}}(M) = n + 2$*.*

The paper of Singhof and Vogt [**44**] proves that category is upper semi-continuous function on the space of foliations in the C^1 topology. They show how to calculate the tangential category of a number of examples by perturbing the given foliation to one with a compact leaf, and then bounding the category from below by that of this leaf. This novel technique produces a systematic calculation of category for many classes of foliations. In §6 of this paper, we apply the techniques of this paper to obtain lower bound estimates for several classes of foliations. These estimates are exact, and yield their transverse categories. The examples typically do not admit perturbations to foliations with compact leaves – for example, the foliations are structurally stable in some cases. Thus, the transverse categories of the examples cannot be obtained using the perturbation method of Singhof and Vogt. The list of examples is not exhaustive, but chosen to illustrate the techniques.

We conclude the paper in §7 by listing various open problems about the tangential category.

2. Tangential category

Let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be foliated manifolds. A map $f\colon (M,\mathcal{F}) \to (M',\mathcal{F}')$ is said to be foliated if it sends leaves into leaves. A homotopy $H\colon M\times\mathbb{R}\to M'$ is said to be *integrable* if H is a foliated map, considering $M\times\mathbb{R}$ to be foliated with leaves $L\times\mathbb{R}$, $L\in\mathcal{F}$. The notation $\simeq_{\mathcal{F}}$ will denote integrable homotopy. Given an integrable homotopy H, for all $t\in\mathbb{R}$ we have a foliated map $H_t\colon (M,\mathcal{F})\to (M',\mathcal{F}')$. Moreover, for each $x\in M$ the curve $t\mapsto H_t(x)$ is a leafwise curve in M'. Thus, an integrable homotopy is exactly a homotopy for which all of the "traces" are leafwise curves. As a consequence, it is easy to see that if $f\simeq_{\mathcal{F}} g$ then f and g induce the same map between the spaces of leaves.

An open subset U of M is tangentially categorical if the inclusion map $(U,\mathcal{F}_U)\hookrightarrow (M,\mathcal{F})$ is integrably homotopic to a foliated map $c\colon U\to M$ which is constant on each leaf of $\mathcal{F}_U$. Here U is regarded as a foliated manifold with the foliation $\mathcal{F}_U$ induced by $\mathcal{F}$ on U. The leaves of $\mathcal{F}_U$ are the connected components of $L\cap U$, where L is a leaf of $\mathcal{F}$. A major technical point about this definition of a categorical set is that while the map c is constant on the leaves of $\mathcal{F}_U$, it need not be constant on the sets $L\cap U$ unless these sets are connected.

Definition 2.1. The tangential category $\mathrm{cat}_{\mathcal{F}}(M)$ of a foliated manifold $(M,\mathcal{F})$ is the least number of tangentially categorical open sets required to cover M. If no such covering exists, let $\mathrm{cat}_{\mathcal{F}}(M)=\infty$.

When $\mathcal{F}$ is a foliation by a single leaf, an open subset is tangentially categorical if and only if it is categorical, so $\mathrm{cat}_{\mathcal{F}}(M)=\mathrm{cat}\, M$. For a foliation by points, we have $\mathrm{cat}_{\mathcal{F}}(M)=1$.

A distinguished open set of a foliated chart is always categorical, so $\mathrm{cat}_{\mathcal{F}}(M)$ is finite if $\mathcal{F}$ is a foliation of a compact manifold.

Tangential category is an invariant of integrable homotopy between manifolds.

Each leaf L of a foliation supports two different topologies: the submanifold topology τ_L which has the plaques as a basis, and the relative topology $\tau\subset\tau_L$ induced by the ambient manifold. We have that $\mathrm{cat}(L,\tau_L)\le\mathrm{cat}(L,\tau)$. In this paper, unless otherwise specified, we will assume $\mathrm{cat}\, L=\mathrm{cat}(L,\tau_L)$, the category of the leaf as a submanifold of M.

Since the integrable homotopy on a tangentially categorical subset U restricted to a leaf of $\mathcal{F}_U$ gives a contraction on the corresponding leaf of $\mathcal{F}$, we have

Proposition 2.2 ([**9**]). *For any leaf $L\in\mathcal{F}$, $\mathrm{cat}\, L\le\mathrm{cat}_{\mathcal{F}}(M)$.*

We will say that a foliated manifold is *tangentially contractible* if $\mathrm{cat}_{\mathcal{F}}(M)=1$. In this case every leaf is contractible and closed. Thus, if M is a compact manifold, $(M,\mathcal{F})$ is tangentially contractible if and only if $\mathcal{F}$ is a foliation by points. The contractibility of every leaf is not sufficient to contract tangentially the manifold, as shown by the example of the linear foliation on the torus with irrational slope. For an example on a non-compact manifold, the Reeb foliation of the plane $\mathbb{R}^2$ also has contractible leaves and $\mathrm{cat}_{\mathcal{F}}(M)=2$.

2.1. Fibrewise category. We now compare the tangential category with the *fibrewise category* introduced by I.M. James and J.R. Morris in [**28**]. Recall that a fibrewise space X over B is a topological space X together with a map $p\colon X\to B$. An open set $U\subset X$ is said to be fibrewise categorical if there exists a *global* section $s\colon B\to X$ such that the inclusion $i_U\colon U\to M$ and the map $c=s\circ p\,|_U\colon U\to M$

are fibrewise homotopic (i.e. integrably homotopic in the continuous sense). The fibrewise category $\mathrm{cat}_B X$ is the least number of fibrewise categorical open sets required to cover X. If no such covering exists, the fibrewise category is said to be infinite.

Consider a smooth version of the fibrewise category, where we require that all objects and maps are smooth. If the projection $p\colon M \to M/\mathcal{F}$ onto the space of leaves has a global section, we can compare the tangential category with the fibrewise category of M as a fibrewise space over $M/\mathcal{F}$. It is clear that

PROPOSITION 2.3. $\mathrm{cat}_{\mathcal{F}}(M) \leq \mathrm{cat}_{M/\mathcal{F}} M$

Cohomological lower bounds for fibrewise category are obtained by taking the quotient of the reduced cohomology $\tilde{H}^*(X)$ by the ideal $\langle p^*\tilde{H}^*(B)\rangle \subset \tilde{H}^*(X)$ generated by the subring $p^*\tilde{H}^*(B)$. James and Morris proved that

$$\mathrm{cat}_B X \geq \mathrm{nil}\, \frac{\tilde{H}^*(X)}{\langle p^*\tilde{H}^*(B)\rangle}.$$

The existence of a good quotient space $M/\mathcal{F}$, and moreover of a section $M/\mathcal{F} \to M$, are very strong assumptions, so the fibrewise category has limited application for the study of foliations.

2.2. Foliated cohomology. We can obtain cohomological lower bounds for the tangential cohomology by considering the foliated cohomology [**11, 20, 29, 32, 35**]. Let $\Omega^r(\mathcal{F})$ be the space of smooth r-forms along the leaves. That is, an r-form $\omega \in \Omega^r(\mathcal{F})$ is a section of the r^{th} exterior power of the cotangent bundle of the leaves, $\bigwedge^r T\mathcal{F}^*$. The differential along the leaves will be denoted by

$$d_F\colon \Omega^r(\mathcal{F}) \to \Omega^{r+1}(\mathcal{F})$$

The *foliated cohomology* $H_{\mathcal{F}}(M)$ is the cohomology of the complex $\Omega^r(\mathcal{F})$, d_F. If the foliation is by points, the tangential cohomology is 0 in positive degrees.

There is a "geometric" interpretation of foliated cohomology [**32, 17**]. Let $M_{\mathcal{F}}$ denote the set M considered as the union of leaves of $\mathcal{F}$, so is a manifold of dimension m, the dimension of the leaves. That is, $M_{\mathcal{F}}$ is M with the leaf topology [**32**]. Then the identity map $j\colon M_{\mathcal{F}} \to M$ is an immersion, $\Omega^r(\mathcal{F})$ is the image of the "restriction map" $j^*\colon \Omega^r(M) \to \Omega^r(M_{\mathcal{F}})$, and d_F is the restriction of the differential of $\Omega^r(M)$. In this way, $H_{\mathcal{F}}(M)$ is identified with the de Rham cohomology of $M_{\mathcal{F}}$.

A form $\omega \in \Omega^r(\mathcal{F})$ if it can be written locally as

$$\omega = \sum f(x,y) dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

where $(x_1, \ldots, x_p, y)$ is a distinguished open set of a foliated chart, the coordinate 1-forms dx_i are differentials along the leaves.

Let $H^+_{\mathcal{F}}(M) = \displaystyle\bigoplus_{k>0} H^k_{\mathcal{F}}(M)$ denote the foliated cohomology in positive degrees.

PROPOSITION 2.4 ([**9**]). *Let U be a tangentially categorical open subset of M. Then the map $i^*\colon H^+_{\mathcal{F}}(M) \to H^+_{\mathcal{F}_U}(U)$ induced by the inclusion $U \hookrightarrow M$ is null.*

THEOREM 2.5 ([**9**]). *For any foliated manifold,* $\mathrm{cat}_{\mathcal{F}}(M) \geq \mathrm{nil}\, H^+_{\mathcal{F}}(M)$.

The tangential cohomology of a manifold is in general hard to calculate [**32, 11**]. W. Singhof and E. Vogt [**44**] gave a cohomological bound for the tangential category using ordinary cohomology instead of foliated cohomology which allows explicit calculation of the tangential category in many cases. We will generalize their result in the next section.

A fundamental result by the same authors proves a generalization of the classical estimate of category by the dimension of the space.

PROPOSITION 2.6 (Singhof and Vogt [**44**]). $\mathrm{cat}_{\mathcal{F}}(M) \leq \dim \mathcal{F} + 1$

This estimate together with the lower bound by the category of the leaves, give the exact value of the tangential category for a large number of foliations, namely all foliations containing a leaf L such that cat $L =$ dim $L + 1$. Classes of manifolds L verifying this condition have been studied by various authors. For example, J. Oprea and J. Walsh [**33**] showed that these include symplectic manifolds (M^{2n}, ω) with $\omega|_{\pi_2 M} = 0$, and also aspherical and hyper-aspherical manifolds. Here are some other results, whose proofs follow quickly from upper bound estimate $\mathrm{cat}_{\mathcal{F}}(M) \leq \dim \mathcal{F} + 1$.

