Compact foliations with finite transverse LS category*

Steven Hurder and Paweł G. Walczak

Abstract

We prove that if $\mathcal{F}$ is a foliation of a compact manifold with all leaves compact submanifolds, and the transverse category $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ is finite, then the leaf space $M/\mathcal{F}$ is compact Hausdorff. The proof is surprisingly delicate, and is based on some new observations about the geometry of compact foliations. Colman proved in [3, 8] that the transverse category of a compact Hausdorff foliation is always finite, so we obtain a new characterization of the compact Hausdorff foliations among the compact foliations as those with $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ finite.

Contents

*preprint date: February 26, 2005

1 Introduction

A *compact foliation* is a foliation of a manifold M with all leaves compact submanifolds. For codimension one or two, a compact foliation $\mathcal{F}$ of a compact manifold M defines a fibration of M over its leaf space $M/\mathcal{F}$ which is a compact orbifold [27, 11, 12, 33, 10]. For codimension three and above, the leaf space $M/\mathcal{F}$ of a compact foliation need not be a Hausdorff space. This was first shown by an example of Sullivan [30] of a flow on a 5-manifold whose orbits are circles, and the lengths of the orbits are not bounded above. Subsequent examples of Epstein and Vogt [13, 35] showed that for any codimension greater than two, there are examples of compact foliations whose leaf spaces are not Hausdorff.

A compact foliation $\mathcal{F}$ with Hausdorff leaf space is said to be *compact Hausdorff*. The holonomy of each leaf of a compact Hausdorff foliation is a finite group, a property which characterizes them among the compact foliations. If every leaf has trivial holonomy group, then a compact Hausdorff foliation is a fibration. Otherwise, a compact Hausdorff foliation is a "generalized Seifert fibration", where the leaf space $M/\mathcal{F}$ is a "V-manifold" [29, 17, 22].

A compact foliation whose leaf space is non-Hausdorff has a closed, non-empty saturated subset, the *bad set* X_1, which is the union of the leaves whose holonomy group is infinite. The image of X_1 in the leaf space $M/\mathcal{F}$ consists of the non-Hausdorff points. The work by Edwards, Millet and Sullivan [10] established many fundamental properties of the geometry of the leaves of a compact foliation near its bad set, yet there is no general structure theory for compact foliations comparable to what is understood for compact Hausdorff foliations. The results of §§4, 5 and 6 used to prove our main theorem provide new insights and techniques for the study of these foliations.

The transverse Lusternik-Schnirelmann (LS) category of foliations was introduced in the 1998 thesis of H. Colman [3, 8]. The key idea is that of a transversally categorical open set. Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be foliated manifolds. A homotopy $H: M' \times [0,1] \to M$ is said to be *foliated* if for all $0 \leq t \leq 1$ the map H_t sends each leaf L' of $\mathcal{F}'$ into another leaf L of $\mathcal{F}$. An open subset U of M is *transversely categorical* if there is a foliated homotopy $H: U \times [0,1] \to M$ such that $H_0: U \to M$ is the inclusion, and $H_1: U \to M$ has image in a single leaf of $\mathcal{F}$. Here U is regarded as a foliated manifold with the foliation induced by $\mathcal{F}$ on U.

In other words, an open subset U of M is transversely categorical if the inclusion $(U, \mathcal{F}_U) \hookrightarrow (M, \mathcal{F})$ factors through a single leaf, up to foliated homotopy.

DEFINITION 1.1 *The transverse (saturated) category* $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ *of a foliated manifold* $(M, \mathcal{F})$ *is the least number of transversely categorical open saturated sets required to cover* M*. If no such covering exists, then* $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \infty$.

The transverse category $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ of a compact Hausdorff foliation $\mathcal{F}$ of a compact manifold M is always finite [8], as every leaf admits a saturated product neighborhood which is transversely categorical. For a non-Hausdorff compact foliation, there is no known construction of categorical neighborhoods of leaves in the bad set. In fact, as our main result implies, this cannot be done:

THEOREM 1.2 *Let* $\mathcal{F}$ *be a compact* C^1*-foliation of a compact manifold* M *with oriented normal bundle and non-empty bad set* X_1*. Then there exists a dense set of* tame points $X^t \subset X_1$ *such that, for each* $x \in X^t$*, there is no transversely categorical saturated open set containing* x.

Given a foliated manifold M with $\mathrm{cat}_{\pitchfork}(M,\mathcal{F}) < \infty$, the lifted foliation $\widetilde{\mathcal{F}}$ on a finite covering $\widetilde{M}$ of M also has finite category [8]. Moreover, the bad set of $\widetilde{\mathcal{F}}$ is the lift of the bad set of $\mathcal{F}$. Thus, if we apply Theorem 1.2 to the normal orientation double cover of a compact foliation $\mathcal{F}$, we obtain:

COROLLARY 1.3 *Let $\mathcal{F}$ be a compact C^1-foliation of a compact manifold M with* $\mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ *finite, then $\mathcal{F}$ is compact Hausdorff.*

Proof: Suppose that $\mathcal{F}$ is compact foliation with $\mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ finite, and the bad set X_1 is non-empty. Then there is a finite covering $\widetilde{M}$ of M whose lifted foliation $\widetilde{\mathcal{F}}$ has oriented normal and tangent bundles, and there exists transversely categorical open saturated sets $\{U_1,\ldots,U_k\}$ which cover $\widetilde{M}$. By Proposition 5.2 there exists a tame point $x_* \in \widetilde{X}_1$, and there exist some $1 \leq \ell \leq k$ so that $x_* \in U_\ell$. As U_ℓ is transversely categorical, this contradicts Proposition 6.1. □

COROLLARY 1.4 *Let $\mathcal{F}$ be a compact C^1-foliation of a compact manifold M with* $\mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ *finite, then there is an upper bound on the volumes of the leaves of $\mathcal{F}$, the transverse holonomy groups of all leaves of $\mathcal{F}$ are finite groups, and $\mathcal{F}$ admits a transverse Riemannian metric which is holonomy invariant.*

Proof: Millett [22] and Epstein [12] and showed that for a compact Hausdorff foliation $\mathcal{F}$ of a manifold M each leaf has finite holonomy, and thus M admits a Riemannian metric so that the foliation is Riemannian. □

Recall that a foliation is *geometrically taut* if the manifold M admits a Riemannian metric so that each leaf is an immersed minimal manifold [28, 31, 15]. Rummler proved in [28] that a compact foliation is Hausdorff if and only if it is taut, and thus we can conclude:

COROLLARY 1.5 *A compact C^1-foliation of a compact manifold M with* $\mathrm{cat}_{\pitchfork}(M,\mathcal{F}) < \infty$ *is geometrically taut.* □

The idea of the proof of Theorem 1.2 is as follows. The formal definition of the exceptional set X_1 in §3 is that it consist of leaves of $\mathcal{F}$ such that every open neighborhood of the leaf contains leaves of arbitrarily large volume. The bad set X_1 is the center of a dynamical system formed by leaves in a saturated open neighborhood of X_1 and thus by definition is the "calm in the middle of a dynamical storm".

This dynamical characterization of the bad set intuitively suggests that it should be a rigid set. That is, any foliated homotopy of an open neighborhood of a point in the bad set should preserve these dynamical properties, hence the open neighborhood cannot be continuously retracted to a single leaf. The proof of this statement is surprisingly delicate, and requires a very precise understanding of the properties of leaves in an open neighborhood of the bad set. A key result is Proposition 5.2, an extension of the Moving Leaf Lemma in [10], which establishes the existence of "tame points".

The overview of the paper is as follows: The first two sections consist of background material, which we recall to establish notations, and also present certain results in the form that we require. In §2 we give some basic results from foliation theory, and in §3 we recall some basic results about compact foliations, especially the structure theory for the good and the bad sets. Then in §4 we establish the local rigidity properties for compact leaves under deformation by a homotopy. These results are of general interest, as they are part of the general study of the topological properties of compact foliations. The most technical results of the paper are contained in §5, where we prove that tame points are dense in the bad set. Finally, in §6 prove that an open saturated set containing a tame point is not categorical. Theorem 1.2 follows immediately from Propositions 5.2 and 6.1.

2 Foliation preliminaries

We assume that M is a smooth compact Riemannian manifold without boundary of dimension $m = p + q$, that $\mathcal{F}$ is a C^1-foliation of codimension-q with oriented normal bundle, and that the leaves of $\mathcal{F}$ are smoothly immersed submanifolds. This is sometimes referred to as a $C^{1,\infty}$-foliation.

We assume that both $T\mathcal{F}$ and Q are orientable subbundles of TM. This suffices for the proof of Theorem 1.2, as the lift of a categorical open set is a categorical open set [8].

We recall below some well-known facts about foliations, and introduce some conventions of notation. The books [2, 14, 16] provide excellent basic references; our notation is closest to that used in [2].

For each $x \in M$, denote by L_x the leaf of $\mathcal{F}$ containing x.

Note that the analysis of the bad sets in later sections requires careful estimates on the foliation geometry; not just in each leaf, but also for nearby leaves of a given leaf. This requires an explicit description of the local metric geometry of a foliation, which we provide in this section.

2.1 Tangential and normal geodesic geometry

Let $T\mathcal{F}$ denote the tangent bundle to $\mathcal{F}$ and $\Pi: Q \to M$ its normal bundle, identified with the subbundle $T\mathcal{F}^\perp \subset TM$ of vectors orthogonal to $T\mathcal{F}$. The Riemannian metric on TM induces a Riemannian metrics on both $T\mathcal{F}$ and Q by fiberwise restriction. For a vector $\vec{v} \in T_xM$, let $\|\vec{v}\|$ denote its length in the Riemannian metric. Then for $\vec{v} \in T_x\mathcal{F}$ the length in the leafwise metric is again $\|\vec{v}\|$ so we use the same notation.

Let $d_M: M \times M \to [0, \infty)$ be the distance function associated to the Riemannian metric on M.

Given $R > 0$ and $x \in K \subset M$, set

$$\begin{aligned} B_M(x, R) &= \{y \in M \mid d_M(x, y) < R\} \\ B_M(K, R) &= \{y \in M \mid d_M(K, y) < R\} \end{aligned}$$

Let $d_L: L \times L \to [0, \infty)$ be the distance function on the leaf L induced by the restriction of the Riemannian metric to L. That is, for $x, x' \in L$ the distance $d_L(x, x')$ is the length of the shortest leafwise geodesic between x and x'. As M is compact, the manifold L with the metric d_L is a complete metric space. We introduce the notation $d_{\mathcal{F}}$ for the collection of leafwise distance functions, where $d_{\mathcal{F}}(x, y) = d_L(x, y)$ if $x, y \in L$ and otherwise $d_{\mathcal{F}}(x, y) = \infty$. This is a very useful notation when considering paths in M and the leafwise distances to points on the path.

Given $R > 0$ and $x \in K \subset L$, set

$$\begin{aligned} B_{\mathcal{F}}(x, R) &= \{y \in M \mid d_{\mathcal{F}}(x, y) < R\} \subset L \\ B_{\mathcal{F}}(K, R) &= \{y \in M \mid d_{\mathcal{F}}(K, y) < R\} \subset L \end{aligned}$$

Let $\exp = \exp^M: TM \to M \times M$ denote the exponential map for d_M. Denote by $p_1: M \times M \to M$ the projection on the first factor, then $p_1 \circ \exp: TM \to M$ is the bundle projection onto the basepoint. Let $p_2: M \times M \to M$ denote the projection on the second factor, then for $x \in M$, $\exp_x \equiv p_2 \circ \exp: T_xM \to M$ is the exponential map at x.

For $x \in L$, we let $\exp_x^{\mathcal{F}}: T_xL \to L$ denote the leafwise exponential map. Then $\exp_x^{\mathcal{F}}$ maps the ball $B_{T_xL}(0, R)$ of radius R in T_xL onto the set $B_{\mathcal{F}}(x, R)$.

