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Category and compact leaves ☆

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Abstract

We prove that if $\mathcal{F}$ is a C^1–foliation of a compact manifold M with finite transverse saturated LS category, $\mathrm{cat}_s^{\pitchfork}(M,\mathcal{F}) < \infty$, then $\mathcal{F}$ has a compact leaf. In contrast, we show that if $\mathcal{F}$ is expansive on some non-trivial minimal set of $\mathcal{F}$, then $\mathrm{cat}_s^{\pitchfork}(M,\mathcal{F}) = \infty$. Examples of foliations are given to illustrate the main results of the paper.

1. Introduction

A set in a topological space X is said to be *categorical* if it is contractible in X. The Lusternik–Schnirelmann (LS) category of X is the least integer k such that X may be covered by k categorical open sets [20,16,17].

Let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be foliated manifolds. A homotopy $H : M' \times [0,1] \to M$ is said to be *foliated* if for all $0 \leqslant t \leqslant 1$, the map H_t sends each leaf L' of $\mathcal{F}'$ into another leaf L of $\mathcal{F}$. A subset U of M is *transversely categorical* if there is a foliated homotopy $H : U \times [0,1] \to M$ such that $H_0 : U \to M$ is the inclusion, and $H_1 : U \to M$ has image in a single leaf of $\mathcal{F}$. Here, U is regarded as a foliated manifold with the foliation induced by $\mathcal{F}$ on U. In other words, the subset U of M is transversely categorical if the inclusion $(U,\mathcal{F}_U) \hookrightarrow (M,\mathcal{F})$ factors through a leaf up to foliated homotopy. For example, a standard foliation

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chart is transversely categorical. The thesis of Colman [3,7] introduced the following two definitions.

Definition 1.1. The transverse LS category of a foliated manifold $(M,\mathcal{F})$ is the least number of transversely categorical open sets required to cover M.

The transverse LS category of $(M,\mathcal{F})$ is always finite when M is compact, as it is bounded above by the number of sets in an open covering by foliation charts. Moreover, the transverse LS category of a foliation is an invariant of foliated homotopy equivalence. This natural invariant has proven surprisingly difficult to calculate, even in simple examples. In the case of a foliation $\mathcal{F}$ defined by a fibration $\pi: M \to B$, the LS category of the base B is an upper bound for the transverse LS category of $\mathcal{F}$, but it is not known, except in almost trivial examples, if the transverse LS category of $\mathcal{F}$ is equal to the LS category of B.

A subset $U \subset M$ is *saturated* if it is a union of leaves of $\mathcal{F}$.

Definition 1.2. The transverse *saturated* LS category of a foliated manifold $(M,\mathcal{F})$ is the least number $\mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ of transversely categorical open saturated sets required to cover M. If no such covering exists, then $\mathrm{cat}_{\pitchfork}(M,\mathcal{F}) = \infty$.

The transverse *saturated* LS category $\mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ is also an invariant of foliated homotopy type, and for a foliation $\mathcal{F}$ defined by a fibration $\pi: M \to B$, it is almost by definition that $\mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ equals the LS category of the leaf (base) space $B = M/\mathcal{F}$. The transverse saturated LS category has other nice properties (cf. [3–7]) relating the number $\mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ to the topology of $\mathcal{F}$.

In this paper, we will consider only the transverse saturated LS category for foliations, and for notational convenience refer to this simply as "transverse category".

One of the most interesting problems about transverse category, is to determine which foliations have $\mathrm{cat}_{\pitchfork}(M,\mathcal{F}) < \infty$. That is, when can a given foliation $\mathcal{F}$ be covered by open saturated sets which are foliated homotopic to a single leaf? For example, a foliation is *compact* if all leaves of $\mathcal{F}$ are compact, and $\mathcal{F}$ is said to be *compact Hausdorff* if $\mathcal{F}$ is compact, and the quotient space $M/\mathcal{F}$ is Hausdorff. Colman proved in [7] that if $\mathcal{F}$ is a compact Hausdorff foliation of a compact manifold, then $\mathrm{cat}_{\pitchfork}(M,\mathcal{F}) < \infty$. The converse is also true, that if $\mathcal{F}$ is a compact foliation of a compact manifold M with $\mathrm{cat}_{\pitchfork}(M,\mathcal{F}) < \infty$, then $M/\mathcal{F}$ is Hausdorff [14].

Recall that $K \subset M$ is a minimal set for $\mathcal{F}$ if K is closed, saturated (a union of leaves) and there are no closed, saturated proper subsets of K. Equivalently, K is saturated, and each leaf $L \subset K$ is dense in K. Every compact saturated subset of M contains a minimal subset; in particular, for M compact, there always exists at least one minimal set for $\mathcal{F}$.

Our first result was suggested by the study of the examples in the last section of this paper, and shows the stability of compact minimal sets under deformation by foliated homotopy.

Theorem 1.3. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. Let K be a minimal set and $H: K \times [0,1] \to M$ a foliated homotopy, with H_0 the inclusion. Then $K_t = H_t(K)$ is a minimal set for all* $0 \leqslant t \leqslant 1$.

Corollary 1.4. *Let L be a compact leaf, and $H : L \times [0,1] \to M$ a foliated homotopy with H_0 the inclusion. Then $L_t = H_t(L)$ is a compact leaf for all* $0 \leqslant t \leqslant 1$.

Theorem 1.3 is used to prove the main result of this paper, an existence theorem for compact leaves in a C^1-foliation with finite category.

Theorem 1.5. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. If* $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) < \infty$, *then $\mathcal{F}$ has a compact leaf.*

The proof of Theorem 1.3 also suggests another property of foliations with finite transverse category. Recall that a foliation is said to be *expansive* if there is an $\varepsilon > 0$ so that for any two distinct leaves L, L' of $\mathcal{F}$ there is an element of holonomy separating them by a distance of at least ε. (This intuitive definition is made precise in Section 5.) We say $\mathcal{F}$ is expansive on a saturated subset $K \subset M$ if this property holds for $L \subset K$, $L' \subset M$. An expansive foliation is expansive on every minimal set.

Theorem 1.6. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. If $\mathcal{F}$ is expansive on some non-trivial minimal set of $\mathcal{F}$, then* $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \infty$.

The conclusion of Theorem 1.6 is related to a result of Langevin and Walczak. Corollary 2 of [18] shows that $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \infty$ for a codimension-one, C^2-foliation with exceptional minimal set. The C^2-hypotheses and Sacksteder's Theorem implies that $\mathcal{F}$ is expansive on such a minimal set.

The C^1 hypotheses is assumed in this paper throughout, but it seems likely that Theorems 1.3 and 1.6 can be proved for topological foliations as well by adapting the techniques of [11,12,23].

2. Foliation basics

We assume that M is a smooth, compact Riemannian manifold without boundary of dimension $m = p + q$, and $\mathcal{F}$ is a C^1-foliation of dimension p and codimension q. Given $x \in M$ we will denote by L_x the leaf of $\mathcal{F}$ containing x. We assume that each leaf L_x is a smoothly immersed submanifold, and the family of immersed submanifold $x \to L_x$ depends C^1 on x for the C^∞-topology on the leaves. This is sometimes referred to as a $C^{1,\infty}$-foliation.

We recall below some well-known facts about foliations, and include a discussion of Riemannian metrics and the geodesic exponential maps, as the proof of Proposition 3.1 in Section 3 requires explicit descriptions of these various metrics and maps.

Throughout this section, we will define (and redefine) a constant $\varepsilon_0 > 0$ by successively imposing conditions which must be satisfied. Thus, ε_0 is the minimum of a finite set of positive constants, but for notational convenience, we just keep redefining ε_0. Initially, assume that $0 < \varepsilon_0 < 1$.

Let $d_M : M \times M \to [0, \infty)$ be the distance function associated to the Riemannian metric g on M. For $R > 0$ we set $B_M(x, R) = \{y \in M \mid d_M(x, y) < R\}$. As M is compact, for $\varepsilon_0 > 0$ sufficiently small, we can assume that for every $x \in M$, $B_M(x, 10\varepsilon_0)$ is a totally normal neighborhood of x (cf. [8, p. 72].) This means that for any pair of points $y, z \in B_M(x, 10\varepsilon_0)$ there is a unique geodesic joining them which lies in $B_M(x, 10\varepsilon_0)$. In particular, $B_M(x, 10\varepsilon_0)$ is geodesically convex.

Let $T\mathcal{F}$ denote the tangent bundle to $\mathcal{F}$ and $\pi : Q \to M$ its normal bundle, identified with the subbundle $T\mathcal{F}^{\perp} \subset TM$ of vectors orthogonal to $T\mathcal{F}$. The Riemannian metric on TM induces a metric by fiberwise restriction on Q. Let TM^{ε} denote the disk subbundle of TM of vectors with length less than ε, and Q^{ε} the corresponding disk subbundle of Q.

Let $\exp_M : TM \to M \times M$ denote the total exponential map. If $p_1 : M \times M \to M$ is projection of the first factor, then $p_1 \circ \exp_M : TM \to M$ is the bundle projection onto the basepoint. For $x \in M$ and $p_2 : M \times M \to M$ the projection of the second factor, set $\exp = p_2 \circ \exp_M : TM \to M$ and $\exp_x \equiv p_2 \circ \exp_M : T_x M \to M$, which is the exponential map at x.

For each $x \in M$, the differential $D_{\vec{0}} \exp_x : TM_x \cong T(TM_x)_{\vec{0}} \to TM_x$ is the identity map. It follows that $\exp_x$ is a diffeomorphism in a sufficiently small neighborhood of $\vec{0} \in TM_x$. As M is compact, for $\varepsilon_0 > 0$ sufficiently small, the restriction $\exp_M : TM^{\varepsilon_0} \to M \times M$ is a diffeomorphism into.

2.1. Leafwise geometry

Let $d_L : L \times L \to [0, \infty)$ be the distance function on the leaf L induced by the restriction of the Riemannian metric to L. That is, for $x, x' \in L$ the distance $d_L(x, x')$ is the length of the shortest leafwise geodesic between x and x'. As M is compact, (L, d_L) is a complete metric space. We introduce the notation $d_{\mathcal{F}}$ for the collection of leafwise distance functions, where $d_{\mathcal{F}}(x, y) = d_L(x, y)$ if $x, y \in L$, and $d_{\mathcal{F}}(x, y) = \infty$ otherwise.

Given $x \in L$ and $R > 0$, let $B_{\mathcal{F}}(x, R) = \{y \in M \mid d_{\mathcal{F}}(y, x) < R\} = B_L(x, R) \subset L$.