PROPOSITION 2.7 (Singhof and Vogt [**44**]). *Let M be a compact manifold, and $\mathcal{F}$ a 1-dimensional foliation defined by a flow. Then* $\mathrm{cat}_{\mathcal{F}}(M) = 2$.

Proof: If $\mathcal{F}$ is a foliation of a compact manifold M, we have that $(M, \mathcal{F})$ is tangentially contractible if and only if $\mathcal{F}$ is a foliation by points. Then, for any flow $\mathcal{F}$, $1 < \mathrm{cat}_{\mathcal{F}}(M) \leq 2$. □

PROPOSITION 2.8 (Colman [**7**]). *Any 2-dimensional (non-trivial) Seifert fibration on a compact manifold has* $\mathrm{cat}_{\mathcal{F}}(M) = 3$.

Proof: Since the leaves are 2-dimensional compact manifolds, they are either 2-spheres or surfaces of category 3. If $\mathcal{F}$ is a non-trivial Seifert fibration, then it contains a leaf L with non-trivial holonomy. The regular leaves of $\mathcal{F}$ are non-trivial coverings of L. In particular, $\pi_1(L) \neq 0$ and L is not a 2-sphere. Then there exists a leaf of maximal category. □

PROPOSITION 2.9 (Colman [**7**]). $\mathrm{cat}_{\mathcal{F}}(M) = 3$ *for every 2-dimensional foliation of the 3-sphere.*

Proof: As a consequence of Novikov's theorem, we have that any codimension one foliation of S^3 has a compact leaf homeomorphic to a 2-dimensional torus. Then any 2-dimensional foliation of the 3-sphere contains a leaf of maximal category. □

3. Spectral sequence of $\mathcal{F}$

Let $T\mathcal{F}$ be the tangent bundle of the foliation $\mathcal{F}$ and $Q = TM/T\mathcal{F}$ the normal bundle. Choose a Riemannian metric on TM, and identify Q with the orthogonal distribution $T\mathcal{F}^{\perp}$, so $TM = T\mathcal{F} \oplus Q$. This induces an embedding of the leafwise cotangent bundle $T\mathcal{F}^* \subset TM^*$ complementing the natural inclusion $Q^* \subset TM^*$.

Let $\Omega^q(\mathcal{F})$ denote the smooth q-forms given by sections of the exterior subbundle $\bigwedge^q T\mathcal{F}^*$, and $\Omega^p(Q^*)$ the smooth p-forms given by sections of the exterior subbundle $\bigwedge^p Q^*$.

A form $\omega \in \Omega^s(M)$, $s = p+q$, is said to have type (p,q) if it is locally a sum of forms $\alpha \wedge \beta$ with $\alpha \in \Omega^q(\mathcal{F})$ and $\beta \in \Omega^p(Q^*)$. The bigraded differential complex $\Omega^{p,q}(M,\mathcal{F})$ is the space of differential forms of type (p,q). Note that $\Omega^{p,0}(M,\mathcal{F}) \cong \Omega^p(Q^*)$.

The exterior differential on $\Omega^*(M)$ splits in three operators $d' = d_{0,1}$, $d'' = d_{1,0}$ and $\delta = d_{2,-1}$ of bidegrees $(0,1)$, $(1,0)$ and $(2,-1)$ respectively. The identity $d \circ d = 0$ is equivalent to $d' \circ d' = 0$, $d' \circ d'' + d'' \circ d' = 0$, $\delta^2 = 0$, and $d'' \circ d'' + d' \circ \delta + \delta \circ d' = 0$. The identity $d' \circ d' = 0$ implies the subalgebra $\Omega^{0,*}(M,\mathcal{F})$ is a differential subcomplex. The other identities arise in the definition and calculation of the spectral sequence of the foliation.

The *defining ideal* of $\mathcal{F}$ is the differential ideal $\mathcal{J}(M,\mathcal{F})$ generated by $\Omega^1(Q^*)$. A form $\omega \in \Omega^1(M)$ is in $\Omega^1(Q^*)$ exactly when $\iota(X)\omega = 0$ for every vector field X tangent to $\mathcal{F}$.

Recall that $M_{\mathcal{F}}$ is the set M with the leaf topology, and $j\colon M_{\mathcal{F}} \to M$ is the identity map, considered as an immersion. Then $\Omega^1(Q^*)$ is the kernel of the restriction map $j^*\colon \Omega^1(M) \to \Omega^1(M_{\mathcal{F}})$, and $\mathcal{J}(M,\mathcal{F})$ is the kernel of the restriction map on all forms, $j^*\colon \Omega^*(M) \to \Omega^*(M_{\mathcal{F}})$.

The powers of $\mathcal{J}(M,\mathcal{F})$ define a multiplicative differential filtration of $\Omega^*(M)$ where $F^p\Omega^{p+q}(M) = \mathcal{J}(M,\mathcal{F})^p \wedge \Omega^q(M)$. The associated spectral sequence $E_r^{p,q}$ is an important tool for the study of geometric properties of foliations. We will use the notation $E_r^{p,q}(M,\mathcal{F})$ when there is a need to indicate the foliated manifold $(M,\mathcal{F})$.

The memoir by M. Mostow [**32**] gives a fundamental treatment of "continuous cohomology", which includes the foliated cohomology as a special case. There are several excellent surveys of the definition and properties of the foliation spectral sequence, notably those by El Kacimi [**11**], Roger [**35**], and Chapter 4 of Tondeur's monograph [**47**]. We recall the main features.

The 0^{th}-order term of the spectral sequence is the quotient space

$$\begin{aligned} E_0^{p,q} &\equiv F^p\Omega^{p+q}(M)/F^{p+1}\Omega^{p+q}(M) \\ &= \mathcal{J}(M,\mathcal{F})^p \wedge \Omega^q(M)/\mathcal{J}(M,\mathcal{F})^{p+1} \wedge \Omega^{q-1}(M) \\ &\cong \Omega^{p,q}(M,\mathcal{F}) \end{aligned}$$

The subsequent terms are defined as usual by $E_r^{p,q} = H(E_{r-1}^{p,q})$ with differential $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$.

Note that $E_0^{p,q} = 0$ if $p > n$ or $q > m$. Thus, there is $s \geq 0$ such that $E_{r+1}^{p,q} \cong E_r^{p,q}$ for all $r \geq s$, and we write $E_\infty^{p,q} = E_s^{p,q}$ for such s. The spectral sequence of the foliation converges to the de Rham cohomology of the manifold:

$$\bigoplus_{p,q} E_r^{p,q} \Longrightarrow \bigoplus_{p,q} E_\infty^{p,q} \cong H(M)$$

The differential $d_{0,1}^q\colon \Omega^{p,q}(M,\mathcal{F}) \to \Omega^{p,q+1}(M,\mathcal{F})$ is called the foliated differential, and variously denoted by d_F^q, d_F, $d_{0,1}$ or just d'. The space

$$E_1^{p,q} = H^{p,q}(M,\mathcal{F}) = \frac{\ker d_F^q}{\operatorname{im} d_F^{q-1}}$$

is the *cohomology of type* (p,q) of the foliation $\mathcal{F}$.

For $p = 0$, $E_1^{0,q} = H^{0,q}(M,\mathcal{F})$ is the *foliated cohomology* of $(M,\mathcal{F})$, and there is a natural isomorphism with the group defined in section 2, $H^{0,q}(M,\mathcal{F}) \cong H^q_{\mathcal{F}}(M)$.

For $q = 0$, $E_1^{p,0}$ is the space of basic forms $\Omega_b^p(M,\mathcal{F})$. The differential $d_{1,0}^p\colon \Omega^{p,q}(M,\mathcal{F}) \to \Omega^{p+1,q}(M,\mathcal{F})$ of bidegree $(1,0)$ induces on $\Omega_b^p(M,\mathcal{F})$ the basic differential, and $E_2^{p,0} = H(E_1^{p,0})$ is the *basic cohomology* of $(M,\mathcal{F})$.

We have a product of forms of type (p,q) induced by the regular multiplicative structure of $\Omega^*(M)$:

$$\wedge\colon \Omega^{p,q}(M,\mathcal{F}) \times \Omega^{p',q'}(M,\mathcal{F}) \to \Omega^{p+p',q+q'}(M,\mathcal{F})$$

The definition of the complex $\Omega^{p,q}(M)$ requires the choice of a splitting $TM = T\mathcal{F} \oplus Q$ which is not natural; that is, the splitting need not be preserved by a foliated map $f\colon (M,\mathcal{F}) \to (M',\mathcal{F}')$. The point is that while the inclusion of the dual $Q^* \subset TM^*$ is canonical, the inclusion $T\mathcal{F}^* \subset TM^*$ need not be preserved, so that $f^*\Omega^{0,1}(M',\mathcal{F}') \subset \Omega^{0,1}(M,\mathcal{F}) + \Omega^{1,0}(M,\mathcal{F})$. However, $f^*\Omega^{p,0}(M',\mathcal{F}') \subset \Omega^{p,0}(M,\mathcal{F})$ and consequently there is a canonical induced map of graded complexes

$$f^*\colon E_0^{'p,q} \to E_0^{p,q}$$

The naturality of the induced map on foliation spectral sequences only applies for the associated graded complexes. This is a subtle point not stressed in the foliation literature, but a well-known issue with morphisms between filtered differential graded algebras (see Theorem 3.2 and the related discussion in §1.5, [**31**].) This point arises in the proof of Proposition 3.2 below.

Proposition 3.1. *If $f,g\colon (M,\mathcal{F}) \to (M',\mathcal{F}')$ are two integrably homotopic maps then they induce the same homomorphism in cohomology*

$$f^* = g^*\colon E_1^{'p,q} \to E_1^{p,q}$$

Proof: This was proved by El Kacimi for the foliated cohomology case [**11**], and the same method applies here.