Throughout this section, we will define (and redefine) a constant $\epsilon_0 > 0$ by successively imposing conditions which must be met. These conditions should be prefaced by the phrase "As M is compact, for $\epsilon_0 > 0$ sufficiently small, ...". To avoid repetition, this phrase will be understood, and so omitted. Initially, we assume that $0 < \epsilon_0 < 1$. Formally, ϵ_0 is the minimum of a finite set of positive constants, but for notational convenience, we just keep redefining ϵ_0.

For $\epsilon > 0$, let $TM^\epsilon \subset TM$ denote the disk subbundle of vectors with length less than ϵ, and $TF^\epsilon \subset T\mathcal{F}$ and $Q^\epsilon \subset Q$ the corresponding disk subbundles of $T\mathcal{F}$ and Q, respectively.

For each $x \in M$, the differential $d_0 \exp_x : TM_x \cong T(TM_x)_0 \to TM_x$ is the identity map. It follows that $\exp_x$ is a diffeomorphism in a sufficiently small neighborhood of $0 \in TM_x$. For all $x \in M$, assume that the restriction $\exp : TM^\epsilon \to M \times M$ is a diffeomorphism onto it image. Thus, ϵ_0 is less than the injectivity radius of the Riemannian metric on M.

We also require that for all $x \in M$, $B_M(x, \epsilon_0)$ is a totally normal neighborhood of x for the metric d_M This means that for any pair of points $y, z \in B_M(x, \epsilon_0)$ there is a unique geodesic contained in $B_M(x, \epsilon_0)$ between y and z. In particular, $B_M(x, \epsilon_0)$ is geodesically convex (cf. page 72, [9].)

For the leafwise Riemannian metric, we require that for all $x \in M$, the leafwise exponential map $\exp_x^{\mathcal{F}} : T\mathcal{F}_x^{\epsilon_0} \to L_x$ is a diffeomorphism onto it image.

We also require that for all $x \in M$, $B_{\mathcal{F}}(x, \epsilon_0) \subset L_x$ be a totally normal neighborhood of x for the leafwise metric $d_{\mathcal{F}}$.

Let $\exp_x^Q : Q_x \to M$ denote the restriction of the exponential map to the normal bundle at x. We assume that for all $x \in M$, $\exp_x^Q : Q_x^{\epsilon_0} \to M$ is transverse to $\mathcal{F}$, and that the image $\exp_x^Q(Q_x^{\epsilon_0})$ of the normal disk has angle at least $\pi/4$ with the leaves of the foliation $\mathcal{F}$.

The exponential map $\exp_x^Q : Q_x \to M$ is the restriction of $\exp_x^M$, hence for all $x \in M$, $\exp_x^Q : Q_x^{\epsilon_0} \to M$ is a diffeomorphism onto its image.

We use the normal exponential map to define a normal product neighborhood of a subset $K \subset M$. Given $0 < \epsilon \leq \epsilon_0$, let $Q(K, \epsilon) \to K$ denote the restriction of the ϵ-disk bundle $Q^\epsilon \to M$ to K. The normal neighborhood $\mathcal{N}(K, \epsilon)$ is the image of the map, $\exp^Q : Q(K, \epsilon) \to M$. If $K = \{x\}$ is a point and $0 < \epsilon < \epsilon_0$, then $\mathcal{N}(x, \epsilon)$ is a uniformly transverse normal disk to $\mathcal{F}$.

2.2 Quasi-isometric geometry

The restriction of the ambient metric d_M to a leaf L need not coincide (locally) with the leafwise geodesic metric $d_{\mathcal{F}}$ – unless the leaves of $\mathcal{F}$ are totally geodesic submanifolds of M. In any case, the Gauss Lemma implies that the two metrics are locally equivalent. We assume that for all $x \in M$ and $y, y' \in B_{\mathcal{F}}(x, \epsilon_0)$, then $d_{\mathcal{F}}$ and d_M are related by

$$d_M(y, y')/2 \ \leq \ d_{\mathcal{F}}(y, y') \ \leq \ 2\, d_M(y, y') \tag{1}$$

Let $dvol$ denote the leafwise volume p-form associated to the Riemannian metric on $T\mathcal{F}$. Given any bounded, Borel subset $A \subset L$ for the leafwise metric, define its leafwise volume as $\mathrm{vol}\,(A) = \int_A \, dvol$.

For $x \in M$, let $\vec{v} \in Q_x^{\epsilon_0}$ and set $y = \exp_x^Q(\vec{v})$. The restriction of Q to the leaf L_x is a flat bundle, as the Bott connection on Q has vanishing leafwise curvatures. Hence, there is a foliation $\widehat{\mathcal{F}}$ of $Q|L_x$ whose leaves are transverse to the fibers $Q|L_x \to L_x$ and for which the bundle projection map induces a covering map on the leaves of $\widehat{\mathcal{F}}$. Thus, for the given $\vec{v}$, there is a local flat section $\sigma_{x,\vec{v}} : B_{\mathcal{F}}(x, \delta) \to Q^{\epsilon_0}$, $\sigma_{x,\vec{v}}(x) = \vec{v}$, for $0 < \delta$ sufficiently small. Compose this section with the normal exponential map to obtain a map

$$\Sigma_{x,\vec{v}} = p_2 \circ \exp^Q \circ \sigma_{x,\vec{v}} \colon B_{\mathcal{F}}(x,\delta) \to M$$

whose image is transverse to Q. Thus, the differential $D\Sigma_{x,\vec{v}} \colon T_x\mathcal{F} \to T_yM$ composed with the orthogonal projection $T_yM \to T_y\mathcal{F}$ yields a linear isomorphism $D\Sigma^{\perp}_{x,\vec{v}} \colon T_x\mathcal{F} \to T_y\mathcal{F}$ which varies continuously with $(x,\vec{v})$. We apply this remark in two ways.

Consider the p-form $\omega_{x,\vec{v}} = \Sigma^*_{x,\vec{v}}(dvol_y)$ on $T\mathcal{F}_x$. When $\vec{v}=0$, unraveling the definitions shows that $\omega_{x,\vec{0}} = dvol_x$. By the continuity of $D\Sigma^{\perp}_{x,\vec{v}}$ we can assume that for all $x \in M$, $\vec{v} \in Q_x$ with $\|\vec{v}\| < \epsilon_0$,

$$dvol/2 \;\leq\; \omega_{x,\vec{v}} \;\leq\; 2\,dvol \tag{2}$$

This condition implies that when we later define local covering maps between sufficiently close adjacent leaves using the inverse of the normal geodesic map, then these covering maps preserve the leafwise volume form, up to a scalar factor of at most 2.

The leafwise Riemannian metric on $T\mathcal{F}$ can be similarly compared on nearby leaves using the normal geodesic projections. We can assume that for all $x \in M$, $\vec{v} \in Q_x$ with $\|\vec{v}\| < \epsilon_0$, and for all $\vec{w} \in T_x\mathcal{F}$

$$\|D\Sigma^{\perp}_{x,\vec{v}}(\vec{w})\|/2 \;\leq\; \|\vec{w}\| \;\leq\; 2\,\|D\Sigma^{\perp}_{x,\vec{v}}(\vec{w})\| \tag{3}$$

This condition implies that if two leaves are "ϵ_0-close" then the normal projection map is a quasi-isometry with scale factor 2.

2.3 Regular Foliation Atlas

A *regular foliation atlas* for $\mathcal{F}$ is a finite collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ so that:

- F1: $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ is a covering of M by $C^{1,\infty}$–coordinate charts $\phi_\alpha : U_\alpha \to (-1,1)^m$ where each U_α is a convex subset of *diameter* at most ϵ_0 with respect to the metric d_M.
- F2: Each coordinate chart $\phi_\alpha : U_\alpha \to (-1,1)^m$ admits an extension to a $C^{1,\infty}$–coordinate chart $\widetilde{\phi}_\alpha : \widetilde{U}_\alpha \to (-2,2)^m$ where $\widetilde{U}_\alpha$ is a convex subset containing the $2\epsilon_0$-neighborhood of U_α, so $B_M(U_\alpha, \epsilon_0) \subset \widetilde{U}_\alpha$. In particular, the closure $\overline{U_\alpha} \subset \widetilde{U}_\alpha$.
- F3: For each $z \in (-2,2)^q$, the preimage $\widetilde{\mathcal{P}}_\alpha(z) = \widetilde{\phi}_\alpha^{-1}((-2,2)^p \times \{z\}) \subset \widetilde{U}_\alpha$ is the connected component containing $\widetilde{\phi}_\alpha^{-1}(\{0\}\times\{z\})$ of the intersection of the leaf of $\mathcal{F}$ through $\phi_\alpha^{-1}(\{0\}\times\{z\})$ with the set $\widetilde{U}_\alpha$.
- F4: $\mathcal{P}_\alpha(z)$ and $\widetilde{\mathcal{P}}_\alpha(z)$ are convex subsets of diameter less than 1 with respect to $d_{\mathcal{F}}$.

The construction of regular coverings is described in chapter 1.2 of [2].

The inverse images

$$\mathcal{P}_\alpha(z) = \phi_\alpha^{-1}((-1,1)^p \times \{z\}) \subset U_\alpha$$

are smoothly embedded discs contained in the leaves of $\mathcal{F}$, called the *plaques* associated to the given foliation atlas. One thinks of the collection of all plaques as "tiling stones" which cover the leaves in a regular fashion. The convexity hypotheses in (F4) implies that if $U_\alpha \cap U_\beta \neq \emptyset$, then each plaque $\mathcal{P}_\alpha(z)$ intersects exactly one plaque of U_β. The analogous statement holds for pairs $\widetilde{U}_\alpha \cap \widetilde{U}_\beta \neq \emptyset$. More generally, an intersection of plaques $\mathcal{P}_{\alpha_1}(z_1) \cap \cdots \cap \mathcal{P}_{\alpha_d}(z_d)$ is either empty, or a convex set.

The closure of each plaque $\overline{\mathcal{P}_\alpha(z)} = \widetilde{\phi}_\alpha^{-1}([-1,1]^p \times \{z\}) \subset \widetilde{U}_\alpha$ is a compact set with interior (for the leaf topology) which depends continuously on the transverse parameter z, hence there exists constants $0 < C_{min} \leq C_{max}$ such that

$$C_{min} \ \leq \ \mathrm{vol}\,(\mathcal{P}_\alpha(z)) \ \leq \ C_{max}, \ \forall \alpha \in \mathcal{A}, \ \forall z \in [-1,1]^q \tag{4}$$

Recall that a Lebesgue number for the covering $\mathcal{U}$ is a constant $\epsilon > 0$ so that for each $x \in M$ there exists $U \in \mathcal{U}$ with $B_M(x,\epsilon) \subset U$. Every covering of a compact manifold admits a Lebesgue number. We assume that ϵ_0 is chosen sufficiently small so that $2\epsilon_0$ is a Lebesgue number for the covering $\mathcal{U}$ of M by foliation charts. Then for any $x \in M$, the restriction of $\mathcal{F}$ to $B_M(x,\epsilon_0)$ is a product foliation, and the leaves of $\mathcal{F} \mid B_M(x,\epsilon_0)$ are convex discs for the metric $d_{\mathcal{F}}$.