For $x \in L$, let $\exp_x^{\mathcal{F}} : T_x L \to L$ denote the leafwise exponential map. Then $\exp_x^{\mathcal{F}}$ maps the ball of radius R in $T_x L$ onto the leafwise ball $B_{\mathcal{F}}(x, R)$, $\exp_x^{\mathcal{F}} : B_{T_x L}(x, R) \to B_{\mathcal{F}}(x, R)$.

For each $x \in M$, there exists a constant $\varepsilon_{\mathcal{F},x} > 0$ so that $B_{\mathcal{F}}(x, \varepsilon_{\mathcal{F},x})$ is totally normal for the metric $d_{\mathcal{F}}$ on L_x. Recall that this means that for any pair of points $y, z \in B_{\mathcal{F}}(x, \varepsilon_{\mathcal{F},x})$ there is a unique geodesic joining them which lies in $B_{\mathcal{F}}(x, \varepsilon_{\mathcal{F},x})$ (cf. [8, p. 72].) In particular, $B_{\mathcal{F}}(x, \varepsilon_{\mathcal{F},x})$ is geodesically convex.

As M is compact and the Riemannian metric on leaves depends continuously (in the leafwise C^{∞}-topology) on x, there exists a constant $\varepsilon_{\mathcal{F}} > 0$ so that we can assume $\varepsilon_{\mathcal{F},x} \geqslant \varepsilon_{\mathcal{F}}$ for all $x \in M$. We assume ε_0 satisfies $10\varepsilon_0 \leqslant \varepsilon_{\mathcal{F}}$.

For $\varepsilon_0 > 0$ sufficiently small, we can assume that for all $y, y' \in B_{\mathcal{F}}(x, 10\varepsilon_0)$,

$$d_M(y, y')/2 \leqslant d_{\mathcal{F}}(y, y') \leqslant 2d_M(y, y'). \tag{1}$$

2.2. Exponential map on the normal bundle

We next consider properties of the exponential map restricted to the normal bundle.

As M is compact, for $\varepsilon_0 > 0$ sufficiently small we can assume that for all $x \in M$, the restriction $\exp_x : Q_x^{\varepsilon_0} \to M$ of the exponential map to the normal ε_0-disk is transverse to $\mathcal{F}$.

Moreover, we can assume that the images of the "vertical fibers" $\exp_x(Q_x^{\varepsilon_0})$ have angle at least $\pi/4$ with the "horizontal" foliation $\mathcal{F}$ for all $x \in M$.

We next introduce a special type of open neighborhoods. Let $x \in L_x$ and $B_{\mathcal{F}}(x, 4\varepsilon) \subset L_x$ denote the leafwise disk of radius 4ε. We let $\mathcal{B}(x, \varepsilon)$ denote the restriction of the ε-disk bundle $\pi : Q^\varepsilon \to M$ to the subset $B_{\mathcal{F}}(x, 4\varepsilon)$. Note that $\mathcal{B}(x, \varepsilon)$ is the product of a leafwise 4ε-disk by a normal ε-disk.

Denote the restriction of the normal bundle projection by $\pi_{\mathcal{B}} : \mathcal{B}(x, \varepsilon) \to B_{\mathcal{F}}(x, 4\varepsilon)$. The restriction of the normal exponential map $p_2 \circ \exp_M : TM \to M$ to $\mathcal{B}(x, \varepsilon)$ is denoted by $\exp_{\mathcal{B}} : \mathcal{B}(x, \varepsilon) \to M$.

The differential $D_{(x,\vec{0})} \exp_M : T_{(x,\vec{0})}TM \to T_xM \times T_xM$ is an isomorphism, so the restriction of $D_{(x,\vec{0})}(p_2 \circ \exp_M) : T_{(x,\vec{0})}TM \to T_xM$ to $T_{(x,\vec{0})}\mathcal{B}(x, \varepsilon)$ is an isomorphism. Thus, for each $x \in M$ there exists $\varepsilon_x > 0$ such that $\exp_{\mathcal{B}} : \mathcal{B}(x, \varepsilon_x) \to M$ is a diffeomorphism onto its image. As M is compact, for $\varepsilon_0 > 0$ sufficiently small, we can assume that for all $x \in M$, $\exp_{\mathcal{B}} : \mathcal{B}(x, \varepsilon_0) \to M$ is a diffeomorphism onto its image.

Let $d_{\mathcal{B}}$ denote the induced Riemannian distance function on $\mathcal{B}(x, \varepsilon_0)$.

The map $\exp_{\mathcal{B}} : \mathcal{B}(x, \varepsilon_0) \to M$ yields an adapted neighborhood of x, consisting of a leafwise disk times normal geodesic coordinates. The unusual choice of $4\varepsilon_0$ for the radius of the leafwise base of the open sets $\mathcal{B}(x, \varepsilon_0)$ is due to its use in the proof of the following lemma.

Lemma 2.1. *For all $x \in M$, $B_M(x, \varepsilon_0) \subset \exp_{\mathcal{B}}(\mathcal{B}(x, \varepsilon_0))$.*

Proof. The point of the lemma is that any path of length at most ε can be "approximated" by a leafwise path of length at most 4ε, followed by a normal path of length at most ε. This would be obvious if the Riemannian metric on M were a product in a neighborhood of $B_{\mathcal{F}}(x, 4\varepsilon)$.

Let $y \in B_M(x, \varepsilon_0)$, then there is a path $\gamma_{x,y}$ from y to x with length less than ε. A point $z \in B_{\mathcal{F}}(x, 5\varepsilon_0) \cap B_M(y, \varepsilon_0)$ can be joined to y by a path of length at most ε_0 also, so $d_M(x, z) < 2\varepsilon_0$ and hence $d_{\mathcal{F}}(x, z) < 4\varepsilon_0$ by (1). It follows that the closure $K = \overline{B_{\mathcal{F}}(x, 5\varepsilon_0) \cap B_M(y, \varepsilon_0)}$ is a compact subset of $B_{\mathcal{F}}(x, 5\varepsilon_0)$. Let $y_* \in K$ be the closest point to y. Note that $d_M(y, x) < \varepsilon_0$ implies that $d_M(y, y*) < \varepsilon_0$. Then by the triangle inequality and (1)

$$d_M(y, x) \leqslant d_M(y, y_*) + d_M(y_*, x) \implies d_M(y_*, x) < 2\varepsilon_0 \implies d_{\mathcal{F}}(y_*, x) < 4\varepsilon_0$$

so that $y_* \in B_{\mathcal{F}}(x, 4\varepsilon_0)$. By the totally normal condition, there is a geodesic from y_* to y with length less than ε_0, hence $y = \exp_{\mathcal{B}}(\vec{v})$ for some $\vec{v} \in Q_{y_*}^{\varepsilon_0}$. That is, $y \in \exp_{\mathcal{B}}(\mathcal{B}(x, \varepsilon_0))$. $\square$

For $x \in M$, let $\vec{v} \in Q_x^{\varepsilon_0}$ so that $(x, \vec{v}) \in \mathcal{B}(x, \varepsilon_0)$. Set $y = \exp_{\mathcal{B}}(\vec{v}) = \exp_x(\vec{v}) \in M$.

The map $\exp_{\mathcal{B}}$ is transverse to $\mathcal{F}$ so induces a foliation $\widehat{\mathcal{F}}$ on $\mathcal{B}(x, \varepsilon_0)$. Let $\widehat{P_{\vec{v}}}$ denote the leaf of $\widehat{F}$ through the point $\vec{v}$. The image $\widehat{L_{\vec{v}}} = \exp_{\mathcal{B}}(\widehat{P_{\vec{v}}}) \subset L_y$ is an open leafwise neighborhood of y. Note that the preimage of $L_y \cap \exp_{\mathcal{B}}(\mathcal{B}(x, \varepsilon_0))$ could possibly consist of more than one leaf of $\widehat{\mathcal{F}}$.

Also note that the restriction of the map $\exp_{\mathcal{B}}$ to the zero section of $\mathcal{B}(x, \varepsilon_0)$ is the identity, and as $B_{\mathcal{F}}(x, 4\varepsilon) \subset L_x$, we have that $\exp_{\mathcal{B}} : \widehat{P_{\vec{0}}} \to B_{\mathcal{F}}(x, 4\varepsilon)$ is the identity map too.

By assumption, the images of the disk fibers $\exp_x(Q_x^{\varepsilon_0})$ have angle at least $\pi/4$ with $\mathcal{F}$, so each leaf $\widehat{P_{\vec{v}}}$ is uniformly transverse to the fibers of $\pi_{\mathcal{B}} : \mathcal{B}(x, \varepsilon_0) \to B_{\mathcal{F}}(x, 4\varepsilon)$. Thus, the restriction $\pi_{\mathcal{B}} : \widehat{P_{\vec{v}}} \to B_{\mathcal{F}}(x, 4\varepsilon)$ is a local diffeomorphism into.

Use the diffeomorphism $\exp_{\mathcal{B}}$ to define the *normal projection map* $\pi_{x,\vec{v}} : \widehat{L_{\vec{v}}} \to L_x$ where

$$\pi_{x,\vec{v}} = \exp_{\mathcal{B}} \circ \pi_{\mathcal{B}} \circ (\exp_{\mathcal{B}} \mid \widehat{P_{\vec{v}}})^{-1} : \widehat{L_{\vec{v}}} \to \widehat{P_{\vec{v}}} \to B_{\mathcal{F}}(x, 4\varepsilon) \to M. \tag{2}$$

The definition (2) is formal, but there is a much simpler geometric description of the maps $\pi_{x,\vec{v}}$. Given a point $y \in \exp_{\mathcal{B}}(\mathcal{B}(x, \varepsilon_0))$ then $\pi_{x,\vec{v}}(y)$ is the closest point in $B_{\mathcal{F}}(x, 4\varepsilon)$.

The map $\pi_{x,\vec{v}}$ allows us to compare the leafwise Riemannian metric on $T\mathcal{F}$ for nearby leaves. Note that the tangent maps $D\pi_{x,\vec{v}} : T_y\mathcal{F} \to T_x\mathcal{F}$ depend continuously on $\vec{v} \in Q_x^{\varepsilon_0}$, and $\pi_{x,\vec{0}}$ is the identity. As M is compact, for $\varepsilon_0 > 0$ sufficiently small we can assume that for all $x \in M$, for all $\vec{v} \in Q_x^{\varepsilon_0}$ and for all $\vec{w} \in T_y\mathcal{F}$

$$\|D\pi_{x,\vec{v}}(\vec{w})\|/2 \leqslant \|\vec{w}\| \leqslant 2\|D\pi_{x,\vec{v}}(\vec{w})\|. \tag{3}$$

This condition implies that if two leaves are "ε_0-close" then the normal projection map (2) is a quasi-isometry with distortion factor at most 2.