Let $f\colon (M,\mathcal{F}) \to (M',\mathcal{F}')$ be foliated maps, and $H\colon M\times\mathbb{R} \to M'$ an *integrable* homotopy between them. Recall this mean that H is a foliated map considering $M\times\mathbb{R}$ foliated by leaves $L\times\mathbb{R}$, $L \in \mathcal{F}$. The point is to define for all p,q a homotopy operator $K\colon \Omega^{p,q}(M',\mathcal{F}') \to \Omega^{p,q-1}(M,\mathcal{F})$ such that $d_{0,1}\circ K \pm K\circ d_{0,1} = f^* - g^*$. Hence, f^* and g^* induce the same maps on $E_1^{'p,q}$.

Given a form $\omega \in \Omega^{p,q}(M',\mathcal{F}')$ note that $H^*\omega \in \Omega^{p,q}(M,\mathcal{F}\times\mathbb{R})$. Let $\partial/\partial t$ be the coordinate vector field along $\mathbb{R}$, which defines a vector field on $M\times\mathbb{R}$ tangent to the leaves of $\mathcal{F}\times\mathbb{R}$. Thus, the contraction operator

$$\iota(\partial/\partial t)\colon \Omega^{p,q}(M,\mathcal{F}\times\mathbb{R}) \to \Omega^{p,q-1}(M,\mathcal{F}\times\mathbb{R})$$

We define

$$K(\omega) = \int_0^1 \iota(\partial/\partial t)H^*(\omega)\ dt$$

and the standard calculation shows $d_{0,1}\circ K \pm K\circ d_{0,1} = f^* - g^*$. $\square$

Proposition 3.2. *Let U be a $\mathcal{F}$-categorical open set and $i^*\colon E_1^{p,q}(M,\mathcal{F}) \to E_1^{p,q}(U,\mathcal{F}_U)$ the map induced in cohomology of type (p,q) by the inclusion map $U \hookrightarrow M$. Then $i^* = 0$ for all $q > 0$.*

Proof: Let $c\colon U \to M$ be a map integrably homotopic to the inclusion, and constant on the leaves of $\mathcal{F}_U$. Then, by Proposition 3.1, $i^* = c^*$. Note that $c_*\colon T(U) \to T(M)$ vanishes on the tangent vectors to the foliation, so $c^*\Omega^s(M) \subset \Omega^{s,0}(U,\mathcal{F}_U)$ for all s. In particular, $c^*\Omega^{p,q}(M,\mathcal{F}) \subset \Omega^{p+q,0}(U,\mathcal{F}_U)$.

Consider now the induced map on graded complexes $c^*\colon E_0^{p,q}(M,\mathcal{F}) \to E_0^{p,q}(U,\mathcal{F}_U)$ for $q > 0$. For $\omega \in \Omega^{p,q}(M)$ representing the graded class $[\omega] \in E_0^{p,q}(M,\mathcal{F})$, then

$$c^*\omega \in \Omega^{p+q,0}(U,\mathcal{F}_U) \subset \mathcal{J}^{p+q}(U,\mathcal{F}_U) \subset \mathcal{J}(U,\mathcal{F}_U)^{p+1} \wedge \Omega^{q-1}(U)$$

hence $c^*[\omega] = [c^*\omega] = 0 \in E_1^{p,q}(U,\mathcal{F}_U)$ as a graded class. $\square$

Remark 3.3. The induced map $i^*\colon E_1^{p,0}(M,\mathcal{F}) \to E_1^{p,0}(U,\mathcal{F}_U)$ is not necessarily zero. As noted already, $E_1^{p,0}$ is the complex of basic forms of $(M,\mathcal{F})$. Suppose that H deforms U into a transversal T, where T is considered as foliated by points. Let $\Phi\colon U \to T$ denote the map H_1 mapping to its image, and $i_T\colon T \to M$ the inclusion. Then $i^* = c^* = i_T^* \circ \Phi^*$, where $i_T^*\colon E_1^{p,0}(M,\mathcal{F}) \to E_1^{p,0}(T) = \Omega^p(T)$ is just the restriction of the basic forms to T, and $\Phi^*\colon E_1^{p,0}(T) \to E_1^{p,0}(U,\mathcal{F})$ is the isomorphism from p-forms on T to basic p-forms for $(U,\mathcal{F}_U)$. Thus, if $\mathcal{F}$ has a non-trivial basic p-form, then for some categorical U the map i^* will be non-trivial.

3.1. Relative cohomology of type (p,q). We develop a relative theory of type (p,q) by explicitly defining the natural notions of *relative* foliated complex $E_1^{*,*}(M,U)$ and a cup product on it.

Following the usual definition of the mapping cone [**5**], if U is an open subset of M, we consider the complex

$$\Omega^{p,q}(M,U) = \Omega^{p,q}(M) \oplus \Omega^{p,q-1}(U)$$

with differential

$$d(\omega,\theta) = (d_F\omega, \omega|_U - d_F\theta).$$

We define the relative cohomology of type (p,q), $E_1^{p,q}(M,U)$, to be the cohomology of the complex above. Here U is regarded as a foliated manifold with the foliation induced by $\mathcal{F}$.

We have the short exact sequence

$$0 \to \Omega^{p,q-1}(U) \xrightarrow{\alpha} \Omega^{p,q}(M,U) \xrightarrow{\beta} \Omega^{p,q}(M) \to 0 \tag{3.1}$$

with $\alpha(\eta) = (0,\eta)$ and $\beta(\omega,\theta) = \omega$. There is a long exact sequence in cohomology of type (p,q)

$$\cdots \to E_1^{p,q-1}(U) \xrightarrow{\alpha^*} E_1^{p,q}(M,U) \xrightarrow{\beta^*} E_1^{p,q}(M) \xrightarrow{i^*} E_1^{p,q}(U) \longrightarrow \cdots$$

3.2. Products in relative cohomology of type (p,q)**.** Let $U, V \subset M$ be open sets with the respective induced foliations. We take a smooth partition $\{f, g\}$ of unity subordinate to the covering $\{U, V\}$ of $U \cup V$.

Then we define a product of relative forms of type (p,q)

$$\bullet \colon \Omega^{p,q}(M,U) \times \Omega^{p',q'}(M,V) \to \Omega^{p+p',q+q'}(M, U \cup V)$$

by

$$(\omega, \theta) \bullet (z, t) = (\omega \wedge z, \eta) \tag{3.2}$$

where $\eta \in \Omega^{p+p',q+q'-1}(U \cup V)$ is given as follows:

$$\eta \mid_U = \theta \wedge z \mid_U + (-1)^p d(\theta \wedge gt) + g\Big((-1)^p(\omega - d\theta) \wedge t - \theta \wedge (z - dt)\Big) \mid_{U \cap V}$$

and

$$\eta \mid_V = (-1)^p \omega \mid_V \wedge t - (-1)^p d(f\theta \wedge t) - f\Big((-1)^p(\omega - d\theta) \wedge t - \theta \wedge (z - dt)\Big) \mid_{U \cap V} .$$

The form η is well defined in the intersection $U \cap V$ and the induced product in cohomology of type (p,q) does not depend on the partition of the unity. Then we have a well defined product:

$$E_1^{p,q}(M,U) \times E_1^{p',q'}(M,V) \to E_1^{p+p',q+q'}(M, U \cup V)$$

3.3. Index of nilpotency of $E_1^{*,+}$**.** The ring A is said to be nilpotent if there exists some integer $k > 0$ such that $a_1 \cdots a_k = 0$ for any elements (not units) $a_1, \ldots, a_k \in A$. The least integer $k = \text{nil } A$ with this property is called the *index of nilpotency* of A.

THEOREM 3.4. $\text{cat}_{\mathcal{F}}(M) \geq \text{nil } E_1^{*,+}$.

Proof: Let $\{U_1, \ldots, U_k\}$ be a covering of M by $\mathcal{F}$-categorical open sets. For each $U = U_j$ we consider the long exact sequence of the pair (M,U),

$$\cdots \to E_1^{p,q-1}(U) \xrightarrow{\alpha^*} E_1^{p,q}(M,U) \xrightarrow{\beta^*} E_1^{p,q}(M) \xrightarrow{i^*} E_1^{p,q}(U) \longrightarrow \cdots$$

where β^* is onto for $q > 0$ due to Proposition 3.2. Let $x_1, \ldots, x_k$ be arbitrary elements of $E_1^{*,+}$. For each x_j, there exists $z_j \in E_1^{p,q}(M,U)$ such that $\beta^*(z_j) = x_j$. Moreover,

$$\beta_{U \cup V}(x \bullet y) = \beta_U(x) \wedge \beta_V(y).$$

Thus the product $x_1 \cdots x_k = \beta^*(z_1 \bullet \cdots \bullet z_k) = 0$ because $z_1 \bullet \cdots \bullet z_k \in E_1^{*,+}(M,M)$. This means that nil $E_1^{*,+} \leq k$. $\square$

3.4. Other lower bounds. The reduced foliated cohomology of a foliated manifold, $H_{\mathcal{F}}^+(M)$, is contained in $E_1^{*,+}$, so

$$\text{nil } H_{\mathcal{F}}^+(M) \leq \text{nil } E_1^{*,+} \leq \text{cat}_{\mathcal{F}}(M)$$

Thus, Theorem 3.4 generalizes the lower bound in terms of the foliated cohomology given in [**9**].

The r-term of the spectral sequence is $E_r^{p,q} = H(E_{r-1}^{p,q})$, hence nil $E_r^{*,+} \leq$ nil $E_1^{*,+}$, for all $r \geq 1$, so we have

$$\text{nil } E_\infty^{*,+} \leq \text{nil } E_2^{*,+} \leq \text{nil } E_1^{*,+} \leq \text{cat}_{\mathcal{F}}(M)$$

While the calculation of nil $E_1^{*,+}$ gives the strongest estimate for $\mathrm{cat}_{\mathcal{F}}(M)$ via the methods of this section, the invariant nil $E_2^{*,+}$ may prove easier to compute in examples.