For each $\alpha \in \mathcal{A}$, the extended chart $\widetilde{\phi}_\alpha$ defines a C^1–embedding

$$t_\alpha = \phi_\alpha^{-1}(\{0\} \times \cdot) : (-2,2)^q \to \widetilde{U}_\alpha \subset M$$

whose image is denoted by $\widetilde{\mathcal{T}}_\alpha$. We can assume that the images $\widetilde{\mathcal{T}}_\alpha$ are pairwise disjoint, and also that each submanifold $\widetilde{\mathcal{T}}_\alpha$ is everywhere perpendicular to the leaves of $\mathcal{F}$ by adjusting the given Riemannian metric on M in an open tubular neighborhood of each $\widetilde{\mathcal{T}}_\alpha$. Furthermore, assume that each $\widetilde{\mathcal{T}}_\alpha$ has diameter at most 1.

Define $\mathcal{T}_\alpha = \phi_\alpha^{-1}(\{0\} \times (-1,1)^q)$. The local coordinate on $\mathcal{T}_\alpha$ is again denoted by $t_\alpha : (-1,1)^q \to \mathcal{T}_\alpha$. We use this coordinate to identify each transversal $\mathcal{T}_\alpha$ with $(-1,1)^q$.

We assume that the coordinates t_α are positively oriented, mapping the positive orientation for the normal bundle to $T\mathcal{F}$ to the positive orientation on $\mathbb{R}^q$.

The collection of all plaques for the foliation atlas is indexed by the *complete transversal*

$$\mathcal{T} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha$$

For a point $x \in \mathcal{T}$, let $\mathcal{P}_\alpha(x) = \mathcal{P}_\alpha(t_\alpha^{-1}(x))$ denote the plaque containing x.

The Riemannian metric on M induces a Riemannian metric and corresponding distance function $\mathbf{d}_{\mathcal{T}}$ on each extended transversal $\widetilde{\mathcal{T}}_\alpha$. For $\alpha \neq \beta$ and $x \in \mathcal{T}_\alpha$, $y \in \mathcal{T}_\beta$ we set $\mathbf{d}_{\mathcal{T}}(x,y) = \infty$.

Given $x \in \widetilde{\mathcal{T}}_\alpha$ and $R > 0$, let $\mathbf{B}_{\mathcal{T}}(x,R) = \{y \in \widetilde{\mathcal{T}}_\alpha \mid \mathbf{d}_{\mathcal{T}}(x,y) < R\}$.

Given a subset $\mathcal{Z} \subset U_\alpha$ let $\mathcal{Z}_{\mathcal{P}}$ denote the union of all plaques in U_α having non-empty intersection with $\mathcal{Z}$. We set $\mathcal{Z}_{\mathcal{T}} = \mathcal{Z}_{\mathcal{P}} \cap \mathcal{T}_\alpha$. If $\mathcal{Z}$ is an open subset of U_α, then $\mathcal{Z}_{\mathcal{P}}$ is open in U_α and $\mathcal{Z}_{\mathcal{T}}$ is an open subset of $\mathcal{T}_\alpha$.

Given any point $w \in (-1,1)^p$, we can define a transversal $\mathcal{T}_\alpha(w) = \phi_\alpha^{-1}(\{w\} \times (-1,1))$. There is a canonical map $\psi_w \colon \mathcal{T}_\alpha(w) \to \mathcal{T}_\alpha(0) = \mathcal{T}_\alpha$ defined by, for $\vec{v} \in (-1,1)^q$,

$$\psi_w(\phi_\alpha^{-1}(w \times \vec{v}) = \phi_\alpha^{-1}(0 \times \vec{v}) \tag{5}$$

The Riemannian metric on M induces also induces a Riemannian metric and distance function on each transversal $\mathcal{T}_\alpha(w)$. By mild abuse of notation we denote all such transverse metrics by $\mathbf{d}_{\mathcal{T}}$. Then by the uniform extension property of the foliation charts, there exists a constant $C_T \geq 1$ so that for all $\alpha \in \mathcal{A}$, $w \in (-1,1)^p$ and $x, y \in \mathcal{T}_\alpha(w)$,

$$\mathbf{d}_{\mathcal{T}}(x,y)/C_T \ \leq \ \mathbf{d}_{\mathcal{T}}(\psi_w(x), \psi_w(y)) \ \leq \ C_T\, \mathbf{d}_{\mathcal{T}}(x,y) \tag{6}$$

We use the maps (5) to translate points in the coordinate charts U_α to the "center" transversal. The constant C_T is a uniform estimate of the normal distortion introduced by any such translation.

In place of the transversals $\mathcal{T}_\alpha(w)$ we can also consider the normal geodesic disk at $y = \phi_\alpha^{-1}(w \times \vec{0})$, $\exp_y^Q: Q_y^\epsilon \to \mathcal{N}(y,\epsilon)$ which for $0 < \epsilon \leq \epsilon_0$ is uniformly transverse to $\mathcal{F}$. If the image $\mathcal{N}(y,\epsilon) \subset U_\alpha$ then we can project it to the transversal $\mathcal{T}_\alpha$ along the plaques in U_α. Denote this projection by $\Pi_\alpha^{\mathcal{F}}: \mathcal{N}(y,\epsilon) \to \mathcal{T}_\alpha$ We will assume that the constant $C_T \geq 1$ is sufficiently large so that for all $y \in M$, for all $0 < \epsilon \leq \epsilon_0$, for all α with $\mathcal{N}(y,\epsilon) \subset U_\alpha$ and for all $z, z' \in \mathcal{N}(y,\epsilon)$ we have

$$\mathbf{d}_{\mathcal{N}}(z,z')/C_T \ \leq \ \mathbf{d}_{\mathcal{T}}(\Pi_\alpha^{\mathcal{F}}(z), \Pi_\alpha^{\mathcal{F}}(z')) \ \leq \ C_T \, \mathbf{d}_{\mathcal{N}}(z,z') \tag{7}$$

2.4 Transverse holonomy

We recall the definition and some properties of the holonomy pseudogroup of $\mathcal{F}$. A pair of indices (α,β) is said to be *admissible* if $U_\alpha \cap U_\beta \neq \emptyset$. Let $\mathcal{T}_{\alpha\beta} \subset \mathcal{T}_\alpha$ denote the open set of plaques of U_α which intersect some plaque of U_β. The holonomy transformation $\mathbf{h}_{\alpha\beta}: \mathcal{T}_{\alpha\beta} \to \mathcal{T}_{\beta\alpha}$ is defined by $y = \mathbf{h}_{\alpha\beta}(x)$ if and only if $\mathcal{P}_\alpha(x) \cap \mathcal{P}_\beta(y) \neq \emptyset$. The finite collection

$$\mathcal{H}_{\mathcal{F}}^1 = \{\mathbf{h}_{\alpha\beta}: \mathcal{T}_{\alpha\beta} \to \mathcal{T}_{\beta\alpha} \mid (\alpha,\beta) \text{ admissible}\} \tag{8}$$

generates the holonomy pseudogroup $\mathcal{H}_{\mathcal{F}}$ of local homeomorphisms of $\mathcal{T}$.

Each generator $\mathbf{h}_{\alpha\beta}: \mathcal{T}_{\alpha\beta} \to \mathcal{T}_{\beta\alpha}$ is a uniformly continuous homeomorphism, in the following strong sense. The charts $\{(U_\alpha, \phi_\alpha)) \mid \alpha \in \mathcal{A}\}$ are a regular foliation atlas, hence $\mathbf{h}_{\alpha\beta}$ is the restriction of $\widetilde{\mathbf{h}}_{\alpha\beta}: \widetilde{\mathcal{T}}_{\alpha\beta} \to \widetilde{\mathcal{T}}_{\beta\alpha}$ defined by the intersection of the extended charts $\widetilde{U}_\alpha \cap \widetilde{U}_\beta$. The domain $\mathcal{T}_{\alpha\beta} \subset \widetilde{\mathcal{T}}_{\alpha\beta}$ is precompact with $\mathbf{B}_{\widetilde{\mathcal{T}}}(\mathcal{T}_{\alpha\beta}, \epsilon_0) \subset \widetilde{\mathcal{T}}_{\alpha\beta}$, so given any $0 < \epsilon < \epsilon_0$, there is a module of continuity $\mu_{\alpha\beta}(\epsilon) > 0$ such that for all $x \in \mathcal{T}_{\alpha\beta}$ then

$$\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \mu_{\alpha\beta}(\epsilon)) \subset \widetilde{\mathcal{T}}_{\alpha\beta} \ \text{ and } \ \widetilde{\mathbf{h}}_{\alpha\beta}(\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \mu_{\alpha\beta}(\epsilon))) \subset \mathbf{B}_{\widetilde{\mathcal{T}}}(\mathbf{h}_{\alpha\beta}(x), \epsilon)$$

For the admissible pairs (α,α) we set $\mu_{\alpha\alpha}(\epsilon) = \epsilon$. Given $0 < \epsilon \leq \epsilon_0$, define

$$\mu(\epsilon) = \min\{\mu_{\alpha\beta}(\epsilon) \mid (\alpha,\beta) \text{ admissible}\} \tag{9}$$

so that $0 < \mu(\epsilon) \leq \epsilon$. For an integer $n > 0$ and $0 < \epsilon \leq \epsilon_0$ set

$$\mu(n,\epsilon) = \min\{\epsilon, \mu(\epsilon), \mu(\mu(\epsilon)), \ldots, \mu^{(n)}(\epsilon)\} \tag{10}$$

where $\mu^{(n)}$ denotes the n-fold composition. Note that $0 < \mu(\epsilon) \leq \epsilon$ implies $\mu(n,\epsilon) = \mu^{(n)}(\epsilon) \leq \epsilon$.

The point of the definition (9) is that for every admissible pair (α,β) and each $x \in \mathcal{T}_{\alpha\beta}$ the holonomy map $\mathbf{h}_{\alpha\beta}$ admits an extension to a local homeomorphism $\widetilde{\mathbf{h}}_{\alpha\beta}$ defined by the holonomy of $\mathcal{F}$, and whose domain includes the closure of the disk $\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \mu(\epsilon))$ about x in $\widetilde{\mathcal{T}}_{\alpha\beta}$, and satisfies $\widetilde{\mathbf{h}}_{\alpha\beta}(\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \mu(\epsilon))) \subset \mathbf{B}_{\widetilde{\mathcal{T}}}(\mathbf{h}_{\alpha\beta}(x), \epsilon)$.

A *plaque chain of length* n, denoted by $\mathcal{P}$, is a collection of plaques

$$\{\mathcal{P}_{\alpha_0}(z_0), \mathcal{P}_{\alpha_1}(z_1), \ldots, \mathcal{P}_{\alpha_n}(z_n)\}$$

satisfying $\mathcal{P}_{\alpha_i}(z_i) \cap \mathcal{P}_{\alpha_{i+1}}(z_{i+1}) \neq \emptyset$ for $0 \leq i < n$. Each pair of indices (α_i, α_{i+1}) is admissible, so determines a holonomy map $\mathbf{h}_{\alpha_i\alpha_{i+1}}$ such that $\mathbf{h}_{\alpha_i\alpha_{i+1}}(z_i) = z_{i+1}$. Let $\mathbf{h}_{\mathcal{P}}$ denote the composition of these maps, so that

$$\mathbf{h}_{\mathcal{P}} = \mathbf{h}_{\alpha_{n-1}\alpha_n} \circ \cdots \circ \mathbf{h}_{\alpha_1\alpha_2} \circ \mathbf{h}_{\alpha_0\alpha_1}$$

The domain of $\mathbf{h}_{\mathcal{P}}$ is not empty, as $\mathbf{h}_{\mathcal{P}}(z_0) = z_n$. In fact, from the definition (10) the domain of $\mathbf{h}_{\mathcal{P}}(z_0)$ contains the ball of radius $\mu(n, \epsilon)$ about z_0. This remark can be given a general formulation. Define $\mathcal{H}^n_{\mathcal{F}}$ to be the holonomy transformations defined by plaque chains of length at most n.