2.3. Regular foliation atlas

The definition of a foliation includes that assumption that for each point $x \in M$ there is a foliated coordinate chart; that is, an open neighborhood V_x and a homeomorphism $\phi_x : V_x \to (-1, 1)^n$ which maps the leaves of $\mathcal{F}$ restricted to V_x to the "horizontal" slices $(-1, 1)^p \times \{y\}$ for $y \in (-1, 1)^q$. It is useful to impose further conditions on these charts to insure that the intersections of the open sets have nice properties. We formulate these conditions as metric properties of the charts.

Definition 2.2. A collection $\{(U_\alpha, \phi_\alpha)) \mid \alpha \in \mathcal{A}\}$ is a *regular foliation atlas* for $\mathcal{F}$ if:

(F1) $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ is an open covering of M by $C^{1,\infty}$-coordinate charts $\phi_\alpha : U_\alpha \to (-1, 1)^n$.

(F2) Each coordinate chart $\phi_\alpha : U_\alpha \to (-1, 1)^n$ admits an extension to a $C^{1,\infty}$-coordinate chart $\tilde{\phi}_\alpha : \widetilde{U}_\alpha \to (-2, 2)^n$, where the closure $\overline{U_\alpha} \subset \widetilde{U}_\alpha$ and $\widetilde{U}_\alpha$ is contained in a convex subset of M for the metric d_M.

(F3) For each $\alpha \in \mathcal{A}$ and $z \in (-2, 2)^q$, set $y = \tilde{\phi}_\alpha^{-1}(\{0\} \times \{z\})$, then the plaque $\widetilde{\mathcal{P}}_\alpha(z) = \tilde{\phi}_\alpha^{-1}((-2, 2)^p \times \{z\}) \subset \widetilde{U}_\alpha$ is the connected component of $L_y \cap \widetilde{U}_\alpha$ containing y.

(F4) For each $\alpha \in \mathcal{A}$ and $z \in (-2, 2)^q$, $\mathcal{P}_\alpha(z)$ and $\widetilde{\mathcal{P}}_\alpha(z)$ are convex subsets of diameter less than $\varepsilon_{\mathcal{F}}$ with respect to $d_{\mathcal{F}}$.

The construction of a regular foliation atlas is discussed by many authors—see for example, Candel and Conlon [2, Chapter 1.2] or Walczak [25, Chapter 1.3], or Tamura [24]. As the above conditions differ somewhat from those in the literature, we sketch the construction of a regular foliation atlas satisfying conditions (F1)–(F4).

For each $x \in M$, let $\varepsilon_x > 0$ be sufficiently small so that $B_M(x, \varepsilon_x)$ is totally normal. Choose a foliation coordinate system $\phi_x : V_x \to (-1,1)^n$ with $V_x \subset B_M(x, \varepsilon_x)$ and $\phi_x(x) = 0$. For each $z \in (-1,1)^q$ let $\delta_z > 0$ be the largest number so that $B_{\mathcal{F}}(\phi_x^{-1}(\{0\} \times \{z\}), \delta_z)$ is totally normal. The continuity of the Riemannian metric on leaves implies there exists $\varepsilon_x > 0$ and $\delta_x > 0$ so that $\|z\| < \varepsilon_x$ implies $\delta_z > \delta_x > 0$. Define the open subsets formed by the union of leafwise convex disks

$$\begin{aligned} U_x &= \bigcup_{\|z\| < \varepsilon_x/2} B_{\mathcal{F}}\bigl(\phi_x^{-1}\bigl(\{0\} \times \{z\}\bigr), \delta_x/2\bigr) \\ &\subset \widetilde{U}_x = \bigcup_{\|z\| < \varepsilon_x} B_{\mathcal{F}}\bigl(\phi_x^{-1}\bigl(\{0\} \times \{z\}\bigr), \delta_x\bigr) \subset V_x. \end{aligned}$$

The restriction of the coordinate map $\phi_x : \widetilde{U}_x \to \mathbb{R}^n$ can then be modified by composing with some $\psi_x : \mathbb{R}^n \to \mathbb{R}^n$ so that $\psi_x \circ \phi_x : \widetilde{U}_x \to (-2,2)^n$ satisfies the hypotheses (F1)–(F4). Finally, as M is compact we can choose a finite collection of points $\{x_\alpha \mid \alpha \in \mathcal{A}\}$ so that the open sets $U_\alpha = U_{x_\alpha}$ form a finite cover of M, and we set $\phi_\alpha = \psi_{x_\alpha} \circ \phi_{x_\alpha}$.

For the remainder of this paper, fix a choice $\{(U_\alpha, \phi_\alpha)) \mid \alpha \in \mathcal{A}\}$ of a regular foliation atlas for $\mathcal{F}$.

Recall that a Lebesgue number for the covering $\mathcal{U}$ is a constant $\varepsilon > 0$ so that for each $x \in M$ there exists $U \in \mathcal{U}$ with $B_M(x, \varepsilon) \subset U$. As M is compact, the covering $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ admits some Lebesgue number, so we can assume that ε_0 is a Lebesgue number for $\mathcal{U}$. In particular, this implies that for any $x \in M$, the restriction of $\mathcal{F}$ to $B_M(x, \varepsilon_0)$ is a product foliation, and the leaves of $\mathcal{F} \mid B_M(x, \varepsilon_0)$ are totally normal discs for the metric $d_{\mathcal{F}}$.

The inverse images $\mathcal{P}_\alpha(z) = \phi_\alpha^{-1}((-1,1)^p \times \{z\}) \subset U_\alpha$ are smoothly embedded discs contained in the leaves of $\mathcal{F}$, called the *plaques* associated to the given foliation atlas. One thinks of the collection of all plaques as "tiling stones" which cover the leaves in a regular fashion. The convexity hypotheses in (F4) implies that if $U_\alpha \cap U_\beta \neq \emptyset$, then each plaque $\mathcal{P}_\alpha(z)$ intersects at most one plaque of U_β. The analogous statement holds for pairs $\widetilde{U}_\alpha \cap \widetilde{U}_\beta \neq \emptyset$. More generally, an intersection of plaques $\mathcal{P}_{\alpha_1}(z_1) \cap \cdots \cap \mathcal{P}_{\alpha_d}(z_d)$ is either empty, or a convex set in the leafwise metric.

The closure of each plaque $\overline{\mathcal{P}_\alpha(z)} = \tilde{\phi}_\alpha^{-1}([-1,1]^p \times \{z\}) \subset \widetilde{U}_\alpha$ is a compact set with interior (for the leaf topology) which depends continuously on the transverse parameter z, hence there exists constants $0 < C_{\min} \leqslant C_{\max}$ such that

$$C_{\min} \leqslant \operatorname{vol}\bigl(\mathcal{P}_\alpha(z)\bigr) \leqslant C_{\max}, \quad \forall \alpha \in \mathcal{A},\ \forall z \in (-1,1)^q. \tag{4}$$

For each $\alpha \in \mathcal{A}$, the extended chart $\tilde{\phi}_\alpha$ defines a C^1-embedding

$$t_\alpha = \phi_\alpha^{-1}(\{0\} \times \cdot) : (-2,2)^q \to \widetilde{U}_\alpha \subset M$$

whose image is denoted by $\widetilde{\mathcal{T}}_\alpha$. We will also assume that these images $\widetilde{\mathcal{T}}_\alpha$ are pairwise disjoint; this can be achieved by a small perturbation of the initial coordinate charts ϕ_x chosen in the construction of the regular charts. We can also assume that each submanifold $\widetilde{\mathcal{T}}_\alpha$ is everywhere perpendicular to the leaves of $\mathcal{F}$ by adjusting the given Riemannian metric on M in an open tubular neighborhood of each $\widetilde{\mathcal{T}}_\alpha$, while leaving the metric on $T\mathcal{F}$ unchanged. We may assume that each $\widetilde{\mathcal{T}}_\alpha$ has diameter at most 1.

Define $\mathcal{T}_\alpha = \phi_\alpha^{-1}(\{0\} \times (-1,1)^q)$. The local coordinate on $\mathcal{T}_\alpha$ is again denoted by $t_\alpha : (-1,1)^q \to \mathcal{T}_\alpha$. We use this coordinate to identify each transversal $\mathcal{T}_\alpha$ with $(-1,1)^q$.

We assume that the coordinates t_α are positively oriented, mapping the positive orientation for the normal bundle to $T\mathcal{F}$ to the positive orientation on $\mathbb{R}$.

The collection of all plaques for the foliation atlas is indexed by the *complete transversal*

$$\mathcal{T} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha.$$

For a point $x \in \mathcal{T}$, let $\mathcal{P}_\alpha(x) = \mathcal{P}_\alpha(t_\alpha^{-1}(x))$ denote the plaque containing x.

The Riemannian metric on M induces a Riemannian metric and corresponding distance function $\mathbf{d}_\mathcal{T}$ on each extended transversal $\widetilde{\mathcal{T}}_\alpha$. For $\alpha \neq \beta$ and $x \in \mathcal{T}_\alpha$, $y \in \mathcal{T}_\beta$ we set $\mathbf{d}_\mathcal{T}(x,y) = \infty$.

Given $x \in \widetilde{\mathcal{T}}_\alpha$ and $R > 0$, let $\mathbf{B}_\mathcal{T}(x,R) = \{y \in \widetilde{\mathcal{T}}_\alpha \mid \mathbf{d}_\mathcal{T}(x,y) < R\}$.

2.4. Homotopy groupoid

Next, recall the construction of the *homotopy groupoid* $\mathcal{G}_\mathcal{F}^h$ of $\mathcal{F}$ [19,21,26]. A point $[\gamma] \in \mathcal{G}_\mathcal{F}^h$ is the equivalence class of a continuous leafwise path $\gamma : [0,1] \to L$, where two leafwise paths γ_1 and γ_2 are equivalent provided $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$ and the paths are leafwise homotopic, relative endpoints. There are natural maps $s, r : \mathcal{G}_\mathcal{F}^h \to M$ defined by $s([\gamma]) = \gamma(0)$, and $r([\gamma]) = \gamma(1)$. The diagonal map $\Delta : M \to \mathcal{G}_\mathcal{F}^h$ maps $x \in M$ to the class $\tilde{x} = [\gamma_x]$ of the constant path γ_x at x.