Set $E_\infty^s = \bigoplus_{p+q=s} E_\infty^{p,q}$. The isomorphism $H^s(M) \cong E_\infty^s$ decomposes the reduced cohomology of the manifold into two parts, $\tilde{H}^*(M) \cong E_\infty^{*,+} \oplus E_\infty^{+,0}$ the latter being the image of the reduced basic cohomology of the foliated manifold by the following morphism

$$p^* \colon \tilde{H}_b^p(M) \cong E_2^{p,0} \longrightarrow \cdots \longrightarrow E_\infty^{p,0} \subset E_\infty^p \cong \tilde{H}^p(M)$$

Note that for all $p \geq 0$, $E_{r+1}^{p,0} = E_r^{p,0} / \operatorname{im}\{d\}$ hence each of the maps $E_r^{p,0} \to E_{r+1}^{p,0}$ is surjective.

We can generalize for foliations the James' estimate (2.1) in terms of the spectral sequence by taking the basic cohomology instead of the cohomology of the base. First note that $\langle p^* \tilde{H}_b(M) \rangle = E_\infty^{+,0} \wedge E_\infty^{+}$ hence we have a surjection

$$\frac{\tilde{H}^*(M)}{\langle p^* \tilde{H}_b^*(M) \rangle} \longrightarrow E_\infty^{1,0} \oplus E_\infty^{0,1} \oplus E_\infty^{0,2} \oplus \cdots \oplus E_\infty^{0,m}$$

with $E_\infty^{1,0} = p^* H_b^1(M)$ and $\bigoplus_{q \geq 1} E_\infty^{0,q} \subset \tilde{H}_{\mathcal{F}}^*(M)$.

4. Fundamental classes

In this section we develop techniques for showing the non-triviality of classes in the cohomology groups $E_r^{p,q}$, and discuss applications to obtaining lower bounds on the tangential category $\mathrm{cat}_{\mathcal{F}}(M)$.

We assume that M and $\mathcal{F}$ are oriented, and M is compact. Choose a Riemannian metric on TM, which restricts also to a Riemannian metric on $T\mathcal{F}$. Let $\theta \in \Omega^m(M_{\mathcal{F}})$ be the positively oriented leafwise volume form. Then $d_F\theta = 0$ and we let $[\theta]_{\mathcal{F}} \in H^m_{\mathcal{F}}(M)$ denote its cohomology class.

Definition 4.1. A fundamental class for $\mathcal{F}$ is a continuous linear map $I\colon \Omega^m(M_{\mathcal{F}}) \to \mathbb{R}$ such that $I \circ d_F = 0$ and $I(\theta) > 0$.

It follows that I induces a map $I_*\colon H^m_{\mathcal{F}}(M) \to \mathbb{R}$ such that $I_*[\theta]_{\mathcal{F}} \neq 0$. In particular, if $\mathcal{F}$ admits a fundamental class then $[\theta]_{\mathcal{F}} \neq 0$. The fundamental class need not be unique. For example, if $\mathcal{F}$ has a compact leaf L then integration defines a map

$$\int_L : \Omega^m(M_{\mathcal{F}}) \;\to\; \mathbb{R} \tag{4.1}$$

which maps the the leafwise volume form θ to the volume of L. Distinct compact leaves yield distinct linear functionals on $\Omega^m(M_{\mathcal{F}})$, and may also induce distinct maps on cohomology. Consider the case where $\mathcal{F}$ is defined by a fibration $M \to M/\mathcal{F}$ with typical compact connected fiber L, then $H^m_{\mathcal{F}}(M) \cong C^\infty(M/\mathcal{F}; H^m(L)) \cong C^\infty(M/\mathcal{F}, \mathbb{R})$. Evaluation of the cohomology class $[\omega]_{\mathcal{F}}$ at the point $[L] \in M/\mathcal{F}$ is given by integrating a representing m-cocycle ω on the fundamental class $\int_L$.

Continuing, assume L is a compact leaf. Suppose there exist classes $x_1, \ldots, x_k \in \Omega^*(M_{\mathcal{F}})$ with $\int_L x_1 \wedge \cdots \wedge x_k \neq 0$. Then

$$0 \neq [x_1 \wedge \cdots \wedge x_k] = [x_1] \bullet \cdots \bullet [x_k] \in H^m_{\mathcal{F}}(M)$$

and hence $\mathrm{cat}_{\mathcal{F}}(M) \geq k$.

Note the inclusion $i\colon L \subset M$ induces a restriction mapping $i^*\colon H^*_{\mathcal{F}}(M) \to H^*(L)$ which preserves products, so if we set $z_i = i^*[x_i]$ then

$$\langle z_1 \bullet \cdots \bullet z_k, [L]\rangle = \int_L x_1 \wedge \cdots \wedge x_k \neq 0$$

so cat $L \geq k$ also. Hence, using a fundamental class defined by a compact leaf to estimate the nilpotency index of $E_1^{*,+}(M, \mathcal{F})$ does not improve upon the basic estimate $\mathrm{cat}_{\mathcal{F}}(M) \geq \mathrm{cat}\ L$ of Proposition 2.2. However, by suitably generalizing the above argument, we can extend the estimate for compact leaves to a more general estimate using foliation currents which yields new results.

4.1. Transverse invariant measures. Recall that a transverse invariant measure μ for a foliation $\mathcal{F}$ is a Borel measure defined on transversals, so that $\mu(T) = \mu(h(T))$ if h is an element of holonomy of $\mathcal{F}$ and T is a transverse manifold in the domain of μ.

Theorem 4.2. *Suppose that μ is non-trivial transverse invariant measure for a foliation $\mathcal{F}$ with leaf dimension m. Then there exists a fundamental class for $\mathcal{F}$*

$$\int_\mu : \Omega^m(M_{\mathcal{F}}) \;\to\; \mathbb{R} \tag{4.2}$$

Proof: We sketch the proof from Ruelle and Sullivan [**36, 45, 17**].

Let $\mathcal{U} = \{U_1, \dots, U_r\}$ be a covering of M by foliation charts $\varphi_i \colon U_i \to [-1,1]^{m+n}$, with local coordinates $\varphi(p) = (\vec{x}, \vec{y}) = (x_1, \dots, x_m, y_1, \dots, y_n)$ where $\vec{x}$ is the leafwise direction and $\vec{y}$ is the transverse coordinate. The local transversal

$$T_i = \varphi_i^{-1}(\vec{0} \times [-1,1]^n) \subset U_i$$

is a submanifold on M, and without loss we can assume T_i are disjoint, and set $\mathcal{T} = \bigcup T_i$.

Let $\mathcal{H}$ denote the holonomy pseudogroup on $\mathcal{T}$ induced by $\mathcal{F}$.

Choose a partition of unity $\{\lambda_1, \dots, \lambda_r\}$ subordinate to $\mathcal{U}$.

Given $\omega \in \Omega^m(M_{\mathcal{F}})$ the form $\omega_i = \lambda_i \omega$ has compact support in U_i. Express ω_i in local coordinates

$$\omega_i = f_i(\vec{x}, \vec{y}) dx_1 \wedge \cdots \wedge dx_m$$

then

$$\int_\mu \omega = \sum_{i=1}^r \int_\mu \omega_i = \sum_{i=1}^r \int_{[-1,1]^n} \left\{ \int_{[-1,1]^m} f_i(\vec{x}, \vec{y}) dx_1 \wedge \cdots \wedge dx_m \right\} d\mu(\vec{y})$$

If $\omega = d_F \phi$ then the leafwise Stokes' Theorem yields $\int_\mu \phi = 0$.

For the volume form θ, $\lambda_i \geq 0$ and $\lambda_1 + \cdots + \lambda_r = 1$ implies

$$\int_\mu \theta = \sum_{i=1}^r \int_{[-1,1]^n} \left\{ \int_{[-1,1]^m} \lambda_i \, |\theta| d\vec{x} \right\} d\mu(\vec{y}) > 0$$

where $|\theta| d\vec{x}$ denotes the restriction of θ to the plaques in U_i. $\square$

A closed form $\eta \in \Omega^n(M)$ yields a transverse measure μ defined by a integration of its absolute value $|\eta|$ with respect to the transverse orientation – see [**34**] – along a transversal: $\mu(T) = \int_T |\eta|$. Then

$$\int_\mu \omega = \int_M \omega \wedge \eta$$

This example illustrates a more general fact, that a transverse holonomy invariant measure μ defines a foliation cycle C_μ with associated homology class $[C_\mu] \in H_m(M)$. In the case where μ is defined by θ then $[C_\mu]$ is the Poincaré dual to the cohomology class $[\eta] \in H^n(M)$ (cf. [**36, 45**].)

The fundamental class (4.2) will be applied in §6 to estimate $\mathrm{cat}_{\mathcal{F}}(M)$ for group actions. However, for some foliations, $H^m_{\mathcal{F}}(M) = 0$ so there are no fundamental classes (see §6.1), and other methods are needed.

4.2. Transverse cohomology. Haefliger defined the holonomy invariant k-currents as a generalization of the foliation fundamental classes [**17**]. We show how these currents can be used to prove non-triviality of classes in $E_r^{*,+}(M, \mathcal{F})$.

Recall the definition of the transverse cohomology and holonomy invariant k-currents. Let $\mathcal{T} = \bigcup T_i$ be the transversal introduced above. Let $\Omega^p_c(\mathcal{T})$ be the vector space of smooth p-forms on $\mathcal{T}$ with compact support. Denote by $\Theta^p_c \subset \Omega^p_c(\mathcal{T})$ the subspace generated by elements of the form $\alpha - h^*\alpha$, where $h \in \mathcal{H}$ and α is a p-form with compact support in the range of h. The spaces Θ^*_c are closed under exterior differentiation. Denote by $\Omega^p_c(\mathcal{T}/\mathcal{H})$ the quotient vector space $\Omega^p_c(\mathcal{T})/\Theta^p_c$. The topology on $\Omega^p_c(\mathcal{T}/\mathcal{H})$ is the quotient topology, which is in general not Hausdorff.