LEMMA 2.1 *Given a plaque chain $\mathcal{P}$ of length n, and $0 < \epsilon \leq \epsilon_0$ set $\delta = \mu(n, \epsilon)$. Then for any x in the domain of $\mathbf{h}_{\mathcal{P}}$, there is an extension to a local homeomorphism $\widetilde{h}_{\mathcal{P}}$ defined by the holonomy of $\mathcal{F}$ whose domain includes the closure of the disk $\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \delta)$ about x in $\widetilde{\mathcal{T}}$, and*

$$\widetilde{h}_{\mathcal{P}}(\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \delta)) \subset \mathbf{B}_{\widetilde{\mathcal{T}}}(\mathbf{h}_{\alpha\beta}(x), \epsilon) \tag{11}$$

That is, $\mu(n, \epsilon)$ is a module of uniform continuity for all elements of $\mathcal{H}^n_{\mathcal{F}}$.

Proof: For each $0 \leq i < n$, $\mu(n, \epsilon) \leq \mu(i, \epsilon)$ hence there is an extension of

$$\mathbf{h}_i = \mathbf{h}_{\alpha_{i-1}\alpha_i} \circ \cdots \circ \mathbf{h}_{\alpha_0\alpha_1}$$

to $\widetilde{\mathbf{h}}_i$ whose domain includes the disk $\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \delta)$ about x. The image $\mathbf{h}_i(\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \delta)$ has size at most $\mu(n-1, \epsilon)$, so that we can continue the extension process to $\mathbf{h}_{i+1}$. $\square$

2.5 Plaque length and metric geometry

We make two observations about the metric geometry of foliations [26].

Let $\gamma \colon [0, 1] \to L$ be a leafwise C^1-path. Its leafwise Riemannian length is denoted by $||\gamma||_{\mathcal{F}}$.

The *plaque length* of γ, denoted by $||\gamma||_{\mathcal{P}}$, is the least integer n such that the image of γ is covered by a chain of convex plaques

$$\{\mathcal{P}_{\alpha_0}(z_0), \mathcal{P}_{\alpha_1}(z_1), \ldots, \mathcal{P}_{\alpha_n}(z_n)\}$$

where $\gamma(0) \in \mathcal{P}_{\alpha_0}(z_0)$, $\gamma(1) \in \mathcal{P}_{\alpha_1}(z_1)$, and successive plaques $\mathcal{P}_{\alpha_i}(z_i) \cap \mathcal{P}_{\alpha_{i+1}}(z_{i+1}) \neq \emptyset$.

PROPOSITION 2.2 *For any leafwise C^1-path γ, $||\gamma||_{\mathcal{P}} \leq \lceil(||\gamma||_{\mathcal{F}}/\epsilon_0)\rceil$. Moreover, if γ is leafwise geodesic, then $||\gamma||_{\mathcal{F}} \leq ||\gamma||_{\mathcal{P}} + 1$.*

Proof: Let $N = \lceil(||\gamma||_{\mathcal{F}}/\epsilon_0)\rceil$ be the least integer greater than $||\gamma||_{\mathcal{F}}/\epsilon_0$ then there exists points $0 = t_0 < t_1 < \cdots < t_N = 1$ such that the restriction of γ to each segment $[t_i, t_{i+1}]$ has length at most ϵ_0. The diameter of the set $\gamma([t_i, t_{i+1}])$ is at most ϵ_0, hence there is some $U_{\alpha_i} \in \mathcal{U}$ with $\gamma([t_i, t_{i+1}]) \subset U_{\alpha_i}$ hence $\gamma([t_i, t_{i+1}]) \subset \mathcal{P}_{\alpha}(z_i)$ for some z_i. Thus, the image of γ is covered by a chain of convex plaques of length at most N.

Conversely, suppose γ is a leafwise geodesic and $\{\mathcal{P}_{\alpha_0}(z_0), \mathcal{P}_{\alpha_1}(z_1), \ldots, \mathcal{P}_{\alpha_n}(z_n)\}$ is a plaque chain covering the image $\gamma([0, 1])$. Each plaque $\mathcal{P}_{\alpha_i}(z_i)$ is a leafwise convex set of diameter at most 1, so $||\gamma||_{\mathcal{F}} \leq (n+1) \leq ||\gamma||_{\mathcal{P}} + 1$. $\square$

The extension property (F2) implies that for all $\alpha \in \mathcal{A}$ and $z \in (-1, 1)^q$, the closure $\overline{\mathcal{P}_{\alpha}(z)}$ is compact, hence has finite leafwise volume which is uniformly continuous with respect to the parameter z. Hence, there exists constants $A, B > 0$ such that

$$A \leq \mathrm{vol}\,(\mathcal{P}_{\alpha}(z)) \leq B \quad \text{for} \quad \text{all} \quad \alpha \in \mathcal{A}, \quad -1 \leq z \leq 1$$

We note a consequence of this uniformity which is critical to the proof of the main theorem.

PROPOSITION 2.3 *Let M be a compact manifold. Then there exists a monotone increasing function $v\colon [0,\infty) \to [0,\infty)$ such that if L is a compact leaf, then $\mathrm{vol}\,(L) \le v(\mathrm{diam}\,(L))$. Conversely, there exists a monotone increasing function $R\colon [0,\infty) \to [0,\infty)$ such that if L is a compact leaf, then $\mathrm{diam}\,(L) \le R(\mathrm{vol}\,(L))$.*

Proof: The holonomy pseudogroup of $\mathcal{F}$ has a finite set of generators, hence given $r > 0$, there exists a positive integer $e(r)$ such that any subset of a leaf with leaf diameter at most r can be covered by at most $e(r)$ plaques. Thus, if L is a leaf with diameter at most r, then L has volume at most $v(r) = B \cdot e(r)$.

If a leaf L has diameter at least r, then L contains a minimizing geodesic segment γ of length r, and hence contains a leafwise tube $N(\gamma, \epsilon_0)$ around γ of radius ϵ_0. As M is compact, the leafwise sectional curvatures of M are uniformly bounded, hence there is a constant $\lambda > 0$ so that $\mathrm{vol}\,(N(\gamma, \epsilon_0)) \ge \lambda r$. Then set $R(s) = s/\lambda$. $\square$

2.6 Local structure theorem

The "Local Structure Theorem" for foliations due to G. Reeb [27, 1, 2, 16, 32] provides a description of the geometry of $\mathcal{F}$ in an open neighborhood of an arbitrarily large compact subset K of a leaf. The version given below includes a uniform estimate on the diameter of the open foliated neighborhood, which we show has several applications to obtaining effective estimates on leaf stability.

Given $\epsilon > 0$, recall that $\mathcal{N}(K, \epsilon) \subset V$ denotes the normal ϵ-neighborhood of K.

A subset $\widetilde{K} \subset \widetilde{L}$ of a metric space is said to be *C–uniformly simply connected* if given any closed loop $\gamma\colon [0,1] \to \widetilde{K}$ with $\gamma(0) = \gamma(1) = x$, then there is a homotopy $H_s\colon [0,1] \to \widetilde{L}$, $0 \le s \le 1$, with

- $H_0(t) = \gamma(t)$, $H_1(t) = x$, $H_s(0) = H_s(1) = x$,
- for all $0 \le s \le 1$, the path $t \mapsto H_s(t)$ has length at most $C \cdot (\|\gamma\| + 1)$.

For example, if $\widetilde{K}$ is an embedded path in $\widetilde{L}$, then $\widetilde{K}$ is 1–uniformly simply connected. If $\widetilde{K} \subset \widehat{K}$ where $\widehat{K}$ is a geodesically star-like subset of $\widetilde{L}$, then $\widetilde{K}$ is C–uniformly simply connected, where C depends upon the geometry of $\widehat{K}$. For example, if $\mathcal{F}$ is the Reeb foliation of $\mathbb{S}^3$ and $K = \mathbb{S}^1$ is the push-off to a vanishing cycle of a latitude or longitude of the compact toral leaf, then each such closed loop is contained in a star-like region, but C will depend upon the choice of the circle, and may be arbitrarily large.

THEOREM 2.4 *Let M be a compact manifold with $C^{1,\infty}$-foliation $\mathcal{F}$. Let L be a leaf of $\mathcal{F}$, and $\widetilde{K} \subset \widetilde{L}$ a compact, path connected subset of the universal covering $\widetilde{L}$. Then there exists an immersion $\Psi\colon \widetilde{K} \times \mathbb{D}^q \to M$ satisfying:*

1. *for each $\vec{v} \in \mathbb{D}^q$, the restriction $\Psi\colon \widetilde{K} \times \{\vec{v}\} \to M$ has image in a leaf*
2. *for each $y \in \widetilde{K}$, the restriction $\Psi\colon \{y\} \times \mathbb{D}^q \to M$ is uniformly transverse to $\mathcal{F}$*
3. *$\Psi\colon \widetilde{K} \times \{\vec{0}\} \to L \subset M$ coincides with the restriction to $\widetilde{K}$ of the covering map $\pi\colon \widetilde{L} \to L$*

Moreover, if $\widetilde{K}$ is C-uniformly simply connected in $\widetilde{L}$, then there exists $\delta > 0$ which depends only on C and the diameter $R = \mathrm{diam}_{\widetilde{L}}(\widetilde{K})$, so that the immersion $\Psi\colon \widetilde{K} \times \mathbb{D}^q \to V$ satisfies

4. *for each $y \in \widetilde{K}$, $\mathcal{N}(y, \delta) \subset \Psi(\{y\} \times \mathbb{D}^q)$*

Proof: Fix $R > 0$. Let $\widetilde{K} \subset \widetilde{L}$ be a compact subset of diameter at most R. We first construct the map Ψ under the assumption that $\widetilde{K}$ is C-uniformly simply connected, and then indicate how the proof is modified to obtain the the general result.

Choose a basepoint $x \in \widetilde{K}$. Every point $y \in \widetilde{K}$ can be joined to x by a leafwise geodesic path $\gamma_{x,y}$ of length at most R. In the following, both C and R are fixed and we set

$$\begin{aligned} \delta_1 &= \mu(\lceil C(2R+1)/\epsilon_0 \rceil + 2, \epsilon_0/C_T) \\ \epsilon_1 &= \delta_1/C_T \end{aligned} \tag{12}$$

where $C_T \geq 1$ was introduced in (6) and (7). Recall that $\mu(n, \epsilon_0) \leq \epsilon_0$ for all $n > 0$, so $\epsilon_1 < \delta_1 < \epsilon_0$.

The construction of $\Psi: \widetilde{K} \times \mathbb{D}^q \to M$ begins with the definition of $\Psi: \{x\} \times \mathbb{D}^q \to M$. Choose an orthonormal framing $\{\vec{e}_1, \ldots, \vec{e}_q\}$ of Q_x which we use to identify $\mathbb{R}^q \cong Q_x$. Then for $\vec{v} \in \mathbb{D}^q$, set

$$\Psi(x, \vec{v}) = \exp_x^Q(\epsilon_1 \cdot \vec{v}) \tag{13}$$

Let $\alpha \in \mathcal{A}$ be an index such that $\mathbf{B}_M(x, \epsilon_0) \subset U_\alpha$ so that $\Psi(\{x\} \times \mathbb{D}^q) \subset \mathbf{B}_M(x, \epsilon_1) \subset U_\alpha$. Define $\widehat{\Psi}_\alpha(x, \cdot) = \Pi_\alpha^{\mathcal{F}} \circ \Psi(x, \cdot): \mathbb{D}^q \to \mathcal{T}_\alpha$. That is, for $\vec{v} \in \mathbb{D}^q$, $z = \widehat{\Psi}_\alpha(x, \vec{v}) \in \mathcal{T}_\alpha$ is the point such that $\Psi(x \times \vec{v}) \in \mathcal{P}_\alpha(z)$. As $\|\vec{v}\| < 1$, $\Psi(\{x\} \times \vec{v})$ can be joined to x by a geodesic of length at most ϵ_1, the estimate (7) implies that

$$\mathbf{d}_{\mathcal{T}}(\widehat{\Psi}_\alpha(x, \vec{0}), \widehat{\Psi}_\alpha(x, \vec{v})) \leq C_T \, \mathbf{d}_{\mathcal{N}}(\Psi(\{x\} \times \vec{0}), \Psi(\{x\} \times \vec{v})) \leq C_T \epsilon_1 = \delta_1 \tag{14}$$

and hence

$$\widehat{\Psi}_\alpha(\{x\} \times \mathbb{D}^q) \subset \mathbf{B}_{\mathcal{T}}(\widehat{\Psi}_\alpha(x \times \vec{0}), \delta_1) \tag{15}$$

To define $\Psi: \widetilde{K} \times \mathbb{D}^q \to M$ in general, let $y \in \widetilde{K}$, and choose a leafwise geodesic path $\gamma_{x,y}: [0,1] \to \widetilde{L}$ of length at most R with $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$.