The topology on $\mathcal{G}_\mathcal{F}^h$ is generated by the following basic open sets. Let $[\gamma] \in \mathcal{G}_\mathcal{F}^h$ with initial and terminal points $s([\gamma]) = x$ and $r([\gamma]) = y$, respectively. Let U_α be an open foliation chart containing x, and U_β be an open foliation chart containing y. Choose open neighborhoods $x \in U \subset U_\alpha$ and $y \in V \subset U_\beta$. Let γ be a leafwise path representing the equivalence class $[\gamma]$. The set $W = (U, \gamma, V)$ consists of all equivalence classes of leafwise paths which start in U, end in V, and which are homotopic to γ through a homotopy of leafwise paths whose endpoints remain in U and V respectively. The set W can also be described as

$$W \cong \bigcup_{x \in \mathcal{T}} P_x \times P'_{\gamma(x)}$$

where T is a transversal in U, P_x is the plaque in U containing x, $P'_{\gamma(x)}$ is the plaque in V containing $\gamma(x)$, $\gamma : \mathcal{T} \to \mathcal{T}'$ is the holonomy along γ and $\mathcal{T}'$ is a transversal in V.

The product map $s \times r : \mathcal{G}_\mathcal{F}^h \to M \times M$ is continuous for the topology on $\mathcal{G}_\mathcal{F}^h$ generated by the above basic open sets. As $M \times M$ is Hausdorff, the topology on $\mathcal{G}_\mathcal{F}^h$ is Hausdorff if and only if, given any pair of points $x, y \in L$ on the same leaf, we can separate points in the preimage $(r \times s)^{-1}(x,y)$. This is equivalent to requiring that given two leafwise paths $\gamma_0, \gamma_1 : [0,1] \to L$ from x to y, if $[\gamma_0] \neq [\gamma_1]$ then there are disjoint open neighborhoods $W_0 = (U_0, \gamma_0, V_0)$ and $W_1 = (U_1, \gamma_1, V_1)$.

If the separation property fails for two paths in a fiber, then there is a closed loop in a leaf L which is not contractible, and a sequence of leaves $\{L'_n \mid n = 1, 2, \ldots\}$ arbitrarily close to L so that the push-off of γ to L'_n is contractible in L'_n. If this condition never happens, then $\mathcal{G}^h_{\mathcal{F}}$ is said to have the "no vanishing loops" condition, and $\mathcal{G}^h_{\mathcal{F}}$ is Hausdorff [19,21] (see also [10]).

The basic open sets $W = (U, \gamma, V)$ give $\mathcal{G}^h_{\mathcal{F}}$ the structure of a smooth $(2p + q)$-dimensional manifold, such that the maps s and r are differentiable, and the product $s \times r : \mathcal{G}^h_{\mathcal{F}} \to M \times M$ is an immersion. The Riemannian metric on M induces a Riemannian metric on $M \times M$ which we lift back via the immersion $s \times r$ to a Riemannian metric on $\mathcal{G}^h_{\mathcal{F}}$.

The fibers of $s : \mathcal{G}^h_{\mathcal{F}} \to M$ form a foliation $\widehat{\mathcal{F}}$ of $\mathcal{G}^h_{\mathcal{F}}$ whose leaves have trivial holonomy.

For $x \in M$, let $\widetilde{L}_x = s^{-1}(x)$, then the restriction $r_x : \widetilde{L}_x \to L_x$ is the universal covering of L_x, where we consider L_x with the leaf topology induced by the leaf metric $d_{\mathcal{F}}$.

Give each fiber $\widetilde{L}_x$ the Riemannian metric $\tilde{d}_x$ induced by the inclusion into $\mathcal{G}^h_{\mathcal{F}}$. Then $r_x : \widetilde{L}_x \to L_x$ is a local isometry. We let $\tilde{d}_{\mathcal{F}}$ denote the family of leafwise metrics, defined by $\tilde{d}_{\mathcal{F}}(\tilde{x}, \tilde{y}) = \tilde{d}_x(\tilde{x}, \tilde{y})$ if $\tilde{x}, \tilde{y} \in \widetilde{L}_x$ and $\tilde{d}_{\mathcal{F}}(\tilde{x}, \tilde{y}) = \infty$ otherwise. Then $r : \mathcal{G}^h_{\mathcal{F}} \to M$ is a local isometry between the metrics $\tilde{d}_{\mathcal{F}}$ and $d_{\mathcal{F}}$.

3. Minimal sets

The main result of this section, Theorem 3.12, states that minimal sets are invariant under a foliated homotopy. This is a type of generalized stability theorem. It implies, in particular, that compact leaves are preserved by a foliated homotopy. The next proposition is the key technical fact needed for the proof of Theorem 3.12, and almost all of this section is devoted to its proof.

Proposition 3.1. *Let $\mathcal{F}$ be a C^1-foliation, K a compact saturated set, and $H : K \times [0, 1] \to M$ a foliated homotopy, with H_0 the inclusion. Then for each $L_0 \subset K$ and for all $0 \leqslant t \leqslant 1$, $H_t : L_0 \to L_t$ is onto the leaf L_t of $\mathcal{F}$.*

Proof. For a leaf L, we use $\iota : L \to M$ (the letter "iota") to denote the inclusion of L with the leaf topology (that is, the $d_{\mathcal{F}}$-metric topology) into M with the d_M-metric topology.

Since K is a compact set, the map $H : K \times [0, 1] \to M$ is uniformly continuous:

$$\begin{aligned} &\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0, \quad |t - t'| < \delta(\varepsilon) \text{ and } d_M(x, x') < \delta(\varepsilon) \\ &\quad \Longrightarrow d_M\big(H(x, t), H(x', t')\big) < \varepsilon. \end{aligned} \tag{5}$$

Set $\delta_0 = \min\{\delta(\varepsilon_0/2)/2, \varepsilon_0\}$. The following estimates follow from the definitions and (5):

$$\forall x \in L, \quad |t - t'| < \delta_0 \Longrightarrow d_M\big(H(x, t), H(x, t')\big) < \varepsilon_0/2, \tag{6}$$

$$\begin{aligned} &\forall t, 0 \leqslant t \leqslant 1, \forall x, x' \in L, \quad d_{\mathcal{F}}(x, x') < \delta_0 \\ &\quad \Longrightarrow d_{\mathcal{F}}\big(H(x, t), H(x', t)\big) < \varepsilon_0. \end{aligned} \tag{7}$$

Fix a leaf $L_0 \subset K$ and choose a basepoint $x_0 \in L_0$. For $0 \leqslant t \leqslant 1$, set $x_t = H_t(x_0)$ and let L_t denote the leaf of $\mathcal{F}$ containing x_t so that $H_t(L_0) \subset L_t$.

Recall that the restriction of r to the fibers of $s : \mathcal{G}^h_{\mathcal{F}} \to M$ are the universal coverings of the leaves. We denote $\widetilde{L}_t = s^{-1}(x_t)$ and the restriction of r to the fiber $s^{-1}(x_t)$ by r_t so that $r_t : \widetilde{L}_t \to L_t$ is the universal covering of L_t where $\widetilde{L}_t$ and L_t have the leaf topology induced by the leaf metric $d_{\mathcal{F}}$.

There is a "diagonal map" $\Delta : M \to \mathcal{G}^h_{\mathcal{F}}$ where $\Delta(x) = \tilde{x}$ is the constant path at x, so that $s(\tilde{x}) = r(\tilde{x}) = x$. Let $\tilde{x}_t = \Delta(x_t) \in \widetilde{L}_t$ be the lifting of the path x_t to $\mathcal{G}^h_{\mathcal{F}}$.

The next lemma shows that the homotopy H_t can be lifted to $\mathcal{G}^h_{\mathcal{F}}$.

Lemma 3.2. *For $L \subset K$, there exists a continuous map $\widetilde{H} : \widetilde{L}_0 \times [0,1] \to \mathcal{G}^h_{\mathcal{F}}$ such that $s \circ \widetilde{H}_t = x_t$, $\widetilde{H}_t(\tilde{x}) = \tilde{x}_t$ and $r_t \circ \widetilde{H}_t = H_t \circ r_0 : \widetilde{L}_0 \to L_t$.*

Proof. A point in the fiber $\widetilde{L}_t = s^{-1}(x_t)$ is a homotopy class $[\gamma]$ of paths in L_t starting at x_t with fixed endpoint $r([\gamma])$. Given a path $\gamma : [0,1] \to L_0$ with $\gamma(0) = x_0$ the image $H_t(\gamma)$ is a path in L_t starting at x_t. Define $\widetilde{H}_t([\gamma]) = [H_t(\gamma)]$. The map $\widetilde{H}_t$ is well-defined, as the image of leafwise homotopic paths are leafwise homotopic. (This property may fail for the lift of a homotopy to the holonomy groupoid Γ, which is why we work with $\mathcal{G}^h_{\mathcal{F}}$ instead.) Continuity is clear from the description of the local coordinate charts for $\mathcal{G}^h_{\mathcal{F}}$.

Note that $\widetilde{H}_t(\tilde{x})$ is represented by the constant path at x_t hence $\widetilde{H}_t(\tilde{x}) = \tilde{x}_t$.

For any $[\gamma] \in \widetilde{L}_0$, the path $H_t(\gamma)$ starts at $H_t(x_0) = x_t$ so $s \circ \widetilde{H}_t([\gamma]) = x_t$.

The endpoint of $H_t(\gamma)$ is $H_t(r_0[\gamma])$ hence $r_t \circ \widetilde{H}_t([\gamma]) = H_t \circ r_0([\gamma])$. $\square$

Observe that if $\widetilde{H}_t : \widetilde{L}_t \to \widetilde{L}_t$ is onto, then $H_t : L \to L_t$ is onto. Thus, the proof of Proposition 3.1 follows by proving that $\widetilde{H}_t$ is onto for all $0 \leqslant t \leqslant 1$. The key point is that H is uniformly continuous on $K \times [0,1]$, hence restricting to $L \times [0,1]$, we have $|t' - t| < \delta_0$ implies $H_{t'}(x)$ is $\varepsilon_0/2$ close to $H_t(x)$ for all $x \in L$. Hence, the same holds for the lift of H_t to the universal coverings – that is, $|t' - t| < \delta_0$ implies $\widetilde{H}_{t'}(x)$ is $\varepsilon_0/2$ close to $\widetilde{H}_t(x)$ for all $x \in \widetilde{L}$. This allows us to compare the two maps $\widetilde{H}_{t'}$ and $\widetilde{H}_t$ for $|t' - t| < \delta_0$ using the "normal bundle structure" of the leaves.