The exterior differential $d_{\mathcal{T}} \colon \Omega^p_c(\mathcal{T}) \to \Omega^{p+1}_c(\mathcal{T})$ induces continuous differentials

$$d_{\mathcal{T}} \colon \Omega^p_c(\mathcal{T}/\mathcal{H}) \to \Omega^{p+1}_c(\mathcal{T}/\mathcal{H})$$

Haefliger proves that the graded differential vector spaces $\{\Omega^*_c(\mathcal{T}/\mathcal{H}), d_{\mathcal{T}}\}$ is independent of the choice of the complete transversal $\mathcal{T}$ up to isomorphism. This isomorphism class is denoted by $\Omega^*_c(Tr\ \mathcal{F})$.

The *transverse cohomology* $H^*(Tr\ \mathcal{F})$ is the cohomology of $\Omega^*_c(Tr\ \mathcal{F})$.

A continuous linear functional $c\colon \Omega^p_c(Tr\ \mathcal{F}) \to \mathbb{R}$ is called a holonomy invariant p-current. A holonomy invariant transverse measure μ corresponds to a 0-current via integration on transversals, $\int_\mu \colon \Omega^0_c(T_i) \to \mathbb{R}$. This induces a map $\int_\mu \colon H^0(Tr\ \mathcal{F}) \to \mathbb{R}$. More generally, a holonomy invariant p-current c induces a linear functional $c_* \colon H^p(Tr\ \mathcal{F}) \to \mathbb{R}$.

For example, there is always a transverse fundamental class $\int_{\mathcal{T}} \colon \Omega^n_c(Tr\ \mathcal{F}) \to \mathbb{R}$ defined by integration of the n-form $\phi \in \Omega^n_c(\mathcal{T})$ over the transversal $\mathcal{T}$. This is a closed current which vanishes on Θ^n_c so induces the fundamental current $[\mathcal{T}] \colon H^n(Tr\ \mathcal{F}) \to \mathbb{R}$.

One key point about $H^*(Tr\ \mathcal{F})$ is that it is often infinite dimensional when non-zero [**17, 19**], because the subspace Θ^p_c need not be closed in $\Omega^p_c(\mathcal{T})$. Note that a continuous linear functional on $\Omega^p_c(\mathcal{T})$ which vanishes on Θ^p_c must vanish on its closure $\overline{\Theta^p_c}$, so this suggests the motivation for Hector's definition of the reduced transverse cohomology [**19**]. Define

$$\overline{\Omega}^p_c(\mathcal{T}/\mathcal{H}) = \Omega^p_c(\mathcal{T})/\overline{\Theta^p_c}$$

which is the Hausdorff quotient of $\Omega^p_c(\mathcal{T}/\mathcal{H})$. The *reduced transverse cohomology*, denoted by $H^*(tr\ \mathcal{F})$, is the cohomology of $\overline{\Omega}^*_c(\mathcal{T}/\mathcal{H})$. The reduced cohomology $H^*(tr\ \mathcal{F})$ is a quotient of $H^*(Tr\ \mathcal{F})$, and often more calculable. For example, Hector proved in Theorem 1.11 [**19**] that the natural restriction map $\Omega^p_b(M,\mathcal{F}) \to \Omega^*_c(\mathcal{T}/\mathcal{H})$ induces an isomorphism $E^{p,0}_2(M,\mathcal{F}) \cong H^p(tr\ \mathcal{F})$ if $\mathcal{F}$ is a Riemannian foliation.

The transverse cohomology can be used to detect elements of $E^{p,m}_2(M,\mathcal{F})$. The following result is essentially in [**17**], though the realtion with the foliation spectral sequence is not stated there. We recall the proof and this relation.

THEOREM 4.3. *There is a natural map*

$$\chi_* \colon E^{p,m}_2(M,\mathcal{F}) \to H^p(Tr\ \mathcal{F}) \tag{4.3}$$

Proof: Haefliger defined the "integration along the leaves" map

$$\int_{\mathcal{F}} \colon\ \Omega^{p+m}_c(M) \to \Omega^p_c(Tr\ \mathcal{F}) \tag{4.4}$$

which commutes with the differential (Theorem 3.1, [**17**].) We use this to define a map which commutes with the indicated differentials,

$$\chi\colon\ (E^{p,m}_1(M,\mathcal{F}), d'') \to (\Omega^p(M_{\mathcal{F}}), d_{\mathcal{T}}) \tag{4.5}$$

The map χ_* is the induced map on cohomology.

The definition of $\int_{\mathcal{F}}$ is similar to that of the fundamental class from a transverse measure, and we use the same notation as in the proof of Theorem 4.2. Let $\omega \in \Omega_c^{p+m}(M)$ and write $\omega_i = \lambda_i \omega$. Then in coordinates

$$\omega_i = \sum_J f_J(\vec{x},\vec{y})\, dx_1 \wedge \cdots \wedge dx_m \wedge dy_J \ + \ \text{terms of degree} < m \ \text{ in } dx_1, \ldots , dx_m$$

where $J = (j_1, \ldots , j_p)$ and $dy_J = dy_{j_1} \wedge \cdots \wedge dy_{j_p}$. Then

$$\int_{\mathcal{F}} \omega = \sum_{i=1}^{r} \int_{\mathcal{F}} \omega_i \text{ where } \int_{\mathcal{F}} \omega_i = \sum_J \left\{ \int_{[-1,1]^m} f_J(\vec{x},\vec{y})\, dx_1 \wedge \cdots \wedge dx_m \right\} dy_J$$

According to Haefliger, $d_{\mathcal{T}} \int_{\mathcal{F}} \omega = \int_{\mathcal{F}} d\omega$.

For $\omega \in \Omega_c^{p,m}(M,\mathcal{F})$ we have $d\omega = d_{0,1}\omega + d_{1,0}\omega + d_{2,-1}\omega$. Observe that $d_{2,-1}\omega \in \Omega_c^{p+2,m-1}(M,\mathcal{F})$ so the restriction of $d_{2,-1}\omega$ to a leaf of $\mathcal{F}$ vanishes, and thus $\int_{\mathcal{F}} d_{2,-1}\omega = 0$. Also note that $d_{0,1}\omega \in \Omega_c^{p,m+1}(M,\mathcal{F}) = 0$ so $\int_{\mathcal{F}} d_{0,1}\omega = 0$. Hence we have

$$d_{\mathcal{T}} \int_{\mathcal{F}} \omega = \int_{\mathcal{F}} d\omega = \int_{\mathcal{F}} d''\omega \tag{4.6}$$

Suppose that $\omega = d_{0,1}\phi$ for $\phi \in \Omega_c^{p,m-1}(M,\mathcal{F})$. Then

$$\int_{\mathcal{F}} \omega = \int_{\mathcal{F}} d_{0,1}\phi = \int_{\mathcal{F}} d\phi = d_{\mathcal{T}} \int_{\mathcal{F}} \phi = 0$$

where the last integral vanishes for dimension reasons. We conclude that the map (4.4) descends to a map on quotients $E_0^{p,m}$ and as (4.4) also vanishes on the image of $d_{0,1}$ further descends to $E_1^{p,m} = E_0^{p,m}/d_{0,1}E_0^{p,m-1}$ yielding the map χ of (4.5). By (4.6) the map χ commutes with differentials, so we obtain the map χ_* of (4.4). □

Corollary 4.4. *A holonomy invariant p-current* $c \colon \Omega_c^p(Tr\,\mathcal{F}) \to \mathbb{R}$ *induces a natural map*

$$\chi_c \colon E_1^{p,m}(M,\mathcal{F}) \to \mathbb{R} \tag{4.7}$$

which vanishes on the image of $d'' \colon E_1^{p-1,m}(M,\mathcal{F}) \to E_1^{p,m}(M,\mathcal{F})$. □

If μ is a transverse invariant measure with associated holonomy invariant 0-current c_μ then its evaluation on $E_1^{0,m}(M,\mathcal{F})$ agrees with the evaluation of the fundamental class defined by the current as given in Theorem 4.2.

5. Secondary classes

H. Shulman defined the *covering dimension* for a foliation, $\mathrm{cd}(M, \mathcal{F})$, to be the minimal number of open sets in a covering of M by foliation charts [**40**]. Clearly, $\mathrm{cd}(M, \mathcal{F}) \geq \mathrm{cat}_{\mathcal{F}}(M)$. The open sets in the covering are coordinate charts, so diffeomorphic to $\mathbb{R}^{m+n}$, hence are categorical in M so $\mathrm{cd}(M, \mathcal{F}) \geq \mathrm{cat}\ M$. The covering dimension $\mathrm{cd}(M, \mathcal{F})$ is thus clearly a distinct invariant from $\mathrm{cat}_{\mathcal{F}}(M)$, though similar in spirit.

The secondary classes of a foliation $\mathcal{F}$ of codimension n are determined by a characteristic map $\Delta\colon H^*(WO_n) \to H^*(M)$, which factors through the classifying map for the foliation $h_{\mathcal{F}}\colon M \to B\Gamma_n$ (see [**3, 29**].) In his thesis, Shulman used the Milnor join realization of $B\Gamma_n$ and the semi-simplicial de Rham theory for $B\Gamma_n$ to prove that the secondary classes "vanish below the diagonal" in $B\Gamma_n$ (see [**39, 4, 41**]). This was key to his proof of the following

THEOREM 5.1 ([**40**]). *If $\Delta(z) \neq 0$ for $z \in H^{n+k}(WO_n)$, then $cd(M, \mathcal{F}) > k$.*

In fact, this theorem remains true as well for $\mathrm{cat}_{\mathcal{F}}(M)$.

THEOREM 5.2 ([**24**]). *If $\Delta(z) \neq 0$ for $z \in H^{n+k}(WO_n)$, then $\mathrm{cat}_{\mathcal{F}}(M) > k$.*

The idea of the proof is that the classifying space of the holonomy groupoid $(\mathcal{T}, \mathcal{H})$ can be realized by a complex $B\Gamma_{\mathcal{F}}$ with dimension $\leq n + \mathrm{cat}_{\mathcal{F}}(M) - 1$, and that the characteristic map Δ factors through $H^*(B\Gamma_{\mathcal{F}})$. This gives the lower bound $n+k+1$ on the dimension of $B\Gamma_{\mathcal{F}}$, which we prove in [**24**] is bounded above by $\mathrm{cat}_{\mathcal{F}}(M) + n$.