Proposition 2.2 implies that the path $\gamma_{x,y}$ determines a plaque chain $\mathcal{P}_{x,y}$ of length at most $\lceil R/\epsilon_0 \rceil$. (Recall that $\lceil x \rceil$ denotes the least integer n with $n \geq x$.)

We adopt the notation $\widehat{x}_\alpha = \widehat{\Psi}_\alpha(x, \vec{0})$, and similarly let $\widehat{y}_\beta$ denote a point on the transversal $\mathcal{T}_\beta$ whose corresponding plaque contains the point y. Thus, $\gamma_{x,y}$ determines a plaque chain from $\widehat{x}_\alpha$ to $\widehat{y}_\beta$ of length $\lceil R/\epsilon_0 \rceil + 2$. The holonomy of this plaque chain will be denoted by $\mathbf{h}_{\widehat{x},\widehat{y}}$.

Note that $\lceil R/\epsilon_0 \rceil + 2 \leq \lceil 2C(R+1)/\epsilon_0 \rceil + 2$, hence by Lemma 2.1 and estimate (14) the set $\widehat{\Psi}_\alpha(\{x\} \times \mathbb{D}^q)$ is contained in the domain of the holonomy transformation $\mathbf{h}_{\widehat{x},\widehat{y}}$.

The notation $\mathbf{h}_{\widehat{x},\widehat{y}}$ is justified as the map does not depend upon the choice of the path. To see this, let $\gamma'_{x,y}$ be another leafwise geodesic path from x to y of length at most R with induced holonomy map $\mathbf{h}'_{\widehat{x},\widehat{y}}$. The composition $\gamma = \gamma_{x,y}^{-1} \circ \gamma'_{x,y}$ is a closed loop of length at most $2R$, so by the hypothesis that $\widetilde{K}$ is C-uniformly simply connected, there is homotopy $H_s(t)$ of γ to a constant such that each path $t \mapsto H_s(t)$ has length at most $C(2R+1)$. Proposition 2.2 implies that a path in $\widetilde{L}$ of leafwise length $C(2R+1)$ can be covered by a plaque chain of length at most $\lceil (C(2R+1)/\epsilon_0) \rceil$. Thus, each path $t \mapsto H_s(t)$ can be covered by a plaque chain of length at most $\lceil (C(2R+1)/\epsilon_0) \rceil$. By the choice of ϵ_1, the set $\widehat{\Psi}_\alpha(\{x\} \times \mathbb{D}^q)$ is contained in the domain of the holonomy transformation associated to each path $t \mapsto H_s(t)$. Thus

$$\mathbf{h}_{\widehat{x},\widehat{y}}^{-1} \circ \mathbf{h}'_{\widehat{x},\widehat{y}} = \mathbf{h}_{H_0} = \mathbf{h}_{H_1} = \mathrm{Id}$$

so the holonomy maps $\mathbf{h}_{\widehat{x},\widehat{y}}$ and $\mathbf{h}'_{\widehat{x},\widehat{y}}$ are equal on the domain $\widehat{\Psi}_\alpha(\{x\} \times \mathbb{D}^q)$.

Let $L_{\vec{v}}$ denote the leaf of $\mathcal{F}$ containing the point $\Psi(\{x\} \times \vec{v})$. As the set $\widehat{\Psi}_\alpha(\{x\} \times \mathbb{D}^q)$ is contained in the domain of the holonomy transformation $\mathbf{h}_{\widehat{x},\widehat{y}}$, this implies the holonomy along the path

$\gamma_{x,y}\colon [0,1] \to \widetilde{L}$ is well-defined for all points in $\widehat{\Psi}_\alpha(\{x\} \times \mathbb{D}^q)$. Thus, the path $\gamma_{x,y}\colon [0,1] \to \widetilde{L}$ lifts to a path $\gamma_{\vec{v}}\colon [0,1] \to L_{\vec{v}}$ such that $\gamma_{\vec{v}}(t) \in L_{\vec{v}} \cap \mathcal{N}(\gamma(t), \epsilon_1)$. Set $\Psi(y, \vec{v}) = \gamma_{\vec{v}}(1)$.

The point $\Psi(y, \vec{v})$ does not depend upon the choice of the $\gamma_{x,y}$, as given two paths $\gamma_{x,y}$ and $\gamma'_{x,y}$ as above, the fact that $\Psi(\{x\} \times \mathbb{D}^q)$ is contained in the domain of the holonomy transformation associated to each path $t \mapsto H_s(t)$ also implies that the homotopy from $\gamma_{x,y}$ to $\gamma'_{x,y}$ lifts to the leaf through $\Psi(\{x\} \times \vec{v})$ hence the lifted paths have the same endpoints.

The conditions (2.4.1– 2.4.3) follow immediately from the definition of $\Psi\colon \widetilde{K} \times \mathbb{D}^q \to M$.

To obtain the constant δ in condition (2.4.4) we first note that given any $z \in \widetilde{K} \cap \mathcal{T}$ there is a leafwise geodesic path $\sigma\colon [0,1] \to \widetilde{K}$ from z to x of length at most R. Recall from Lemma 2.1 the definition of the module of continuity function $\mu^{(n)}(\epsilon)$. Set $n = \lceil R/\epsilon_0 \rceil$ and then let $\delta = \mu(n, \delta_1)/2$. Then a point in the transversal $w \in \mathcal{N}(x, \delta)$ determines a point $\widehat{w} \in \mathcal{T}$ which is within $2\delta = \mu(n, \delta_1)$ of z. Then the holonomy $\mathbf{h}_\sigma$ defined by the path σ contains $\widehat{w}$ in its domain, and moreover

$$\mathbf{h}_\sigma(\mathbf{B}_{\widetilde{\mathcal{T}}}(z, 2\delta)) \subset \mathbf{B}_{\widetilde{\mathcal{T}}}(x, \delta_1) \subset \widehat{\Psi}(\{x\} \times \mathbb{D}^q)$$

Thus, $\mathcal{N}(z, \delta) \subset \Psi(\{z\} \times \mathbb{D}^q)$ for $z \in \widetilde{K} \cap \mathcal{T}$.

To prove the general case, note that the compact set $\widetilde{K} \subset \widetilde{L}$ can be covered by a finite collection of foliation plaques, say $\{\mathcal{P}_{\alpha_1}(z_1), \mathcal{P}_{\alpha_1}(z_1), \ldots, \mathcal{P}_{\alpha_n}(z_n)\}$. As above, each point $z_i \in \mathcal{T}$ can be joined to the basepoint $\widehat{x}$ by a chain of plaques of length at most $\lceil R/\epsilon_0 \rceil + 2$. As the manifold $\widetilde{L}$ is simply connected, each plaque chain determines a well-defined transverse holonomy map

$$\mathbf{h}_i\colon \mathbf{B}_{\mathcal{T}}(\widehat{x}, \delta_i) \to \mathbf{B}_{\mathcal{T}}(z_i, \epsilon_0)$$

where the diameter of the domain $\delta_i > 0$. Set $\delta = \min\{\delta_1, \ldots, \delta_n\} > 0$, then proceed as above. □

Chapter V of Tamura [32] gives the proof of the Local Structure Theorem in much greater detail for the interested reader. In this paper, we need the effective estimate above in the case of paths, so the above proof is complete for our purposes.

The proof of Theorem 2.4 introduced the quantity $\mu(\lceil C(2R{+}1)/\epsilon_0 \rceil{+}2, \epsilon/C_T)/C_T$ which represents a fundamental property of the geometry of $\mathcal{F}$, as it is a "uniform modulus of continuity" for the transverse holonomy along paths. We introduce a more compact terminology for a special case of this quantity, applicable for the case of leafwise paths:

DEFINITION 2.5 *For $0 < \epsilon \leq \epsilon_0$ and $R > 0$,*

$$\Delta(R, \epsilon) \equiv \mu(\lceil R/\epsilon_0 \rceil + 2, \epsilon/C_T)/C_T \tag{16}$$

COROLLARY 2.6 *Given $\epsilon > 0$ and any leafwise path $\gamma\colon [0,1] \to L$ of length at most R, the transverse holonomy along γ defines a smooth embedding*

$$\mathbf{h}_\gamma\colon \mathcal{N}(\gamma(0), \Delta(R, \epsilon)) \to \mathcal{N}(\gamma(1), \epsilon)\ ,\ \mathbf{h}_\gamma(\gamma(0)) = \gamma(1)$$

Proof: Let $\widetilde{L}$ be the universal covering of L, and $\widetilde{\gamma}\colon [0,1] \to \widetilde{L}$ a lift of γ. Then $\widetilde{\gamma}$ has length at most R, so is homotopic relative endpoints to a length minimizing geodesic path $\widehat{\gamma}$ from $\widetilde{\gamma}(0)$ to $\widetilde{\gamma}(1)$. The length of $\widehat{\gamma}$ is bounded above by R, and its image is 1-uniformly simply connected, so from the proof of Theorem 2.4, there is a well-defined holonomy map along $\widehat{\gamma}$ with domain including the disk $\mathcal{N}(\gamma(0), \Delta(R, \epsilon))$ and image in the disk $\mathcal{N}(\gamma(1), \epsilon)$. □

2.7 Normal tubular neighborhoods

A compact leaf L_0 represents a finite orbit for the transverse holonomy pseudogroup, and the Local Structure Theorem 2.4 has many applications to the study of the structure of $\mathcal{F}$ near L_0. We will first establish a notation used throughout this paper, then give an essential application.

For $\epsilon > 0$, recall that $\exp^Q \colon Q(L_0, \epsilon) \to M$ is the normal geodesic map along L_0 restricted to the ϵ-disk subbundle of $Q|L_0 \to L_0$. As L_0 is compact, there is an $\epsilon > 0$ such that this map is a diffeomorphism into an open neighborhood $\mathcal{N}(L_0, \epsilon)$ of L_0. Then $\exp^Q \colon Q(L_0, \epsilon) \to \mathcal{N}(L_0, \epsilon)$ is a diffeomorphism onto, so the bundle projection $\Pi \colon Q(L_0, \epsilon) \to L_0$ induces a projection map on the image, also denoted by $\Pi \colon \mathcal{N}(L_0, \epsilon) \to L_0$. Thus, L_0 is a strong deformation retract of $\mathcal{N}(L_0, \epsilon)$.

If ϵ_* is such that $\epsilon = 2\epsilon_*$ has the above properties, then the induced projection $\Pi \colon \mathcal{N}(L_0, \epsilon_*) \to L_0$ has the additional property that the map extends to the closure, $\Pi \colon \overline{\mathcal{N}(L_0, \epsilon_*)} \to L_0$ and thus is also uniformly continuous.