There is a technical problem that must be considered, which makes the following proof more involved. Though we use $\mathcal{G}^h_{\mathcal{F}}$ to define the maps $\widetilde{H}_t$ we do not use the groupoid to define the induction which will prove $\widetilde{H}_1 : \widetilde{L}_1 \to \widetilde{L}_1$ is onto. The problem is that if the space $\mathcal{G}^h_{\mathcal{F}}$ is non-Hausdorff, then the topology on $\mathcal{G}^h_{\mathcal{F}}$ is not metric. In fact, the metric geometries of the fibers are not continuous near a non-Hausdorff leaf $\widetilde{L}$—the distances in a path of leaves through $\widetilde{L}$ can decrease discontinuously when the covering "collapses". For this reason, we formulate an induction process in terms of a homotopy in a tubular neighborhood of the normal bundle to $\widetilde{L}_t$ rather than through a homotopy in a tubular neighborhood of $\widetilde{L}_t$ in $\mathcal{G}^h_{\mathcal{F}}$. This approach involves greater technical complexities, but avoids the problem of non-Hausdorff leaves. In fact, a corollary of the proof is that the path $\widetilde{L}_t$ cannot meet any non-Hausdorff points of $\mathcal{G}^h_{\mathcal{F}}$. We need two definitions.

Definition 3.3. Let (X, d_X) and (Y, d_Y) be complete metric spaces. A set map $f : X \to Y$ is *proper* if given any $x_0 \in X$ and $r > 0$, there exists $R > 0$ such that

$$f^{-1}\bigl(\bigl\{y \in Y \mid d_Y\bigl(y, f(x_0)\bigr) \leqslant r\bigr\}\bigr) \subset \{x \in X \mid d_X(x, x_0) \leqslant R\}.$$

Definition 3.4. Let (X, d_X) and (Y, d_Y) be connected smooth manifolds of dimension p with complete Riemannian metrics. A continuous proper map $f : X \to Y$ has $\mathbb{Z}/2\mathbb{Z}$-degree one if for any $y \in Y$, the image of the orientation class is non-zero under the map

$$f_* : H_p\bigl(X, X - f^{-1}(y); \mathbb{Z}/2\mathbb{Z}\bigr) \to H_p\bigl(Y, Y - \{y\}; \mathbb{Z}/2\mathbb{Z}\bigr).$$

Note that Y connected implies that $H_p(Y, Y - \{y\}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for all $y \in Y$.

The degree of the map is independent of the choice of point $y \in Y$, and depends only on the proper homotopy class of f. For details, see for example Dold [9, Chapter VIII]. The proof of the following is an elementary consequence of the definition:

Lemma 3.5. *If $f : X \to Y$ is a continuous proper map with $\mathbb{Z}/2\mathbb{Z}$-degree one, then f is onto.*

Given these remarks, we now set up an induction on the parameter t. Recall that $x_o \in L_0 \subset K$ is the chosen basepoint, which is used to define the lift of the homotopy $\widetilde{H}_t : \widetilde{L}_0 \to \widetilde{L}_t$ from Lemma 3.2. Fix $0 < s \leqslant 1$, choose an integer $N > 0$ with $1/N < \delta_0$, set $\Delta s = s/N$, and let $s_i = i \cdot \Delta s$. Set $L_i = L_{s_i}$, $H_i = H_{s_i}$ and $\widetilde{H}_i = \widetilde{H}_{s_i}$. We use induction on i to prove that $\widetilde{H}_s : \widetilde{L}_0 \to \widetilde{L}_s$ is onto.

For $i = 0$, $\widetilde{H}_0 : \widetilde{L}_0 \to \widetilde{L}_0$ is the identity, which is a proper, continuous map of degree one.

Claim 3.6. *For $0 \leqslant i < N$, if $\widetilde{H}_i : L_0 \to \widetilde{L}_i$ is a proper, continuous onto map of degree one, then $\widetilde{H}_{i+1} : \widetilde{L}_0 \to \widetilde{L}_{i+1}$ is a proper, continuous map of degree one, and hence is onto.*

Proof. Let $\xi_i = \iota \circ r_i : \widetilde{L}_i \to L_i \to M$ denote the immersion of the covering $\widetilde{L}_i$ into M.

Let $\tilde{\pi}_i : \widetilde{Q}_i = \xi_i^! Q \to \widetilde{L}_i$ denote the pull-back of bundle $Q \to M$ via ξ_i, and let $\eta_i : \widetilde{Q}_i \to TM$ be the vector bundle map which covers ξ_i, then set $\nu_i = \exp \circ \eta_i : \widetilde{Q}_i \to M$. We have the commutative diagram:

$$\begin{tikzcd}
\widetilde{Q}_i^{\varepsilon_0} \arrow[r, phantom, "\subset"] & \widetilde{Q}_i \arrow[r, "\eta_i"] \arrow[d, "\tilde{\pi}_i"] & TM \arrow[r, "\exp_M"] \arrow[d, "\pi"] \arrow[dr, "\exp"] & M \times M \arrow[d, "p_2"] \\
 & \widetilde{L}_i \arrow[r, "\xi_i"] \arrow[ul, "z_i"] & M & M
\end{tikzcd}$$

The restriction $\nu_i : \widetilde{Q}_i^{\varepsilon_0} \to M$ is a submersion as $\exp_x^Q : Q_x^{\varepsilon_0} \to M$ is transverse to $\mathcal{F}$ for all $x \in M$, hence ν_i induces a foliation $\widetilde{\mathcal{F}}_i$ on $\widetilde{Q}_i^{\varepsilon_0}$ which is everywhere transverse to the fibers of $\tilde{\pi}_i : \widetilde{Q}_i^{\varepsilon_0} \to \widetilde{L}_i$.

Let $z_i : \widetilde{L}_i \to \widetilde{Q}_i^{\varepsilon_0}$ denote the zero section with image denoted by $\widehat{L}_i$. The composition $\nu_i \circ z_i : \widetilde{L}_i \to M$ equals the composition $\xi_i : \widetilde{L}_i \to L_i \to M$, and so $\widehat{L}_i$ is a leaf of $\widetilde{\mathcal{F}}_i$.

Define a Riemannian metric $\tilde{g}_i$ on $\widetilde{Q}_i^{\varepsilon_0}$ by lifting the Riemannian metric g on M via ν_i, with the associated path-length metric on $\widetilde{Q}_i^{\varepsilon_0}$ denoted by $\tilde{d}_i$. Define a leafwise metric $\tilde{d}_{\mathcal{F}}$ for $\widetilde{\mathcal{F}}_i$ by lifting the Riemannian metric on $\mathcal{F}$ via ν_i.

For a leaf $\widehat{L} \subset \widetilde{Q}_i^{\varepsilon_0}$ of $\widetilde{\mathcal{F}}_i$ the restriction $\nu_i : \widehat{L} \to L$ is a local isometry onto its image, but it need not be onto the leaf L. In the case of the zero section, $\widehat{L_i}$ is always onto the leaf L_i as this is essentially the identity map. For the other leaves of $\widetilde{\mathcal{F}}_i$ the map $\nu_i : \widehat{L} \to L$ can fail to be a covering, as the leaf of $\widehat{F}$ could "escape" out of the open bundle $\widetilde{Q}_i^{\varepsilon_0}$ so not be onto the corresponding leaf of $\mathcal{F}$. We must show this does not happen for the leaves which are in the image of $\widehat{H}_i$. The idea is to show that these leaves are trapped in the open set $\widetilde{Q}_i^{\varepsilon_0}$.

Recall from Section 2 that given $x \in L_i$ the "box neighborhood" $\mathcal{B}_i(x, \varepsilon_0) = \tilde{\pi}_i^{-1}(B_{\mathcal{F}}(x, 4\varepsilon_0)) \subset \widetilde{Q}_i^{\varepsilon_0}$ is the normal ε_0-disk fiber over the leafwise disk $B_{\mathcal{F}}(x, 4\varepsilon_0) \subset L_i$, and the restriction of the normal exponential map, $\nu_i : \mathcal{B}_i(x, \varepsilon_0) \to M$, is an isometry onto its image with $B_M(x, \varepsilon_0) \subset \nu_i(\mathcal{B}_i(x, \varepsilon_0))$.

Lemma 3.7. *For all $s_i \leqslant t \leqslant s_{i+1}$, there exists a continuous map $\widehat{H}_t : \widetilde{L}_0 \to \widetilde{Q}_i^{\varepsilon_0}$ such that for all $\tilde{x}, \tilde{y} \in \widetilde{L}_0$,*

$$\nu_i \circ \widehat{H}_t = H_t \circ r_0 : \widetilde{L}_0 \to M, \qquad \widehat{H}_{s_i}(\tilde{x}) = z_i \circ \widetilde{H}_{s_i}(\tilde{x}), \tag{8}$$

$$\begin{aligned} &\tilde{d}_{\mathcal{F}}(\tilde{x}, \tilde{y}) < \delta_0, s_i \leqslant t, t' \leqslant s_{i+1} \\ &\quad \Longrightarrow \tilde{d}_i\big(\widehat{H}_t(\tilde{x}), \widehat{H}_{t'}(\tilde{y})\big) = d_M\big(H_t\big(r_0(\tilde{x})\big), H_{t'}\big(r_0(\tilde{y})\big)\big). \end{aligned} \tag{9}$$

Proof. Let $\tilde{x} \in \widetilde{L}_0$, $x = r_0(\tilde{x})$, and $s_i \leqslant t \leqslant s_{i+1}$, then $|t - s_i| \leqslant s_{i+1} - s_i = \Delta s < \delta_0$. So by Eq. (6),

$$d_M\big(H_t\big(r_0(\tilde{x})\big), H_{s_i}\big(r_0(\tilde{x})\big)\big) < \varepsilon_0/2 \tag{10}$$

That is, the path $\{H_t(x) \mid s_i \leqslant t \leqslant s_{i+1}\}$ is contained in the ball $B_M(H_{s_i}(x), \varepsilon_0/2) \subset M$.

The map $\nu_i : \mathcal{B}_i(x, \varepsilon_0) \to M$ is an isometry onto a set containing $B_M(H_{s_i}(x), \varepsilon_0/2)$ so we can use the relation (8) to define $\widehat{H}_t(\tilde{x})$. The equality (9) follows as ν_i is a local isometry.