The cup length estimate for spectral sequence cohomology of §3 can be used to give a quick proof of Theorem 5.2 for the case of the Godbillon-Vey class. Let ω be a defining n-form for $\mathcal{F}$. That is, ω is a non-vanishing section of the ideal $\mathcal{J}^n(M, \mathcal{F})$. Then there exists a 1-form η such that $d\omega = \omega \wedge \eta$. Then $\omega \wedge d\eta = 0$, and so $d(\eta \wedge (d\eta)^n) = (d\eta)^{n+1} = 0$. The Godbillon-Vey class of $\mathcal{F}$ is defined as the cohomology class of $\eta \wedge (d\eta)^n$

$$GV(\mathcal{F}) = [\eta \wedge (d\eta)^n] \in H^{2n+1}(M)$$

This is a well-defined, and very well studied invariant of $\mathcal{F}$ [**14, 13, 23**].

THEOREM 5.3. *Suppose that $GV(\mathcal{F}) \neq 0$, then $\mathrm{cat}_{\mathcal{F}}(M) \geq n+2$.*

Proof: Since $\omega \wedge d\eta = 0$ we have that $j^*\eta \in \Omega^r(\mathcal{F})$ is closed, and its image $[\eta] \in E_1^{0,1}(M, \mathcal{F})$ is called the *Reeb class* of $\mathcal{F}$. The identity $\omega \wedge d\eta = 0$ also implies that $d\eta \in \mathcal{J}(M, \mathcal{F})$ so we can consider its image $[d\eta] \in E_1^{1,1}(M, \mathcal{F})$. The $(n+1)$-fold product

$$[\eta] \bullet [d\eta] \bullet \cdots \bullet [d\eta] \in E_1^{n,n+1}(M, \mathcal{F})$$

is represented by the closed form $\eta \wedge (d\eta)^n \in \Omega^{2n+1}(M)$ so survives to the limiting term

$$[\eta \wedge (d\eta)^n] \in E_\infty^{n,n+1}(M, \mathcal{F}) \subset E^{*,*}(M, \mathcal{F}) \cong H^{2n+1}(M)$$

which is non-zero by assumption. Then $\mathrm{cat}_{\mathcal{F}}(M) \geq n+2$ by Theorem 3.4. $\square$

For example, if M is a compact manifold of dimension $2n+1$ with a codimension n foliation with non-trivial Godbillon-Vey class, then $\mathrm{cat}_{\mathcal{F}}(M) = n+2$.

6. Applications and Examples

6.1. $\mathbb{R}^m$ actions. A smooth action $\alpha\colon \mathbb{R}^m \times M \to M$ is *locally free* if for all $z \in M$ the stabilizer $\Lambda_z = \{\vec{x} \in \mathbb{R}^m \mid \alpha(\vec{x})z = z\}$ is a discrete subgroup of $\mathbb{R}^m$. The orbits of α define a foliation $\mathcal{F}_\alpha$ of dimension m.

THEOREM 6.1. $\mathrm{cat}_{\mathcal{F}_\alpha}(M) = m + 1$.

Proof: Let $x_1, \dots, x_m$ denote the standard coordinates on $\mathbb{R}^m$ with corresponding coordinate 1-forms dx_i. Then the Lie algebra cohomology $H^*(\mathbb{R}^m) \cong \bigwedge^*(dx_1, \dots, dx_m)$.

Given $z \in M$ the leaf through z is identified with the orbit, $L_z \cong \mathbb{R}^m \cdot z \cong \mathbb{R}^m/\Lambda_z$. The forms dx_i are invariant under the action of the stabilizers Λ_z so descend on each orbit to yield closed leafwise 1-forms, again denoted by $dx_i \in \Omega^1(M_{\mathcal{F}})$. The exterior product $\theta = dx_1 \wedge \cdots \wedge dx_m$ is the leafwise volume form. Thus, in $E_1^{0,*}(M, \mathcal{F})$ we have

$$[dx_1] \bullet \cdots \bullet [dx_m] = [\theta] \in E_1^{0,m}(M, \mathcal{F})$$

The group $\mathbb{R}^m$ has polynomial growth, so the choice of a basepoint $z \in M$ yields a transverse invariant measure $\mu(z)$ by asymptotic averaging over the orbit $\mathbb{R}^m \cdot z$ (cf. [**34**].) Clearly, $\int_{\mu(z)} \theta = 1$ so $[\theta] \neq 0$ and we can apply Theorem 3.4 to get $\mathrm{cat}_{\mathcal{F}}(M) \geq m + 1$. The estimate $\mathrm{cat}_{\mathcal{F}}(M) \leq m + 1$ follows from Singhof and Vogt [**44**]. □

Locally free $\mathbb{R}^m$ actions occur naturally in many geometric contexts. Arraut and dos Santos have studied their geometry using a combination of Lie algebra and foliation techniques [**2**]. The authors have also given a more general construction of characteristic classes for group actions, and studied the relations with geometry and dynamics (see [**37, 38**]).

The simplest example is that of a locally free $\mathbb{R}^1$ action on a compact manifold, which is just a non-singular flow α_1 on M. The category of the flow is always 2 by Proposition 2.7. Given a collection of non-singular flows $\alpha_i\colon \mathbb{R} \times M_i \to M_i$ for $1 \leq i \leq m$, their product is an $\mathbb{R}^m$ flow on $M = M_1 \times \cdots \times M_m$. If $\mathcal{F}$ is the resulting foliation, then by Theorem 6.1 we have $\mathrm{cat}_{\mathcal{F}}(M) = m + 1$.

Locally free $\mathbb{R}^m$ actions frequently arise in the study of Lie groups, and have an important role in the study of hyperbolic dynamical systems. Let G be a connected Lie group of real rank m. Then the maximal $\mathbb{R}$-split torus is a semi-simple subgroup $\mathbb{R}^m \subset G$. Consequently, every space with a locally free G action also has a locally free $\mathbb{R}^m$-action. The simplest example is to consider a torsion-free uniform lattice $\Lambda \subset SL(m+1, \mathbb{R})$, and let $M = SL(m+1, \mathbb{R})/\Lambda$ be the compact quotient manifold. The subgroup of diagonal matrices with determinant 1 in $SL(m+1, \mathbb{R})$ is isomorphic to $\mathbb{R}^m$, and its action on M is locally free. There are many more examples of this type; see for example §7 of [**22**].

6.2. Anosov weak-stable foliations. Consider the codimension one foliation on a compact 3-manifold defined as the weak-stable foliation associated to the geodesic flow of a closed surface of constant negative curvature ([**14**], page 8 of [**30**]). This has an elementary description using Lie groups. Let $\Gamma \subset SL(2,\mathbb{R})$ be a uniform lattice, and set $M = \Gamma\backslash SL(2,\mathbb{R})$. Consider the elements of the Lie algebra of left-invariant vector fields $\mathfrak{sl}(2,\mathbb{R})$, identified with the matrices of trace 0,

$$X = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}, Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

which satisfy $[X,Y] = Y$, $[X,Z] = -Z$ and $[Y,Z] = 2X$.

The vector fields $\{X, Z\}$ define a Lie subalgebra, corresponding to the subgroup of lower triangular matrices $T(2) \subset SL(2,\mathbb{R})$. The leaves of $\mathcal{F}$ are the right cosets of $T(2)$ acting on M on the left, so the tangent bundle to $\mathcal{F}$ is spanned by the vector fields X, Z. The vector field Z is transverse to $\mathcal{F}$. Define the Riemannian metric on M by declaring the vector fields $\{X, Y, Z\}$ everywhere orthogonal. Introduce the dual forms X^*, Y^*, Z^*. The leafwise volume form is $\theta = X^* \wedge Z^*$, and Y^* spans the defining ideal $\mathcal{J}(M,\mathcal{F})$.

Note that $dX^* = -2Y^* \wedge X^*$, so $d_F X^* = 0$. In fact, $[X^*]_{\mathcal{F}} \in H^1_{\mathcal{F}}(M)$ is non-zero, since the form X^* can be integrated against a closed orbit of the flow of X – which is just the geodesic flow for the Riemannian surface $\Sigma = \Gamma\backslash SL(2,\mathbb{R})/O(2)$ – to get the length of the orbit, which is non-zero.

The relation $[X,Z] = -Z$ implies $dZ^* = X^* \wedge Z^*$, so $[X^* \wedge Z^*]_{\mathcal{F}} = 0$. For this example, we have more generally:

Proposition 6.2. *$H^2_{\mathcal{F}}(M) = 0$.*

Proof: Let φ_t denote the flow of X on M. Then $(\varphi_t)_*X = X$ and $(\varphi_t)_*Z = e^{-t}Z$ and hence $d\varphi_t X^* = X^*$ and $d\varphi_t Z^* = e^t Z^*$

Let $\theta = fX^* \wedge Z^*$. Set

$$g(z) = \int_{-\infty}^{0} e^t f(\varphi_t(z))\, dt, \quad \eta = gZ^*$$

The integral defining g exists as f is a bounded function. A much more subtle point is that g is actually smooth – it is as smooth as the foliation $\mathcal{F}$, so is C^∞. (See Guillemin and Kazhdan [**15**].) Then $d_F\eta = (Xg + g)X^* \wedge Z^*$. Now calculate

$$\begin{aligned}
Xg(z) &= \lim_{s\to 0} \frac{1}{s}\{g(\varphi_s(z)) - g(z)\} \\
&= \lim_{s\to 0} \frac{1}{s}\left\{\int_{-\infty}^{0} e^t f(\varphi_t(\varphi_s(z)))\, dt) - \int_{-\infty}^{0} e^t f(\varphi_t(z))\, dt\right\} \\
&= \lim_{s\to 0} \frac{1}{s}\left\{\int_{-\infty}^{s} e^{t-s} f(\varphi_t(z))\, dt) - \int_{-\infty}^{0} e^t f(\varphi_t(z))\, dt\right\} \\
&= \lim_{s\to 0} \frac{1}{s}\left\{e^{-s} \cdot \int_{0}^{s} e^t f(\varphi_t(z))\, dt) + (e^{-s} - 1) \cdot \int_{-\infty}^{0} e^t f(\varphi_t(z))\, dt\right\} \\
&= g(z) - f(z)
\end{aligned}$$

Hence, $d_F\eta = (Xg + g)X^* \wedge Z^* = fX^* \wedge Z^*$. $\square$

This example is a special case of a more general result, which is proved similarly:

THEOREM 6.3. *Let $\varphi\colon \mathbb{R}\times M \to M$ be a smooth Anosov flow on a compact manifold M of dimension $2n+1$. Let $\mathcal{F}$ denote the weak stable foliation of codimension n. Then $H^{n+1}_{\mathcal{F}}(M)=0$.*

Proof: First, there is the caveat that the foliation $\mathcal{F}$ is generically only Hölder continuous transversely, but with each leaf smoothly immersed in M [**1, 25**]. Thus, the complex of leafwise forms $\Omega^*_{\mathcal{F}}(M)$ is assumed to be leafwise smooth, but the transverse regularity can only be assumed continuous.