DEFINITION 2.7 *We say that ϵ is L_0–normal if $0 < \epsilon \leq \epsilon_0$ and*

$$\exp^Q \colon Q(L_0, 2\epsilon) \to \mathcal{N}(L_0, 2\epsilon)$$

is a diffeomorphism onto. The induced map $\Pi \colon \mathcal{N}(L_0, \epsilon) \to L_0$ is called the normal projection.

PROPOSITION 2.8 *Let L_0 be a compact leaf of $\mathcal{F}$, and suppose that ϵ is L_0–normal. If L_1 is a leaf with $L_1 \subset \mathcal{N}(L_0, \epsilon)$, then the normal projection $\Pi \colon \mathcal{N}(L_0, \epsilon) \to L_0$ restricts to a 2–quasi-isometric covering map $\Pi \colon L_1 \to L_0$.*

Proof: Let $y \in L_1$ and $x = \Pi(y) \in L_0$. By the definition of Π, there exists $\vec{v} \in Q_x^\epsilon$ so that $\exp_x^Q(\vec{v}) = y$. Then the transversal hypothesis of section 2.2 implies that the differential $D\Sigma_{x,\vec{v}} \colon T_x\mathcal{F} \to T_y M$ of the map $\Sigma_{x,\vec{v}} = p_2 \circ \exp^Q \circ \sigma_{x,\vec{v}} \colon B_{\mathcal{F}}(x, \delta) \to M$ is injective. Let $B_{\mathcal{F}}(y, \delta) \subset L_1$ be a sufficiently small disk, then the restriction $\Pi \colon B_{\mathcal{F}}(y, \delta) \to L_0$ is a diffeomorphism into. Moreover, by the hypothesis (3) the map $\Pi \colon L_1 \to L_0$ is a local quasi-isometry for the metric $d_{\mathcal{F}}$ on L_1 and L_0. Thus, $\Pi \colon L_1 \to L_0$ has the path lifting property, so is a covering map. $\square$

When the leaf L_0 in Proposition 2.8 is assumed to be compact, a stronger statement can be proved. The following says that a compact leaf with an *a priori* bound on its volume that is sufficiently close to another compact leaf, is "captured" by the holonomy of the nearby leaf.

PROPOSITION 2.9 *Let L_0 be a compact leaf of $\mathcal{F}$ and let ϵ be L_0-normal. Given $\Lambda > 0$, there exists $\delta > 0$ so that if L_1 is a compact leaf with volume $\mathrm{vol}(L_1) < \Lambda$ and $L_1 \cap \mathcal{N}(L_0, \delta) \neq \emptyset$, then $L_1 \subset \mathcal{N}(L_0, \epsilon)$ and the normal projection restricts to a 2–quasi-isometric covering map $\Pi \colon L_1 \to L_0$.*

Proof: Let $\mathfrak{R} = R(\Lambda)$ be the constant introduced in the proof of Proposition 2.3. Then $\mathrm{vol}(L_1) < \Lambda$ implies that $\mathrm{diam}\,(L_1) \leq \mathfrak{R}$. Set $\delta = \Delta(4\,\mathfrak{R}, \epsilon/2)$.

Assume there exists $x \in L_0$ and $y \in L_1 \cap \mathcal{N}(x, \delta)$, then it suffices to show that $L_1 \subset \mathcal{N}(L_0, \epsilon)$.

Suppose not, and that there exists $z \in L_1$ but $z \notin \mathcal{N}(L_0, \epsilon)$. As $\mathrm{diam}\,(L_1) \leq \mathfrak{R}$, there exists a smooth path $\sigma \colon [0, 1] \to L_1$ with $\sigma(0) = y$, $\sigma(1) = z$ and $\|\sigma\| \leq \mathfrak{R}$. Let $t_* = \inf\{t \mid \sigma(t) \notin \mathcal{N}(L_0, \epsilon)\}$. As $\sigma(0) = y \in \mathcal{N}(L_0, \delta) \subset \mathcal{N}(L_0, \epsilon)$, there is $\lambda > 0$ with $\sigma[0, \lambda) \subset \mathcal{N}(L_0, \epsilon)$, and hence $0 < t_* \leq 1$.

For any $0 < s < t_*$, the image $\sigma[0, s] \subset \mathcal{N}(L_0, \epsilon)$, so by (3) the path $\gamma = \Pi \circ \sigma[0, s] \to L_0$ has length at most 2 times the length of the restricted path $\sigma \colon [0, s] \to L_1$, or length at most $2\,\mathfrak{R}$. Thus, the holonomy along γ defines a map $\mathbf{h}_\gamma \colon \mathcal{N}(x, \delta) \to \mathcal{N}(\gamma(s), \epsilon/2)$, $\mathbf{h}_\gamma(y) = \sigma(s)$. This implies that

$\sigma(t) \in \overline{\mathcal{N}(L_0, \epsilon/2)}$ for all $0 \leq t < t_*$. By continuity, $\sigma(t_*) \in \overline{\mathcal{N}(L_0, \epsilon/2)} \subset \mathcal{N}(L_0, \epsilon)$ so there is $s_* > t_*$ such that $\sigma[0, s_*) \subset \mathcal{N}(L_0, \epsilon)$. This contradicts the minimality of t_*. It follows that a minimal t_* does not exist, hence $\{t \mid \sigma(t) \notin \mathcal{N}(L_0, \epsilon)\}$ must be the empty set, hence $z = \sigma(1) \in \mathcal{N}(L_0, \epsilon)$ as was to be shown. $\square$

COROLLARY 2.10 *Let L_0 be a compact leaf of $\mathcal{F}$. Given $\Lambda > 0$, there exists $\delta > 0$ so that if L_1 is a compact leaf with volume $\mathrm{vol}(L_1) < \Lambda$ and $L_1 \cap \mathcal{N}(L_0, \delta) \neq \emptyset$, then $\mathrm{vol}\,(L_1) \leq 2\, d_*\, \mathrm{vol}\,(L_0)$ where d_* is the homological degree of the covering map $\Pi \colon L_1 \to L_0$.*

Proof: Let ϵ be L_0–normal, and choose $\delta = \Delta(4\,\mathfrak{R}, \epsilon/2)$. as before. Then by Proposition 2.9, if $L_1 \cap \mathcal{N}(L_0, \delta) \neq \emptyset$ then $L_1 \subset \mathcal{N}(L_0, \epsilon)$ and $\Pi \colon L_1 \to L_0$ is a covering map.

The homological degree of a covering map equals its covering degree, so the volume of the covering in the lifted metric, satisfies $\mathrm{vol}'(L_1) = d_*\, \mathrm{vol}\,(L_0)$. Then by (2) the volume of L_1 in the leaf metric $d_{\mathcal{F}}$ satisfies

$$\mathrm{vol}\,(L_1) \leq 2\, \mathrm{vol}'(L_1) = 2\, d_*\, \mathrm{vol}\,(L_0) \qquad \square$$

We mention also the well-known "Reeb Stability Theorem" [27] which follows as a corollary of the Local Structure Theorem 2.4.

PROPOSITION 2.11 *Let L be a compact leaf with transverse germinal holonomy group $\mathcal{H}_x$ at $x \in L$ is finite. Then $\mathcal{H}_x$ can be represented by a finite group of homeomorphisms acting on an open neighborhood $U_x \subset \mathcal{T}$ of x. Thus, L admits a open saturated neighborhood $V \to L$ such that the leaves of $\mathcal{F}$ in V are all finite coverings of L.*

Proof: Pick a basepoint $x \in L$ and let $\mathcal{H}_x$ denote the transverse germinal holonomy group at x. Note that "germinal" means that the germ of each element $\mathbf{h} \in \mathcal{H}_x$ is well-defined.

By assumption, $\mathcal{H}_x$ is a finite group, with order denoted by N. The holonomy construction defines a map $\mathbf{h} \colon \pi_1(L, x) \to \mathcal{H}_x$ whose kernel $\mathcal{K}_x \subset \pi_1(L, x)$ is a finite index subgroup.

Let $\pi_h \colon \widetilde{L}_h \to L$ be the holonomy covering of L; that is, the covering associated to $\mathcal{K}_x$.

Let $R = \mathrm{diam}\,(L)$. Then any pair of preimages $y, z \in \pi_h^{-1}(x)$ can be joined by a path $\widetilde{\sigma}_{y,z} \colon [0,1] \to \widetilde{L}_h$ of length $||\widetilde{\sigma}_{y,z}||_{\mathcal{F}} \leq 2NR$ so each element $g \in \mathcal{H}_x$ is represented by the holonomy h_g along a path $\sigma_{y,z} = \pi_h \circ \sigma_{y,z} \colon [0,1] \to L$ also of length at most $2NR$ with $\sigma_{y,z}(0) = \sigma_{y,z}(1) = x$. The domain of h_g contains $\mathbf{B}_{\mathcal{T}}(x, \Delta(2NR, \epsilon))$ and has image in $\mathbf{B}_{\mathcal{T}}(z, \epsilon)$.

Let ϵ be L–normal. Then the open set

$$U_x = \bigcap_{g \in \mathcal{H}_x} h_g\left(\mathbf{B}_{\mathcal{T}}(x, \Delta(2NR, \epsilon))\right)$$

is invariant under each h_g for $g \in \mathcal{H}_x$. $\square$

3 Properties of compact foliations

In this section, $\mathcal{F}$ is assumed to be a compact foliation of a manifold M without boundary. The geometry of compact foliations has been extensively analyzed by Epstein [11, 12], Millett [22], Vogt [33, 34, 35] and Edwards, Millet and Sullivan [10]. We recall some of their results.

3.1 The good and the bad sets

Let $\mathrm{vol}\,(L)$ denote the volume of a leaf L with respect to the Riemannian metric induced from M. Define the volume function on M by setting $v(x) = \mathrm{vol}\,(L_x)$. Clearly, $x \mapsto v(x)$ is constant along leaves of $\mathcal{F}$. However, $v(x)$ need not be continuous on M.

The *bad set* X_1 of $\mathcal{F}$ consists of the points $y \in M$ where $x \mapsto v(x)$ is not bounded in any open neighborhood of y. By its definition, the bad set X_1 is saturated. Note also that

$$X_1 = \bigcup_{n=1}^{\infty} X_1 \cap \mathrm{vol}^{-1}(0, n]$$

The leaves in the intersection $X_1 \cap \mathrm{vol}^{-1}(0, n]$ have volume at most n, while $v(x)$ is not locally bounded in any open neighborhood of $y \in X_1$, therefore each set $X_1 \cap \mathrm{vol}^{-1}(0, n]$ has no interior. By the Baire category theorem, X_1 has no interior.

The complement $G = M - X_1$ is called the *good set*. The holonomy of every leaf $L \subset G$ is finite, thus by the Reeb Stability Theorem, L has an open saturated neighborhood consisting of leaves with finite holonomy. Hence, G is an open set, X_1 is closed, and the leaf space $G/\mathcal{F}$ is Hausdorff.

Inside the good set is the open dense saturated subset $G_e \subset G$ consisting of leaves without holonomy. Its complement $G_h = G - G_e$ consists of leaves with finite holonomy.

3.2 The Epstein filtration

The restriction of the volume function $v(x)$ to X_1 again need not be locally bounded, and the construction of the bad set can be iterated to obtain the *Epstein filtration*:

$$M = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_\alpha \supset \cdots$$

The definition of the sets X_α proceeds inductively: Let $\alpha > 1$ be a countable ordinal, and assume that X_β has been defined for $\beta < \alpha$. If α is a successor ordinal, let $\alpha = \gamma + 1$ and define X_α to be the closed saturated set of $y \in X_\gamma$ where $x \mapsto v(x)$ is not bounded in any relatively open neighborhood in X_γ of $y \in X_\gamma$.

If α is a limit ordinal, then define $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$.