Continuity of $\widehat{H}_t$ follows from the continuity of $H_t \circ r_0 : \widetilde{L}_0 \to M$ and the metric estimate (9). $\square$

Lemma 3.8. *Let $\widehat{L}_t \subset \widetilde{Q}_i^{\varepsilon_0}$ be the leaf of $\widetilde{\mathcal{F}}_i$ containing $\widehat{H}_t(\tilde{x}_0)$. Then $\widehat{H}_t(\widetilde{L}_0) \subset \widehat{L}_t$.*

Proof. By the condition (8) the image

$$\nu_i \circ \widehat{H}_t(\widetilde{L}_0) = H_t \circ r_0(\widetilde{L}_0) = H_t(L_0) \subset L_t.$$

Thus, $\widehat{H}_t(\widetilde{L}_0) \subset \nu_i^{-1}(L_t)$ so $\widehat{H}_t(\widetilde{L}_0)$ is contained in a union of leaves of $\widetilde{\mathcal{F}}_i$. The manifold $\widetilde{L}_0$ is path connected, so the image $\widehat{H}_t(\widetilde{L}_0)$ lies in the path component of the leaf of $\widetilde{\mathcal{F}}_i$ containing $\widehat{H}_t(\tilde{x}_0)$ which is $\widehat{L}_t$ by definition. $\square$

Lemma 3.7 implies that the image $\widehat{H}_t(\widetilde{L}_0)$ is contained in $\widetilde{Q}_i^{\varepsilon_0}$ for all $s_i \leqslant t \leqslant s_{i+1}$ and $\widehat{H}_t$ is a lifting of the homotopy H_t for $s_i \leqslant t \leqslant s_{i+1}$ such that at time $t = s_i$ it agrees with $\widetilde{H}_i$. In order to study the properties of the family $\{\widehat{H}_t \mid s_i \leqslant t \leqslant s_{i+1}\}$ of maps, we form the composition with the bundle projection $\tilde{\pi}_i$ to obtain a homotopy into the fixed leaf $\widetilde{L}_i$

$$\mathcal{H}_t = \tilde{\pi}_i \circ \widehat{H}_t : \widetilde{L}_0 \to \widetilde{L}_i, \quad s_i \leqslant t \leqslant s_{i+1}.$$

Note that $\mathcal{H}_{s_i} = \widetilde{H}_{s_i}$.

Lemma 3.9. *If $\widetilde{H}_i : \widetilde{L}_0 \to \widetilde{L}_i$ is a proper, continuous, degree-one map, then $\mathcal{H}_t : \widetilde{L}_0 \to \widetilde{L}_i$ is a proper, continuous, degree-one map for all $s_i \leqslant t \leqslant s_{i+1}$. In particular, $\mathcal{H}_t : \widetilde{L}_0 \to \widetilde{L}_i$ is onto.*

Proof. We first show that if $\widetilde{H}_i : \widetilde{L}_0 \to \widetilde{L}_i$ is proper, then $\mathcal{H}_t : \widetilde{L}_0 \to \widetilde{L}_i$ is proper for all $s_i \leqslant t \leqslant s_{i+1}$.

Recall that $\tilde{x}_0 \in \widetilde{L}_0$ is the basepoint. Given $r > 0$, let $\tilde{y} \in \widetilde{L}_0$ satisfy $\tilde{d}_{\mathcal{F}}(\mathcal{H}_t(\tilde{x}_0), \mathcal{H}_t(\tilde{y})) < r$. We need to obtain a uniform estimate on $\tilde{d}_{\mathcal{F}}(\tilde{x}_0, \tilde{y})$. Start with the equalities

$$\tilde{d}_{\mathcal{F}}\bigl(\widetilde{H}_i(\tilde{x}_0), \widetilde{H}_i(\tilde{y})\bigr) = \tilde{d}_{\mathcal{F}}\bigl(\widetilde{H}_{s_i}(\tilde{x}_0), \widetilde{H}_{s_i}(\tilde{y})\bigr) = \tilde{d}_{\mathcal{F}}\bigl(\mathcal{H}_{s_i}(\tilde{x}_0), \mathcal{H}_{s_i}(\tilde{y})\bigr) \tag{11}$$

and apply the triangle inequality to the last expression

$$\begin{aligned}\tilde{d}_{\mathcal{F}}\bigl(\mathcal{H}_{s_i}(\tilde{x}_0), \mathcal{H}_{s_i}(\tilde{y})\bigr) &\leqslant \tilde{d}_{\mathcal{F}}\bigl(\mathcal{H}_{s_i}(\tilde{x}_0), \mathcal{H}_t(\tilde{x}_0)\bigr) + \tilde{d}_{\mathcal{F}}\bigl(\mathcal{H}_t(\tilde{x}_0), \mathcal{H}_t(\tilde{y})\bigr)\\ &\quad + \tilde{d}_{\mathcal{F}}\bigl(\mathcal{H}_t(\tilde{y}), \mathcal{H}_{s_i}(\tilde{y})\bigr).\end{aligned} \tag{12}$$

By assumption, the second term on the right-hand side is bounded above by r. Using the estimate (1), the leafwise bi-Lipschitz estimate (3) and the assumption (6) we obtain

$$\begin{aligned}\tilde{d}_{\mathcal{F}}\bigl(\mathcal{H}_{s_i}(\tilde{x}_0), \mathcal{H}_t(\tilde{x}_0)\bigr) &= \tilde{d}_{\mathcal{F}}\bigl(\tilde{\pi}_i \circ \widehat{H}_{s_i}(\tilde{x}_0), \tilde{\pi}_i \circ \widehat{H}_t(\tilde{x}_0)\bigr)\\ &\leqslant 2d_{\mathcal{B}}\bigl(\tilde{\pi}_i \circ \widehat{H}_{s_i}(\tilde{x}_0), \tilde{\pi}_i \circ \widehat{H}_t(\tilde{x}_0)\bigr)\\ &\leqslant 4d_{\mathcal{B}}\bigl(\widehat{H}_{s_i}(\tilde{x}_0), \widehat{H}_t(\tilde{x}_0)\bigr)\\ &\leqslant 4\varepsilon_0/2 = 2\varepsilon_0.\end{aligned}$$

A similar estimate holds for $\tilde{y}$. Combine these estimates with (12) to obtain

$$\tilde{d}_{\mathcal{F}}\bigl(\widetilde{H}_i(\tilde{x}_0), \widetilde{H}_i(\tilde{y})\bigr) \leqslant 2\varepsilon_0 + r + 2\varepsilon_0 = r + 4\varepsilon_0. \tag{13}$$

By the assumption that $\widetilde{H}_i$ is proper, given the constant $r + 4\varepsilon_0$ there exists $R > 0$ so that

$$\bigl\{\tilde{y} \in \widetilde{L}_0 \mid \tilde{d}_{\mathcal{F}}\bigl(\widetilde{H}_i(\tilde{x}_0), \widetilde{H}_i(\tilde{y})\bigr) < r + 4\varepsilon_0\bigr\} \subset \bigl\{\tilde{y} \in \widetilde{L}_0 \mid \tilde{d}_{\mathcal{F}}(\tilde{x}_0, \tilde{y}) < R\bigr\}.$$

Thus, $\tilde{d}_{\mathcal{F}}(\tilde{x}_0, \tilde{y}) < R$, where R depends only on $r > 0$ and not on the parameter t. This shows that $\mathcal{H}_t$ is a uniformly proper family of maps.

Thus, the continuous family of continuous maps $\mathcal{H}_t : \widetilde{L}_0 \to \widetilde{L}_i$ for $s_i \leqslant t \leqslant s_{i+1}$ is a proper homotopy. Degree is constant under proper homotopy, so $\mathcal{H}_t$ has degree one for all $s_i \leqslant t \leqslant s_{i+1}$. $\square$

Lemma 3.8 showed that $\widehat{H}_t(\widetilde{L}_0) \subset \widehat{L}_t$. We next prove that the map is onto.

Lemma 3.10. *$\widehat{H}_t : \widetilde{L}_0 \to \widehat{L}_t$ is a continuous, proper, degree-one onto map for $s_i \leqslant t \leqslant s_{i+1}$.*

Proof. Lemma 3.9 implies that the map $\mathcal{H}_t = \tilde{\pi}_i \circ \widehat{H}_t : \widetilde{L}_0 \to \widetilde{L}_i$ is continuous, proper, degree-one and onto. We need to prove that the lift via the map $\tilde{\pi}_i : \widehat{L}_t \to \widetilde{L}_i$ has the same properties.

First, we prove that $\widehat{H}_t : \widetilde{L}_0 \to \widehat{L}_t$ is proper. Let $x, y \in \widehat{L}_t$. By the assumption (3), the differential $D_x\tilde{\pi}_i : T_x\widehat{L}_t \to T_x\widetilde{L}_i$ is bi-Lipshitz with constant 2. The path-length distance functions on $\widehat{L}_t$ and $\widetilde{L}_i$ are thus related by

$$\tilde{d}_{\mathcal{F}}\big(\tilde{\pi}_i(x), \tilde{\pi}_i(y)\big) \leqslant 2\tilde{d}_{\mathcal{F}}(x, y).$$

It follows that given $x \in \widehat{L}_t$ the leafwise disk $B_{\widetilde{\mathcal{F}}}(x, r) \subset \widehat{L}_t$ satisfies $\tilde{\pi}_i(B_{\widetilde{\mathcal{F}}}(x, r)) \subset B_{\widetilde{\mathcal{F}}}(\tilde{\pi}_i(x), 2r)$.

Now, given that $\mathcal{H}_t$ is proper, there exists $R > 0$ such that $\mathcal{H}_t^{-1}(B_{\widetilde{\mathcal{F}}}(\tilde{\pi}_i(x), 2r)) \subset B_{\widetilde{\mathcal{F}}}(x_0, R)$.

Hence $\widehat{H}_t^{-1}(B_{\widetilde{\mathcal{F}}}(x, r)) \subset \tilde{\pi}_i(B_{\widetilde{\mathcal{F}}}(x, r)) \subset B_{\widetilde{\mathcal{F}}}(\tilde{\pi}_i(x), 2r)$ so the map $\widehat{H}_t$ is proper.