Let θ be a leafwise $(m+1)$-form, X the vector field generating φ_t, then define the m-form η

$$\eta = \int_{-\infty}^{0} \iota(X)(\varphi_t^*\theta)dt$$

which exists because the flow φ is exponentially contracting on the leafwise volume elements. The calculation $d\eta=\theta$ is similar to the proof of Proposition 6.2, but far more technical and is omitted. $\square$

The dynamically defined foliations associated to Anosov flows are an important class of examples, serving as a model for interactions of dynamics and the geometry of foliations. It is thus interesting that they are also a key example for this work. Associated to an Anosov flow on an $2n+1$-dimensional manifold are four natural foliations, the weak stable $\mathcal{F}_s$ and weak unstable $\mathcal{F}_u$ foliations of codimension n, and the strong stable $\mathcal{F}_{ss}$ and strong unstable $\mathcal{F}_{su}$ foliations of codimension $n+1$. For the weak stable and weak unstable foliations, the expansiveness of the flow on the leaves implies they have exponential growth. In fact, by Plante [**34**] if there is a leaf of non-exponential growth, then there is a transverse invariant measure for $\mathcal{F}$, so by the results of §4 the leafwise volume form is non-zero. So Theorem 6.3 is the complementary result to Theorem 4.2.

In contrast, the strong stable and strong unstable foliations have polynomial growth, so by Plante [**34**] and Theorem 4.2 there exists fundamental classes for $\mathcal{F}_{ss}$ and $\mathcal{F}_{su}$.

PROPOSITION 6.4. *Let $\Gamma\subset SL(2,\mathbb{R})$ be a uniform lattice, $M=\Gamma\backslash SL(2,\mathbb{R})$ and $\mathcal{F}$ the foliation defined by the left-invariant vector fields X, Z. Then $\mathrm{cat}_{\mathcal{F}}(M)=3$.*

Proof: It was noted that $[X^*]_{\mathcal{F}}\in H^1_{\mathcal{F}}(M)=E^{0,1}(M,\mathcal{F})$ is non-zero. Also note that

$$\iota(X)d(Y^*\wedge Z^*) = L_X(Y^*\wedge Z^*)+d\iota(X)(Y^*\wedge Z^*) = (+Y^*)\wedge Z^* + Y^*\wedge(-Z^*)+0=0$$

so $d_{0,1}(Y^*\wedge Z^*)=0$. Hence $[Y^*\wedge Z^*]_{\mathcal{F}}\in E_1^{1,1}(M,\mathcal{F})$ and the product

$$[X^*]_{\mathcal{F}}\bullet[Y^*\wedge Z^*]_{\mathcal{F}}\in E_1^{1,2}(M,\mathcal{F})$$

is represented by the volume form on M, so is non-trivial. Hence by Theorem 3.4 $\mathrm{cat}_{\mathcal{F}}(M)\geq 3$. As usual, $\mathrm{cat}_{\mathcal{F}}(M)\leq 3$ follows from Singhof and Vogt [**44**]. $\square$

An Anosov flow φ_t with generating vector field X is said to be *contact* if the dual 1-form X^* to the flow satisfies $X^*\wedge(dX^*)^n$ is a nowhere vanishing multiple of the volume form [**1**]. As mentioned above, the leaves of the weak stable foliation $\mathcal{F}_s$ are smoothly immersed in M, though the transverse regularity of $\mathcal{F}_s$ is typically only Hölder. The complex $E_0^{p,q}(M,\mathcal{F})$ can be defined using the p-forms along $\mathcal{F}$ with coefficients in the exterior complex $\bigwedge^q(T\mathcal{F}^{\perp})^*$. Define the differential $d_{0,1}=d_F$ the leafwise differential, noting that the normal bundle $T\mathcal{F}^{\perp}$ is flat when restricted to leaves. Then $[X^*]_{\mathcal{F}}\in E^{0,1}(M,\mathcal{F})$ and $[dX^*]_{\mathcal{F}}\in E^{1,1}(M,\mathcal{F})$ are well-defined.

Theorem 3.4 and Proposition 6.4 extend to this context to give the following:

PROPOSITION 6.5. *Let $\mathcal{F}$ be the weak stable foliation of a contact Anosov flow φ_t on a manifold M of dimension $2n+1$. Then* $\mathrm{cat}_{\mathcal{F}}(M) = n+2$.

Finally, given a collection of Anosov flows $\varphi_i \colon \mathbb{R} \times M_i \to M_i$ where M_i has dimension $2n_i + 1$ for $1 \le i \le k$, let $\mathcal{F}_i$ denote the weak stable foliation of φ_i. Form the product foliation $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$ of $M = M_1 \times \cdots \times M_k$ with leaf dimension $m = m_1 + \cdots m_k$. Then combining the arguments above with Theorem 3.4 and [**44**] yields $\mathrm{cat}_{\mathcal{F}}(M) = m+1$.

An Anosov flow defined by suspension of an Anosov diffeomorphism is never contact, and in fact the dual X^* form is closed. The tangential category for these foliations and their products can also be calculated using Theorem 3.4 and [**44**], with the result that $\mathrm{cat}_{\mathcal{F}}(M)$ is again $m+1$.

6.3. Secondary estimates. Our last examples are of a general nature. There are many constructions of foliations for which the Godbillon-Vey class $GV(\mathcal{F}) \in H^{2n+1}(M)$ is non-zero [**46, 21, 23**]. For example, Thurston showed that for each real $\alpha \in \mathbb{R}$ there is a foliation $\mathcal{F}_\alpha$ of S^3 for which $\langle GV(\mathcal{F}_\alpha), [S^3]\rangle = \alpha$. For $\alpha \neq 0$, by Theorem 5.3 we have $\mathrm{cat}_{\mathcal{F}_\alpha}(S^3) = 3$.

This estimate also follows from Proposition 2.9. However, one also knows that the same result holds for every odd dimensional sphere S^{2n+1}: given $\alpha \in \mathbb{R}$ there is a codimension n foliation $\mathcal{F}_\alpha$ on S^{2n+1} with $\langle GV(\mathcal{F}_\alpha), [S^{2n+1}]\rangle = \alpha$. For $\alpha \neq 0$, $\mathrm{cat}_{\mathcal{F}_\alpha}(S^{2n+1}) = n+2$.

Similarly, for all of the Heitsch examples in [**21**], which are foliations of dimension $n+1$ and codimension n with non-zero Godbillon-Vey invariant, their category is $n+2$.

Given a collection of foliated manifolds $(M_i, \mathcal{F}_i)$ for $1 \le i \le k$, where M_i has dimension $2n_i + 1$ and $\mathcal{F}_i$ has codimension n_i. Assume that $GV(\mathcal{F}_i) \neq 0$ for all i. Form the product foliation $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$ of $M = M_1 \times \cdots \times M_k$ with codimension $n = n_1 + \cdots + n_k$. Then $\mathrm{cat}_{\mathcal{F}}(M) = n+k+1$. To see this, note the proof of Theorem 5.3 used the definition of the Godbillon-Vey class to construct classes in $E_1^{0,1}$ for which we can apply Theorem 3.4 to get a lower bound estimate. This cup-length estimate works as well applied to the forms X_i^* and dX_i^* for each foliation $\mathcal{F}_i$. The resulting product is represented by the product of Godbillon-Vey classes, which is non-zero in cohomology.

This extension to products is similar to the extension of the contact Anosov case to products. In fact, these two classes of examples coincide for weak-stable foliations to the algebraic Anosov flows. But in general, these two cases are completely different. For example, there are no Anosov flows on S^3, while most weak stable foliations are not even smooth [**25**], so have no Godbillon-Vey invariant.

7. Open Questions and Problems

We mention a few of the open questions about the tangent category of foliations:

Problem 7.1. Let $\mathcal{F}$ be foliation of M and $\mathcal{F}'$ foliation of M' , set $M'' = M \times M'$ and $\mathcal{F}'' = \mathcal{F} \times \mathcal{F}'$, then show

$$\operatorname{cat}_{\mathcal{F}''}(M'') \leq \operatorname{cat}_{\mathcal{F}}(M) + \operatorname{cat}_{\mathcal{F}'}(M') - 1 \tag{7.1}$$

The calculations of tangent category in this paper support the estimate (7.1). Note that our lower bound estimates on $\operatorname{cat}_{\mathcal{F}}(M)$ are based on the cuplength estimate, which always satisfies a subadditivity estimate. In general, there is no technique developed for the general estimation of the category of a product in terms of the categories of the factors, beyond the upper bound dimension estimate by Singhof and Vogt.

Problem 7.2. If $\mathcal{F}$ is a subfoliation of $\mathcal{F}'$, what is the relationship between $\operatorname{cat}_{\mathcal{F}}(M)$ and $\operatorname{cat}_{\mathcal{F}'}(M)$?