For $\beta < \alpha$, the set X_α is nowhere dense in X_β. Note that since each set $M - X_\alpha$ is open, the filtration is at most countable. The *filtration length* of $\mathcal{F}$ is the ordinal α such that $X_\alpha \neq \emptyset$ and $X_{\alpha+1} = \emptyset$.

Vogt [35] showed that for any finite ordinal α, there is a compact foliation of a compact manifold with filtration length α. He also remarked that given any countable ordinal α, the construction can be modified to produce a foliation with filtration length α.

Such examples show that the bad set X_1 and the substrata X_α need not be a finite unions or intersections of submanifolds; they may have much more pathological topological structure, especially when the filtration length is an infinite ordinal.

3.3 Regular points

A point $x \in X_1$ is called a regular point if the restricted holonomy of $\mathcal{F}|X_1$ is trivial at x. Equivalently, the regular points are the points of continuity for the restricted volume function $v|X_1\colon X_1 \to \mathbb{R}$. If $X_1 \neq \emptyset$, then the regular points are an open and dense subset of $X_1 - X_2$. We recall a key result of Edwards, Millet, and Sullivan (see § 5 of [10].)

PROPOSITION 3.1 (Moving Leaf) *Let $\mathcal{F}$ be a compact foliation of an oriented manifold M with orientable normal bundle. Suppose that X_1 is compact and non-empty. Let $x \in X_1$ be a regular point. Then there exists a generic leaf $L \subset G_e$, and a smooth isotopy $h\colon L \times [0,1) \to G_e$ such that*

- *For all $0 \leq t < 1$, $h_t\colon L \to L_t \subset M$ is a diffeomorphism onto its image L_t*
- *L_x is in the closure of the leaves* $\displaystyle\bigcup_{t>1-\delta} L_t$ *for any $\delta > 0$*
- $\displaystyle\limsup_{t\to 1}\ \mathrm{vol}\,(L_t) = \infty$

While the "moving leaf" L_t limits on X_1, the moving leaf cannot accumulate on a single compact leaf of X_1. This follows because a compact leaf L admits a relative homology dual cycle, which for $\epsilon > 0$ sufficiently small and $x \in L$, is represented by the transverse disk $\mathbf{B}_{\mathcal{T}}(x,\epsilon)$. This disk intersects L precisely in the point x, hence the relative homology class $[\mathbf{B}_{\mathcal{T}}(x,\epsilon), \partial\mathbf{B}_{\mathcal{T}}(x,\epsilon)]$ is Poincaré dual to the fundamental class $[L]$. Assuming that $\{L_t\}$ limits on L, for $t < 1$ sufficiently close to 1, each $L_t \subset \mathcal{N}(L,\epsilon)$ and so the intersection number $[L_t \cap \mathbf{B}_{\mathcal{T}}(x,\epsilon)] = [L_t] \cap [\mathbf{B}_{\mathcal{T}}(x,\epsilon), \partial\mathbf{B}_{\mathcal{T}}(x,\epsilon)]$ is constant. Thus the leaves $\{L_t\}$ have bounded volume as $t \to 1$, which is a contradiction.

It is precisely this "non-localized limit behavior" for leaves approaching the bad set with volumes unbounded, which makes the study of compact foliations with non-empty bad sets so interesting, and difficult. There are no results describing how these paths of leaves must behave in the limit.

3.4 Structure of the good set

Epstein [12] and Millett [22] showed that for a compact foliation $\mathcal{F}$ of a manifold V, then

$$v(x) \text{ is locally bounded} \Leftrightarrow V/\mathcal{F} \text{ is Hausdorff} \Leftrightarrow \text{ the holonomy of every leaf is finite}$$

By definition, the leaf volume function is locally bounded on the good set G, hence the restriction of $\mathcal{F}$ to G is compact Hausdorff, and all leaves of $\mathcal{F}|G$ have finite holonomy group. Epstein and Millet showed there is a much more precise structure theorem for the foliation $\mathcal{F}$ in an open neighborhood of a leaf of the good set:

PROPOSITION 3.2 *Let V denote an open connected component of the good set G, and $V_e = V \cap G_e$ the set of leaves with no holonomy. There exists a "generic leaf" $L_0 \subset V_e$, such that for each $x \in V$ with leaf L_x containing x,*

1. *there is a finite subgroup H_x of the orthogonal group $\mathbf{O}(\mathbf{q})$ and a free action α_x of H_x on L_0*
2. *there exists a diffeomorphism of the twisted product*
$$\phi_x: L_0 \times_{H_x} \mathbb{D}^q \to V_x \tag{17}$$
onto an open saturated neighborhood V_x of L_x (where $\mathbb{D}^q$ denotes the unit disk in $\mathbb{R}^q$)
3. *the diffeomorphism ϕ_x is leaf preserving, where $L_0 \times_{H_x} \mathbb{D}^q$ is foliated by the images of $L_0 \times \{w\}$ for $w \in \mathbb{D}^q$ under the quotient map $\mathcal{Q}: L_0 \times \mathbb{D}^q \to L_0 \times_{H_x} \mathbb{D}^q$*
4. *ϕ_x maps $L_0/H_x \cong L_0 \times_{H_x} \{0\}$ diffeomorphically to L_x*

In particular, if $x \in V_e$ then H_x is trivial, and ϕ_x is a product structure for a neighborhood of L_x.

The open set V_x is called a standard neighborhood of L_x, and the 4–tuple $(V_x, \phi_x, H_x, \alpha_x)$ is called a *standard local model* for $\mathcal{F}$. Note that, by definition, $V_x \subset G$ hence $V_x \cap X_1 = \emptyset$. Hence, as $G \ni x \to X_1$ the diameter of the transversal image disc $\phi_x(\{x\} \times_{H_x} \mathbb{D}^q)$ tends to zero, so these are geometric models, but not necessarily metric models.

The Hausdorff space $G/\mathcal{F}$ is a Satake manifold; that is, for each point $b \in G/\mathcal{F}$ and $\pi(x) = b$ the leaf L_x has an open foliated neighborhood V_x as above, and $\phi_x: L_0 \times_{H_x} \mathbb{D}^q \to V_x$ induces a coordinate map $\varphi_b: \mathbb{D}^q/H_x \to W_b$, where $W_b = \pi(V_x)$. The open sets $W_b \subset G/\mathcal{F}$ are called basic open sets for $G/\mathcal{F}$. Note also that π is a closed map [12, 22].

Let G_e denote the open set of leaves in the good set G with trivial holonomy. The restricted quotient map $\pi: G_e \to G_e/\mathcal{F}$ is a fibration with fibers diffeomorphic to L_0. Thus, the singularities of the quotient map $\pi: G \to G/\mathcal{F}$ are concentrated on the set of leaves with holonomy, G_h. Millett [22] called the map π a *twisted twisted fibration*, where the fibration $\pi: G_e \to G_e/\mathcal{F}$ has additional "twisting" introduced along the singular set $G_h/\mathcal{F}$.

The existence of a standard neighborhood about every leaf of a compact Hausdorff foliation has further consequences for its global geometry. For example, the holonomy of L_x is given by the composition
$$\pi_1(L_x, x) \to H_x \subset \mathbf{O}(\mathbf{q}) \subset \mathrm{Diffeo}(\mathbb{D}^q, 0). \tag{18}$$

COROLLARY 3.3 *Given $y \in V_x$ let $(z, w) \in L_0 \times \mathbb{D}^q$ have image $\phi_x^{-1}(y)$. Let $H_{xy} \subset H_x$ be the isotropy group at $w \in \mathbb{D}^q$ of the linear action of H_x on $\mathbb{D}^q$. Then the holonomy of L_y is given by*
$$\pi_1(L_y, y) \to H_{xy} \subset \mathbf{O}(\mathbf{q})$$
Hence, there are at most finite number of isomorphism classes of holonomy groups of leaves $L_y \subset V_x$.

We can also use the local models (17) for $\mathcal{F}$ to give a better description of G_h.

COROLLARY 3.4 *G_h is a stratified space. That is, for $x \in G_h$ let $W_x^1, \ldots, W_x^{k(x)} \subset \mathbb{R}^q$ be the collection of linear subspace such that W_x^i is the fixed set for some $g_i \in H_x$. Then the set $W_x = W_x^1 \cap \ldots \cap W_x^{k(x)} \cap \mathbb{D}^q$ is invariant under H_x, and $G_h \cap V_x = \phi_x(L_0 \times_{H_x} W_x)$. Hence, G_h is relatively closed and nowhere dense in G. Moreover, if $\mathcal{F}$ has orientable normal bundle, then each subspace W_x^i has codimension at least 2, and hence the set G_e is open, dense, and locally path connected.*

4 Stability properties of compact leaves

In this section, we prove some technical results about stability under homotopy of compact leaves in a foliated manifold. The results of this section do not assume that $\mathcal{F}$ is a compact foliation.

The main goal of this section is to prove Proposition 4.7, which yields an upper bound on both the volumes of the compact leaves, and the topological degrees of the covering maps, which arise in a homotopy of a compact leaf. This plays an essential role in our proof of Proposition 6.1.

PROPOSITION 4.1 *Let $\mathcal{F}$ be a C^1 foliation of a compact manifold M. Let $H: L \times [0,1] \to M$ be a leafwise homotopy. If L is a compact leaf, then $H_t(L)$ is a compact leaf for all $0 \leq t \leq 1$.*

Proof: Let $L_t = H_t(L)$ and L_t^* be the leaf of $\mathcal{F}$ containing L_t. The image L_t is compact, so it will suffice to show that the map $H_t: L \to L_t^*$ is onto for all $0 \leq t \leq 1$.

Let $H^*(\cdot)$ denote cohomology with real coefficients, so that $H^p(L) = \mathbb{R}$ as L is a compact oriented manifold of dimension p. Also, for each $0 \leq t \leq 1$, L_t^* is a connected manifold of dimension p, hence for any $y \in L_t^*$, we have $H^p(L_t^* - \{y\}) = 0$. It follows by combining these two remarks that if $L_t \neq L_t^*$ then the induced map $H_t^*: H^p(L_t^*) \to H^p(L)$ is zero.

Let $I = \{t \mid H_t^* \neq 0\} \subset [0,1]$. It will suffice to show that $I = [0,1]$ to prove Proposition 4.1.

For $t = 0$, H_0 is the identity, hence $0 \in I$.

We claim that I is both a relatively open and closed subset of $[0,1]$. To see that it is relatively open, fix $t \in I$. Then by assumption, $H_t^* \neq 0$ hence $L_t = L_t^*$.

For each $0 \leq t \leq 1$, let ϵ_t' be L_t^*–normal. Set $V_t = \mathcal{N}(L_t^*, \epsilon_t')$. The open subset $H^{-1}(V_t) \subset L \times [0,1]$ contains the compact set $L \times \{t\}$, hence there exists $\kappa > 0$ such that $L \times (t - \kappa, t + \kappa) \subset V_t$.

Let s satisfy $t - \kappa < s < t + \kappa$. Then the image $L_s \subset V_t$, and hence the composition

$$\iota \circ \Pi \circ H_s: L \to L_t^* \subset V_t$$

is homotopy equivalent to H_t. Thus, the composition $(\Pi \circ H_s)^*: H^p(L_t^*) \to H^p(L)$ is non-zero.

We claim that $L_s^* \subset V_t$. Otherwise, $L_s^* \cap V_t$ is a proper open submanifold of L_s^*, and hence $H^p(L_s^* \cap V_t) = 0$. This implies that $L \to L_s \subset L_s^* \cap V_t \subset V_t \to L_t$ induces the zero map on p-cohomology, contradicting the previous observations.

For $t - \kappa < s < t + \kappa$, apply the cohomology functor $H^p(\cdot)$ to the composition

$$L \to L_s \subset L_s^* \subset V_1 \to L_t$$

The composition $(\Pi \circ H_s)^*: H^p(L_t^*) \to H^p(L)$ is non-zero, hence $H_s^*: H^p(L_s^*) \to H^p(L)$ is non-zero, and we obtain that $s \in I$.