The degree is an integer invariant which multiplies under the composition of maps, so $\mathcal{H}_t : \widetilde{L}_0 \to \widetilde{L}_i$ has degree one implies $\widehat{H}_t$ has degree one. Thus, $\widehat{H}_t : \widetilde{L}_0 \to \widehat{L}_t$ is a continuous, proper map with degree one, hence is onto. □

The proof of the inductive step Claim 3.6 is now almost complete, as Lemma 3.9 implies that $\widehat{H}_{s_{i+1}} : \widetilde{L}_0 \to \widehat{L}_{s_{i+1}}$ is a continuous, proper, degree-one onto map. We need to show the same properties are true for $\widetilde{H}_{s_{i+1}} : \widetilde{L}_0 \to \widetilde{L}_{s_{i+1}}$ and this follows from the following geometric lemma.

Lemma 3.11. *If $\widehat{H}_t : \widetilde{L}_0 \to \widehat{L}_t$ is a continuous proper degree-one onto map for $s_i \leqslant t \leqslant s_{i+1}$, then there is an isometric diffeomorphism $\Phi_t : \widehat{L}_t \to \widetilde{L}_t$ such that $\widetilde{H}_t = \Phi_t \circ \widehat{H}_t$.*

Proof. First we show that the leaf $\widehat{L}_t$ is a complete Riemannian manifold for $s_i \leqslant t \leqslant s_{i+1}$. Let $\{\hat{y}_n \mid n = 1, 2, \ldots\} \subset \widehat{L}_t$ be a Cauchy sequence. Choose points $\tilde{x}_n \in \widetilde{L}_0$ such that $\mathbf{H}_t(\tilde{x}_n) = \hat{y}_n$. As $\widehat{H}_t$ is a proper map, the set $\{\tilde{x}_n \mid n = 1, 2, \ldots\}$ is bounded, so there is a convergent subsequence. Pass to a subsequence and re-label, so we can assume that $\tilde{x}_n \to \tilde{x}_*$. Then

$$\lim_{n\to\infty} \hat{y}_n = \lim_{n\to\infty} \widehat{H}_t(\tilde{x}_n) = \widehat{H}_t(\tilde{x}_*) = \hat{y}_* \in \widehat{L}_t.$$

We noted above that $\nu_i : \widehat{L}_t \to L_t$ is a locally isometric map, so $\widehat{L}_t$ complete implies that ν_i is a covering map.

Lemma 3.9 implies the restriction of the bundle map $\tilde{\pi}_i : \widehat{L}_t \to \widetilde{L}_i$ is onto, and thus this is also a covering map. As $\widetilde{L}_i = \widetilde{L}_{s_i}$ is simply connected, every covering map to $\widetilde{L}_i$ is a diffeomorphism. It follows that the composition

$$\nu_i \circ (\pi_i \mid \widehat{L}_t)^{-1} : \widetilde{L}_i \to L_t$$

is the universal covering map of the leaf L_t.

Define the map $\Phi_t : \widehat{L}_t \to \widetilde{L}_t$ for $s_i \leqslant t \leqslant s_{i+1}$ via the path lifting property, starting with the basepoint $\hat{x}_t = \widehat{H}_t(\tilde{x}_0) \in \widehat{L}_t$ which is mapped to the basepoint $\tilde{x}_t = \widetilde{H}_t(\tilde{x}_0) \in \widetilde{L}_t$. The metrics on $\widehat{L}_t$ and $\widetilde{L}_t$ are both defined as the lift of the Riemannian metric on L_t so Φ_t is locally isometric.

Finally, both maps $\widetilde{H}_t$ and $\widehat{H}_t$ are defined by local path lifting property, starting with the map $\widetilde{H}_{s_i}$, so the identity $\widetilde{H}_t = \Phi_t \circ \widehat{H}_t : \widetilde{L}_0 \to \widetilde{L}_t$ is immediate. □

To complete the proof of Claim 3.6, note that Lemma 3.9 implies $\widehat{H}_{s_{i+1}} : \widetilde{L}_0 \to \widehat{L}_{s_{i+1}}$ is a continuous, proper, degree-one onto map. Lemma 3.11 implies that for the diffeomorphism $\Phi_{s_{i+1}}$ we have

$$\widetilde{H}_{s_{i+1}} = \Phi_{s_{i+1}} \circ \widehat{H}_{s_{i+1}} : \widehat{L}_0 \to \widetilde{L}_{s_{i+1}}$$

where $\Phi_{s_{i+1}}$ is a diffeomorphism. Hence, $\widetilde{H}_{s_{i+1}} : \widetilde{L}_0 \to \widetilde{L}_{s_{i+1}}$ is a continuous, proper, degree-one onto map. This also completes the proof of Proposition 3.1. □

We can now give the main result of this section:

Theorem 3.12. *Let K be a compact saturated set for a C^1-foliation $\mathcal{F}$. Let $H : K \times [0, 1] \to M$ be a foliated homotopy, with H_0 the inclusion. Then for all $0 \leqslant t \leqslant 1$, $K_t = H_t(K)$ is a compact saturated set.*

Proof. For each $0 \leqslant t \leqslant 1$, set $K_t = H_t(K)$ which is compact, as K is compact and H_t is continuous. Let $L \subset K$ be a leaf, and set $X_t = H_t(L)$. Then $X_t \subset L_t$ for some leaf of $\mathcal{F}$, as H_t is a foliated homotopy. By Proposition 3.1, the map $H_t : L_0 \to L_t$ is onto, so $L_t = X_t \subset K_t$ and hence each K_t is saturated. □

Corollary 3.13. *Let K be a compact minimal set for a C^1-foliation $\mathcal{F}$. Let $H : K \times [0, 1] \to M$ be a foliated homotopy, with H_0 the inclusion. Then for all $0 \leqslant t \leqslant 1$, $K_t = H_t(K)$ is a compact minimal set.*

Proof. By Theorem 3.12, $K_t = H_t(K)$ is a compact saturated set. Given a leaf $L_t \subset K_t$, there is some leaf $L \subset K$ with $H_t(L) \subset L_t$. As K is minimal, L is dense in K, hence the image $H_t(L)$ dense in K_t so also the leaf L_t must be dense in K_t. □

We conclude this section with one more observation about minimal sets.

Lemma 3.14. *Let U be a saturated open set for a foliation $\mathcal{F}$ of a compact manifold M. Suppose K is a minimal set for $\mathcal{F}$ with $K \cap U \neq \emptyset$, then $K \subset U$.*

Proof. Suppose there exists a leaf $L \subset K \cap (M - U)$, then as $(M - U)$ is a closed saturated set, the closure of L must be contained in $(M - U)$. Then K minimal implies that every leaf of K is dense, hence $\overline{L} = K \subset M - U$, or $K \cap U = \emptyset$ which is a contradiction. □

4. Existence of a compact leaf

In this section we give the proof of Theorem 1.5.

Assume $\mathcal{F}$ has finite transverse category, with $k = \mathrm{cat}_{\pitchfork}(M, \mathcal{F})$. Choose a covering $\{U_1, \ldots, U_k\}$ of M by transversely categorical saturated open sets, with foliated homotopies $H^i : U_i \times [0, 1] \to M$ such that H^i_1 has image in a leaf $L^i \subset M$. As k is minimal, the union $U_2 \cup \cdots \cup U_k$ does not cover all of M. Therefore, the complement $C_1 = M - U_2 \cup \cdots \cup U_k$ is a non-empty closed saturated set.

For $L \in C_1$, its closure $\overline{L} \subset C_1$ is a compact saturated set, so contains a minimal set $K^1 \subset \overline{L}$. As $C_1 \subset U_1$ we have $K^1 \subset \overline{L} \subset U_1$. By Theorem 3.12, the image $K_t^1 = H_t(K^1)$ is a compact saturated set for all t. Hence, for $t = 1$ we have K_1^1 is a compact saturated minimal set contained in the leaf L^1. Thus, L^1 is a compact leaf.

The above proof easily extends to show that $\mathcal{F}$ has at least $k = \mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ disjoint minimal sets, $K^i \subset U_i$. Moreover, each image $H_1^i(K^i) = L^i$ is a compact leaf in M. The proof does not show that the leaves L^i are distinct, a topic which is discussed in the sequel to this paper [13] where we obtain estimates for the number of distinct compact leaves in terms of $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ and the topology of M.

5. Expansive foliations

Recall the notion of holonomy for the foliation $\mathcal{F}$. Given a leafwise path $\gamma : [0, 1] \to L$, let $\mathcal{P}_\alpha(x_0)$ be a plaque containing $\gamma(0)$ where $x_0 \in \mathcal{T}_\alpha$ and $\mathcal{P}_\beta(x_1)$ be a plaque containing $\gamma(1)$ where $x_1 \in \mathcal{T}_\beta$. The holonomy f_γ along γ is then a local homeomorphism from an open neighborhood of $x_0 \in \mathcal{T}_\alpha$ to an open neighborhood of $x_1 \in \mathcal{T}_\beta$: Given $y_0 \in \mathcal{T}_\alpha$ let L' be the leaf containing y_0. Then for y_0 sufficiently close to x_0 there is a leafwise path $\gamma' : [0, 1] \to L'$ such that $d_M(\gamma(t), \gamma'(t)) < \varepsilon_{\mathcal{U}}$ for all $0 \leqslant t \leqslant 1$. Let $y_1 \in \mathcal{T}_\beta$ be such that $\gamma'(1) \in \mathcal{P}_\beta(y_1)$. Then $h_\gamma(y_0) = y_1$.

A foliation $\mathcal{F}$ is *ε-expansive* [15] if given any transversal $\mathcal{T}_\alpha$ and $x_0, y_0 \in \mathcal{T}_\alpha$ there is a leafwise path γ with $\gamma(0) = x_0$ such that y_0 is in the domain of h_γ and $d_{\mathcal{T}}(h_\gamma(x_0), h_\gamma(y_0)) \geqslant \varepsilon$. If $d_{\mathcal{T}}(x_0, y_0) \geqslant \varepsilon$, then we can choose γ to be the constant path. Given a closed saturated set K, we say that $\mathcal{F}$ is *ε-expansive* on K if the expansive condition holds for all $x_0 \in K \cap \mathcal{T}_\alpha$ and all $y_0 \in \mathcal{T}_\alpha$.

Note that if $0 < \varepsilon' < \varepsilon$ then $\mathcal{F}$ is ε-expansive implies $\mathcal{F}$ is ε'-expansive.