The leaves of $\mathcal{F}$ are all submanifolds of the leaves of $\mathcal{F}'$, but so what? Every foliation $\mathcal{F}$ is a subfoliation of the foliation $\mathcal{F}'$ with one leaf M, so a special case is to ask for the relationship between $\operatorname{cat}_{\mathcal{F}}(M)$ and $\operatorname{cat} M$.

Problem 7.3. Let G be a connected Lie group, with real rank $k = \operatorname{rank}(G)$. Let $\varphi\colon G \times M \to M$ a locally free action on a compact manifold M, and $\mathcal{F}$ the foliation by the orbits of G. Show that $\operatorname{cat}_{\mathcal{F}}(M) \geq k$

There is a subfoliation $\mathcal{F}'$ of $\mathcal{F}$ by the orbits of the maximal $\mathbb{R}$-split torus $\mathbb{R}^k \subset G$. Then $\operatorname{cat}_{\mathcal{F}'}(M) = k$ by Theorem 6.1. Does this imply $\operatorname{cat}_{\mathcal{F}}(M) \geq k$ (cf. Singhof [**42, 43**])?

Problem 7.4. Suppose that $\mathcal{F}$ has a transverse invariant measure μ. Is there a relation between $\operatorname{cat}_{\mathcal{F}}(M)$ and the Morse inequalities for measured foliations of Connes and Fack [**10**]?

One of the most interesting open problems is to understand the relationship between $\operatorname{cat}_{\mathcal{F}}(M)$ and analysis on the leaves of $\mathcal{F}$. The Morse theory for foliations is not well-understood, so perhaps approaching it from the category viewpoint will yield new insights.

Problem 7.5. Give a homotopy-theoretic interpretation of $\operatorname{cat}_{\mathcal{F}}(M)$ corresponding to the Whitehead and Ganea definitions of category.

This is one of the most important open problems in the subject, and was asked by Yuli Rudyak during the evening problem sessions. Discussions at the week-long conference proposed this very natural problem, and several other related questions. Problems 7.4 and 7.5 are part of the general program to extend to the foliated context, the well-developed theory of category for spaces and manifolds.

The following result answers a question we were going to include in this list. It generalizes a theorem of Eilenberg and Ganea [**12**] that the category of a space $X = K(\pi, 1)$ equals the cohomological dimension of π.

Theorem 7.6. *Let M be a compact manifold, and assume the holonomy covering of each leaf of $\mathcal{F}$ is contractible, then* $\operatorname{cat}_{\mathcal{F}}(M) = m + 1$.

The proof of this result and Theorem 5.2 appear in [**24**].

References

[1] D. V. Anosov, *Geodesic Flows on Closed Riemannian Manifolds with Negative Curvature*, Proc. Steklov Inst. Math. vol. 90 (1967), Amer. Math. Soc., Providence, RI, 1969.

[2] J.L. Arraut and N.M. dos Santos, *Actions of R^p on closed manifolds*, Topology Appl. **29** (1988), 41–54.

[3] R. Bott and A. Haefliger, *On characteristic classes of G-foliations*, Bulletin Amer. Math. Soc. **78** (1972), 1039–1044.

[4] R. Bott, H. Shulman and J. Stasheff, *On the de Rham theory of certain classifying spaces*, Advances in Math. **20** (1976), 43–56.

[5] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, Berlin-New York, 1982.

[6] A. Candel and L. Conlon, *Foliations 1*, Amer. Math. Soc., Providence, RI, 2000.

[7] H. Colman, *Categoría LS en foliaciones*, Publicaciones del Departamento de Topología y Geometría , N 93, 1998.

[8] H. Colman, *LS-Categories for foliated manifolds*, Proc. Foliations: Geometry and Dynamics, Warsaw 2000, World Scientific Publishing, 2002, pp. 17–28.

[9] H. Colman and E. Macias, *Tangential Lusternik-Schnirelmann category of foliations*, J. London Math. Soc.**67** (2002), ?–??.

[10] A. Connes and T. Fack, *Morse inequalities for measured foliations*, Preprint, 1991, 1998.

[11] A. El Kacimi-Alaoui, *Sur la cohomologie feuilletée*, Compositio Mathematica, **49** (1983), 195–215.

[12] S. Eilenberg and T. Ganea, *On the Lusternik-Schnirelmann category of abstract groups*, Annals of Math. **(2) 65** (1957), 517–518.

[13] É. Ghys, *Sur l'invariant de Godbillon-Vey*. Seminaire Bourbaki, Mars 1989, Asterisque, vol. 177-178, Société Mathématique de France, 1989, pp. 155–181,

[14] C. Godbillon and J. Vey, *Un invariant des feuilletages de codimension 1*, C.R. Acad. Sci. Paris, **273** (1971), 92–95.

[15] V. Guillemin and D. Kazhdan, *On the cohomology of certain dynamical systems*, Topology **19** (1980), 291–299.

[16] A. Haefliger. *Homotopy and integrability*, Lect. Notes in Math., vol. 197, Springer–Verlag, New York and Berlin, 1971, pp. 133–163.

[17] A. Haefliger, *Some remarks on foliations with minimal leaves*, J. Diff. Geom. **15** (1980), 269–284.

[18] A. Haefliger, *Groupöides d'holonomie et classifiants*, Structures Transverses des Feuilletages. Toulouse 1982, Asterisque, vol. 116, Société Mathématique de France, 1984, pp. 70–97.

[19] G. Hector, *Cohomolgies transversales des feuilletages riemanniens I*, Feuilletgaes Riemanniens, quantification géométrique et mécanique, Travaux en Cours. **26** (1988), 15–40.

[20] J. Heitsch, *A cohomology for foliated manifolds*, Comment. Math. Helv. **50** (1975), 197–218.

[21] J. Heitsch, *Independent variation of secondary classes*, Annals of Math. **108** (1978), 421–460.

[22] S. Hurder, *Rigidity for Anosov actions of higher rank lattices*, Annals of Math. **135** (1992), 361–410.

[23] S. Hurder, *Dynamics and the Godbillon-Vey Classes: A History and Survey*, Proc. Foliations: Geometry and Dynamics, Warsaw 2000, World Scientific Publishing, 2002, pp. 17–28.

[24] S. Hurder, *Tangential category and classifying spaces of foliations*, Preprint, http://www.math.uic.edu/~hurder/publications (2002).

[25] S. Hurder and A. Katok, *Differentiability, rigidity and Godbillon-Vey classes for Anosov flows*, Publ. Math. Inst. Hautes Etudes Sci. **72** (1990), 5–64.

[26] I.M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology **17**(1978), 331–348.

[27] I.M. James, *Lusternik-Schnirelmann Category*, Chapter 27, Handbook of Algebraic Topology, 1995, pp. 1293–1310.

[28] I.M. James and J.R. Morris, *Fibrewise category*, Proc. Royal Soc. Edinburgh **119A** (1991), 177–190.

[29] F. Kamber and Ph. Tondeur, *Foliated bundles and characteristic classes*, Lect. Notes Math., vol. 494, Springer–Verlag, New York and Berlin, 1975.

[30] H.B. Lawson, Jr., *Quantitative Theory of Foliations*, Conf. Board Math. Sciences, vol. 27, Amer. Math. Soc., Providence, RI, 1977.

[31] J. McCleary, *User's Guide to Spectral Sequences*, Publish or Perish, Inc., Wilmington, DE, 1985.
[32] M. Mostow, *Continuous cohomology of spaces with two topologies*, Mem. Amer. Math. Soc., vol. 7, Amer. Math. Soc., Providence, RI 1976.
[33] J. Oprea and J. Walsh, *Quotient maps, group actions and Lusternik-Schnirelmann category*, Topology and its Applications (2002), ? –??.
[34] J. Plante, *Foliations with measure-preserving holonomy*, Annals of Math. **102** (1975), 327–361.
[35] C. Roger, *Cohomologie (p,q) des feuilletages et applications*, Structures Transverses des Feuilletages. Toulouse 1982, Asterisque, vol. 116, Société Mathématique de France, 1984, pp. 195–213.
[36] D. Ruelle and D. Sullivan, *Currents, flows and diffeomorphisms*, Topology **14** (1975), 319–327.
[37] N.M. dos Santos, *Foliated cohomology and characteristic classes*, Differential topology, foliations, and group actions (Rio de Janeiro, 1992), Contemp. Math., vol. 161, Amer. Math. Soc., Providence, RI, 1994, pp. 41–57
[38] N.M. dos Santos, *A survey of a theory of characteristic classes for locally free actions of Lie groups*, XI Brazilian Topology Meeting (Rio Claro, 1998), World Sci. Publishing, River Edge, NJ, 2000, pp. 115–131.
[39] H. Shulman, *Secondary obstructions to foliations*, Topology **13** (1974), 177–183.
[40] H. Shulman, *Covering dimension and characteristic classes for foliations*, Algebraic and geometric topology, Proc. Sympos. Pure Math., Stanford 1976, vol. XXXII Part 2, Amer. Math. Soc., Providence, RI, 1978, pp. 189–190.
[41] H. Shulman and J. Stasheff, *de Rham theory for $B\Gamma$*, Differential Topology, Foliations and Gelfand-Fuks cohomology, Rio de Janeiro (1976), Lect. Notes Math., vol. 652, Springer–Verlag, New York and Berlin, 1978, pp. 62–74.
[42] W. Singhof, *On the Lusternik-Schnirelmann category of Lie groups*, Math. Z., vol. = 145 (1975), 111–116.
[43] W. Singhof, *On the Lusternik-Schnirelmann category of Lie groups. II*, Math. Z., vol. = 151 (1976), 143–148.
[44] W. Singhof and E. Vogt, *Tangential category of foliations*, Preprint (2001).
[45] D. Sullivan, *Cycles for the dynamical study of foliated manifolds and complex manifolds*, Invent. Math. **36** (1976), 225–255.
[46] W. Thurston, *Non-cobordant foliations on S^3*, Bulletin Amer. Math. Soc. **78** (1972), 511–514.
[47] Ph. Tondeur. *Geometry of Foliations*, Monographs in Mathematics, vol. 90, Birkhäuser Verlag, Basel, Boston, Berlin, 1997.

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