The proof of Theorem 1.5 started with a simple observation, that if K is a compact saturated set, then any foliated homotopy H_t restricted to K must be uniformly continuous. Thus, if $L \subset K$ is a leaf and $t < s$ are sufficiently close, then the image leaves $H_t(L)$ and $H_s(L)$ are uniformly close. Since the homotopy maps leaves to leaves, this implies there are leaves which do not "expand" away from each other. We use this observation to prove the following:

Proposition 5.1. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. Let K be a compact saturated set, and $H : K \times [0, 1] \to M$ a foliated homotopy with H_0 the inclusion map. Suppose that there exists a leaf $L_0 \subset K$ and $t > 0$ such that $H_t(L_0)) \not\subset L_0$. Then $\mathcal{F}$ is not ε-expansive on K for any $\varepsilon > 0$.*

Proof. Let $L_0 \subset K$ be a leaf and let L_t denote the leaf of $\mathcal{F}$ containing $H_t(L_0)$. (By Proposition 3.1 we know that $H_t(L_0) = L_t$, but this fact is not used in the proof below.) Let

$$s_0 = \sup\bigl\{s \mid 0 \leqslant t \leqslant s \Longrightarrow H_t(L_0) \subset L_0\bigr\}.$$

Then $H_{s_0}(L_0) \subset L_0$. Without loss of generality we can assume that $s_0 = 0$, so that for all $\delta > 0$ there exists $0 < t < \delta$ with $L_t \neq L_0$.

Given $0 < \varepsilon < \varepsilon_0$ let $\delta(\varepsilon)$ be the modulus of continuity for H on K as defined by Eq. (5) and set $\delta_0 = \min\{\delta(\varepsilon/2)/2, \varepsilon\}$. Choose $0 < t < \delta_0$ with $L_t \neq L_0$.

Let $x_0 \in L_0 \cap \mathcal{T}_\alpha$ and let $\gamma : [0,1] \to L_0$ be any leafwise curve with $\gamma(0) = x_0$, $\gamma(1) = y_0 \in L_0 \cap \mathcal{T}_\beta$. The path $\gamma' = H_t \circ \gamma$ satisfies $d_M(\gamma(s), \gamma'(s)) < \varepsilon/2$ for all $0 \leqslant s \leqslant 1$, so the path γ' on L_t defines the holonomy map h_γ applied to $x_t = H_t(x_0)$. That is, $h_\gamma(x_t) = y_t$ where $\gamma'(1) \in \mathcal{P}_\beta(y_t)$. Thus, $d_{\mathcal{T}}(h_\gamma(x_0), h_\gamma(x_t)) < \varepsilon$. This holds for all leafwise paths γ with $\gamma(0) = x$, so $x_0, x_t \in \mathcal{T}_\alpha$ cannot be ε-separated. It follows that $\mathcal{F}$ is not ε-expansive at $x_0 \in K$. $\square$

Now suppose K is a non-trivial minimal set on which $\mathcal{F}$ is ε-expansive. If $\mathrm{cat}_\pitchfork(M, \mathcal{F}) < \infty$ then there must exist some transversely categorical saturated open set U containing a point of K, and hence by Lemma 3.14 it must be that $K \subset U$. Proposition 5.1 shows that $\mathcal{F}$ cannot be ε'-expansive for $0 < \varepsilon' < \varepsilon_0$. This is a contradiction, so $\mathrm{cat}_\pitchfork(M, \mathcal{F}) = \infty$. This completes the proof of Theorem 1.6.

6. Three examples

We give some examples of foliations of compact manifolds with finite transverse category and non-compact leaves. Recall, that for $\mathrm{cat}_\pitchfork(M, \mathcal{F}) < \infty$, $\mathcal{F}$ must have a compact leaf by Theorem 1.5, and $\mathcal{F}$ cannot be expansive by Theorem 1.6. The basic question is what other limitations on $\mathcal{F}$ are imposed by finite transverse category? The first example is probably the most basic, yet illustrates many important aspects of transversely categorical sets in foliated manifolds.

Example 6.1 *(A proper foliation of T^2).* The following example was suggested during conversations (which included Elmar Vogt and Paul Schweitzer) following the talk by Hellen Colman on "Transverse Category Theory" at the conference *Foliations: Geometry and Dynamics*, Warsaw 2000. Consider the foliation of the 2-torus with two oriented Reeb components [22]: on the vertical strip $\{(x, y) \mid -\pi \leqslant x \leqslant \pi\}$, we identify $x \sim x + 2\pi$ and $y \sim y + 2\pi$. There is a compact leaf L_1 corresponding to the vertical lines $x = -\pi$ and $x = \pi$ and a second compact leaf L_2 corresponding to the vertical line $x = 0$. The other leaves are the vertical translates of the graphs $\{y = \tan(\pi/2 - x) \mid 0 < x < \pi\}$ and $\{y = \tan(x + \pi/2) \mid -\pi < x < 0\}$. Define U_1 to be the complement of L_1 and U_2 the complement of L_2. Then $\{U_1, U_2\}$ is a transversely categorical cover for $(\mathbf{T}^2, \mathcal{F})$, so $\mathrm{cat}_\pitchfork(\mathbf{T}^2, \mathcal{F}) = 2$. See Fig. 1.

Note that $\mathcal{F}$ is a proper foliation with depth one.

The next two examples illustrate a more subtle phenomenon, that a non-trivial minimal set in a compact foliated manifold can be deformed by a foliated homotopy.

Suppose that $\mathrm{cat}_\pitchfork(M, \mathcal{F})$ is finite. Given a compact minimal set K of $\mathcal{F}$, which is not a single compact leaf, hence all leaves of K are non-compact and dense in K. Let $\{U_1, \ldots, U_k\}$ be a finite covering of M by transversely categorical saturated open sets, then $K \subset U_i$ for some i by Lemma 3.14. Let $U = U_i$ and let $H : U \times [0,1] \to M$ be a foliated homotopy to a leaf L_1. The restriction $H : K \times [0,1] \to M$ is uniformly continuous, and

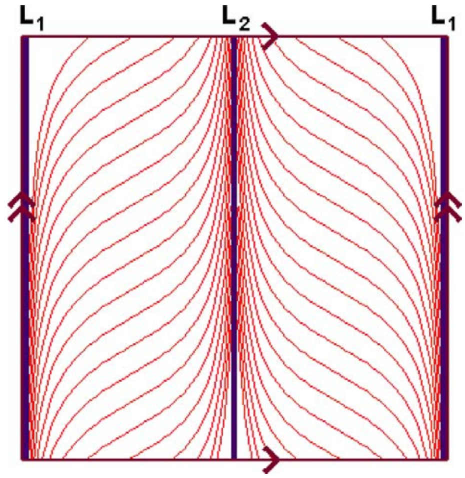

Fig. 1.

for each leaf $L_0 \subset K$ the family of leaves $L_t = H_t(L_0)$ starts at the non-compact leaf L_0 and ends with the leaf L_1 which must be compact by the proof of Theorem 1.5. This very special type of moving non-compact leaf occurs in both of the following examples.

Example 6.2 *(Irrational flow on S^3).* We define a foliation $\mathcal{F}_{a,b}$ of the 3-sphere as the orbits of a locally free action of $\mathbf{R}$. Let $S^3 = \{[z,w] \mid z\bar{z} + w\overline{w} = 1\}$. For non-zero real numbers a, b such that a/b is irrational, set

$$t \cdot [z,w] = \left[e^{2\pi iat} \cdot z, e^{2\pi ibt} \cdot w\right].$$

The orbits of $[z,0]$ and $[0,w]$ are closed circles L_1 and L_2, respectively. The orbit of a point $[z,w]$ with $z \neq 0 \neq w$ is an immersed line L_{zw} whose closure is the 2-torus

$$\overline{L_{zw}} = \left\{[u \cdot z, v \cdot w] \mid \|u\| = 1 = \|v\|\right\}.$$

Thus, $\mathcal{F}$ has a continuous decomposition into non-trivial minimal sets, but clearly $\mathcal{F}$ is a Riemannian foliation and not expansive.

Define open sets $U_1 = \{[z,w] \mid w \neq 0\}$ and $U_2 = \{[z,w] \mid z \neq 0\}$ and coordinate charts $\phi_1[z,w] = w/z \in \mathbf{C}$ and $\phi_2[z,w] = z/w \in \mathbf{C}$. The action of $\mathbf{R}$ in the coordinate ϕ_1 is the circular action $t \cdot w/z = e^{2\pi ibt/a} \cdot w/z$ while the action in the coordinate ϕ_2 is the circular action $t \cdot z/w = e^{2\pi iat/b} \cdot z/w$. The radial contraction of $\mathbf{C}$ to the origin preserves this action in both coordinates, so induces a foliated homotopy of U_1 to L_1 and U_2 to L_2. Thus, $\mathrm{cat}_{\pitchfork}(S^3, \mathcal{F}) = 2$.

Example 6.3 *(Suspension of group actions).* In the topological category, it is easy to construct suspension foliations with finite category. We give one class of examples: Let Γ be a finitely-generated group, and $\phi : \Gamma \times \mathbf{S}^1 \to \mathbf{S}^1$ an action of Γ on the circle. For example, one might take $\Gamma = \mathbf{Z}$ and let the action be determined by a diffeomorphism $f : \mathbf{S}^1 \to \mathbf{S}^1$. Another interesting example is to let Γ be the free group on 2 generators, and the action generated by diffeomorphisms f, g whose dynamics has a unique minimal set which is a Cantor set in $\mathbf{S}^1$.

Consider $\mathbf{S}^1$ as the boundary of the unit disc in $\mathbf{R}^2$, we can extend the action of ϕ to the disk radially, so that 0 is a fixed point. Then glue two such actions together to get a continuous action $\Phi : \Gamma \times \mathbf{S}^2 \to \mathbf{S}^2$ with two fixed points $\{N, S\}$.

To obtain a foliation, we choose a closed Riemann surface Σ_g of sufficiently high genus g such that $\Gamma_g = \pi_1(\Sigma)$ maps onto Γ. Then form the suspension manifold (see [1])

$$M_\phi = \widetilde{\Sigma_g} \times_\Phi S^2$$

with foliation $\mathcal{F}_\phi$ whose leaves are the images of the disks $\widetilde{\Sigma_g} \times \{\theta\}$ for $\theta \in \mathbf{S}^2$. Deleting the leaves corresponding to the two fixed-points yields a transversely categorical covering of M

$$U_1 = \widetilde{\Sigma_g} \times_\Phi \left(S^2 - S\right), \qquad U_2 = \widetilde{\Sigma_g} \times_\Phi \left(S^2 - N\right)$$

hence $\mathrm{cat}_\pitchfork(M, \mathcal{F}_\phi) = 2$. In this way, we can construct foliations with arbitrary complexity in its leaf dynamics. However, these foliations are never expansive.

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