Next we show that I is closed. Let $\overline{I}$ denote its closure in $[0,1]$, then for $t \in \overline{I}$ and $\delta > 0$ the closed set $L \times (\overline{I} \cap [\delta - t, t + \delta])$ is compact in $L \times [0,1]$, hence its image $H(L \times (\overline{I} \cap [\delta - t, t + \delta]))$ is compact. It thus equals the closure in M of the image $H(L \times (I \cap [\delta - t, t + \delta]))$.

The set $H(L \times (I \cap [\delta - t, t + \delta]))$ is saturated by assumption, hence its closure

$$\overline{H(L \times (I \cap [\delta - t, t + \delta]))} = H(L \times (\overline{I} \cap [\delta - t, t + \delta]))$$

is also saturated. The intersection of saturated sets is saturated, so taking the intersection over all $\delta > 0$ we obtain that $H(L \times \{t\})$ is a saturated set. That is, $L_t = L_t^*$, hence the leaf L_t^* is compact.

Finally, we must show that for $t \in \overline{I}$ the map $H_t^*: H^p(L) \to H^p(L_t^*)$ is non-zero. As $t \in \overline{I}$, there exists a sequence $s_n \in I$ with $s_n \to t$, so the sequence of compact leaves $L \times \{s_n\}$ limits to $L \times \{t\}$, and hence $L_{s_n}^*$ limits to L_t^*.

Let $0 < \epsilon_t' < \epsilon_0$ be sufficiently small so that the normal neighborhood $V_t = \mathcal{N}(L_t^*, \epsilon_t')$ is an open neighborhood retract of L_t^*, with $\Pi : V_t \to L_t^*$ the retraction. The open subset $H^{-1}(V_t) \subset L \times [0,1]$ contains the compact set $L \times \{t\}$, hence there exists $\kappa > 0$ such that $L \times (t-\kappa, t+\kappa) \subset V_t$.

Let n be sufficiently large so that $s_n \in (t-\kappa, t+\kappa)$, and then we have $L_{s_n} \subset V_t$. The composition $\iota \circ \Pi \circ H_{s_n}: L \to L_t^* \subset V_t$ is homotopy equivalent to H_t. It is given that $H_{s_n}^*: H^p(L_{s_n}^*) \to H^p(L)$ is non-zero, hence $H_t^* = (\Pi \circ H_s)^*: H^p(L_t^*) \to H^p(L)$ is non-zero. $\square$

Implicit in the above proof is the geometric result that a "small" homotopy of a compact leaf is again a compact leaf, and its image is a covering of the initial leaf. We introduce a formal generalization of this phenomenon, and show in Lemma 4.3 that a homotopy of compact leaves yields an example.

DEFINITION 4.2 *Given $0 < \epsilon \leq \epsilon_0$, we say that two compact leaves $L, L' \subset M$ are* ϵ-commensurable *if there is a sequence of compact leaves $L_0, \mathfrak{L}_0, L_1, \ldots, L_{k-1}, \mathfrak{L}_{k-1}, L_k$ and constants $\epsilon_0'', \epsilon_1'', \ldots, \epsilon_k''$ such that $L = L_0$, $L' = L_k$, and for each $0 \leq \ell \leq k$,*

1. *$\epsilon_\ell'' \leq \epsilon$, and ϵ_ℓ'' is L_ℓ–normal*
2. *$\mathfrak{L}_\ell \subset \mathcal{N}(L_\ell, \epsilon_\ell'') \cap \mathcal{N}(L_{\ell+1}, \epsilon_{\ell+1}'')$.*

The collection of leaves $\mathfrak{L} = \{L_0, \mathfrak{L}_0, L_1, \ldots, L_{k-1}, \mathfrak{L}_{k-1}, L_k\}$ is called an ϵ-chain from L to L'.

Proposition 2.8 implies that the normal projection Π restricted to to the leaf $\mathfrak{L}_\ell$ yields covering maps $\pi: \mathfrak{L}_\ell \to L_\ell$ and $\pi: \mathfrak{L}_\ell \to L_{\ell+1}$ of finite degree; the leaf $\mathfrak{L}_\ell$ is a "geometric correspondence" from L_ℓ to $L_{\ell+1}$. (It may happen in examples that either $\mathfrak{L}_\ell = L_\ell$ or $\mathfrak{L}_\ell = L_{\ell+1}$, in which case the geometric correspondence is just a map.) The sequence of covering maps yields a diagram

$$L = L_0 \longleftarrow \mathfrak{L}_0 \longrightarrow L_1 \longleftarrow \cdots \longrightarrow L_{k-1} \longleftarrow \mathfrak{L}_{k-1} \longrightarrow L_k = L' \tag{19}$$

and the induced maps on the fundamental groups (with respect to appropriate basepoints) yields

$$\pi_1(L, y_0) \longleftarrow \pi_1(\mathfrak{L}_0, y_0^*) \longrightarrow \pi_1(L_1, y_1) \cdots\cdots \pi_1(\mathfrak{L}_{k-1}, y_{k-1}^*) \longrightarrow \pi_1(L', y_k) \tag{20}$$

which we denote by $\pi_1(\mathfrak{L})$. All of the maps in (20) have images of finite index, and it is in this sense that the leaves L and L' are commensurable, even though their fundamental groups $\pi_1(L, y_0)$ and $\pi_1(L', y_k)$ need not be subgroups of finite index in a common group.

LEMMA 4.3 *Let $\mathcal{F}$ be a C^1 foliation of a compact manifold M. Let $H: L \times [0,1] \to M$ be a leafwise homotopy. If L is a compact leaf, and so $L' = H_1(L)$ is also a compact leaf, then for all $0 < \epsilon \leq \epsilon_0$, L and L' are ϵ-commensurable.*

Proof: For each $0 \leq t \leq 1$, Proposition 4.1 implies that the image $L_t = H_t(L)$ is a compact leaf, so we can choose ϵ_t' which is L_t–normal. As L is compact, we can also choose $\xi_t > 0$ so that the image

$$H(L \times [t - \xi_t, t + \xi_t]) \subset \mathcal{N}(L_t, \epsilon_t')$$

The collection of open intervals $\{\mathcal{I}_t = (t - \xi_t, t + \xi_t) \mid 0 \leq t \leq 1\}$ is a cover for $[0,1]$, hence there is a finite set $\{t_0 = 0 < t_1 < \cdots < t_k = 1\}$ so that $\{\mathcal{I}_{t_0}, \mathcal{I}_{t_1}, \ldots, \mathcal{I}_{t_k}\}$ is a cover for $[0,1]$. We assume that the cover has minimal cardinality, which implies that $t_{\ell+1} - \xi_{t_{\ell+1}} < t_\ell + \xi_{t_\ell}$.

For $0 \leq \ell \leq k$, set $L_\ell = L_{t_\ell}$ and $\epsilon''_\ell = \epsilon'_{t_\ell}$. For each $0 \leq \ell < k$, choose $t_{\ell+1} - \xi_{t_{\ell+1}} < s_\ell < t_\ell + \xi_{t_\ell}$ and set $\mathfrak{L}_\ell = L_{s_\ell}$. It is immediate that $\mathfrak{L}_\ell \subset \mathcal{N}(L_\ell, \epsilon''_\ell) \cap \mathcal{N}(L_{\ell+1}, \epsilon''_{\ell+1})$. $\square$

We note the following result, whose elementary proof is a precursor to the proof of Proposition 4.7.

PROPOSITION 4.4 *Let $\mathcal{F}$ be a C^1 foliation of a compact manifold M, and $H: L \times [0,1] \to M$ be a leafwise homotopy such that for all $0 \leq t \leq 1$, H_t is a differentiable map whose derivatives depend continuously on t. If L is a compact leaf, then $t \to \mathrm{vol}\,(L^*_t)$ is a bounded function on $[0,1]$.*

Proof: Let *dvol* denote the leafwise volume p-form on M. This form is not closed, unless restricted to a leaf of $\mathcal{F}$, but is continuous on M. As H_t is a C^1 map whose derivatives depend continuously on t, the pull-back of the volume p-form, $\omega_t = H^*_t(dvol)$ on L, depends continuously on t. Let N_t denote the degree of the map $H_t: L \to L_t$, which is well defined as both L and L_t are oriented closed manifolds. That is, $(H_t)_*([L]) = N_t \cdot [L_t]$. Then

$$\mathrm{vol}\,(L^*_t) = \int_{L^*_t} dvol = \frac{1}{N_t} \cdot \int_{(H_t)_*([L])} dvol = \frac{1}{N_t} \cdot \int_L H^*_t(dvol) \leq \int_L \omega_t$$

which depends continuously t, so there exists a uniform upper bound. $\square$

We need two preliminary results before giving Proposition 4.7.

PROPOSITION 4.5 *Let $\mathcal{F}$ be a C^1 foliation of a compact manifold M, and suppose that L, L' are compact leaves which are ϵ_0-commensurable by an ϵ_0-chain $\mathfrak{L}$ of length k. Then there exist a constant $C(\pi_1(\mathfrak{L}))$, depending only on the algebraic data $\pi_1(\mathfrak{L})$, such that*

$$2^{(1-2k)}\, C(\pi_1(\mathfrak{L})) \cdot \mathrm{vol}\,(L) \;\leq\; \mathrm{vol}\,(L') \;\leq\; 2^{(2k-1)}\, C(\pi_1(\mathfrak{L})) \cdot \mathrm{vol}\,(L) \tag{21}$$

Proof: We are given an ϵ_0-chain $L_0, \mathfrak{L}_0, L_1, \ldots, L_{k-1}, \mathfrak{L}_{k-1}, L_k$ and constants $\epsilon''_0, \epsilon''_1, \ldots, \epsilon''_k$ such that $L = L_0$, $L' = L_k$, and for each $0 \leq \ell \leq k$, ϵ''_k is L_ℓ–normal. Let a_ℓ denote the degree of the covering map $\Pi: \mathfrak{L}_\ell \to L_\ell$, which equals the homological degree $\deg(\Pi)$. The Riemannian metric on the leaf L_ℓ lifts via the covering map Π to a Riemannian metric on $\mathfrak{L}_\ell$ with associated volume form $dvol'$. Then the total volume for this covering metric satisfies $\mathrm{vol}'(\mathfrak{L}_\ell) = a_\ell \cdot \mathrm{vol}\,(L_\ell)$.

As $\epsilon''_\ell < \epsilon_0$ we can apply the estimate (2) to conclude that the two volume forms on the leaf $\mathfrak{L}_\ell$ are related by the inequality $1/2 \cdot dvol \leq dvol' \leq 2 \cdot dvol$ and hence we have

$$a_\ell/2 \cdot \mathrm{vol}\,(L_\ell) \;\leq\; \mathrm{vol}\,(\mathfrak{L}_\ell) \;\leq\; 2\, a_\ell \cdot \mathrm{vol}\,(L_\ell) \tag{22}$$

Similarly, for each $0 \leq \ell < k$, the normal projection $\Pi: \mathcal{N}(L_{\ell+1}, \epsilon''_{\ell+1}) \to L_{\ell+1}$ restricts to a covering map $\Pi: \mathfrak{L}_\ell \to L_{\ell+1}$. Let b_ℓ denote the degree of this covering. Then we obtain in an analogous manner that

$$b_\ell/2 \cdot \mathrm{vol}\,(L_{\ell+1}) \;\leq\; \mathrm{vol}\,(\mathfrak{L}_\ell) \;\leq\; 2\, b_\ell \cdot \mathrm{vol}\,(L_{\ell+1}) \tag{23}$$

Combining these sequences of upper and lower estimates, we obtain

$$2^{(1-2k)}\, \frac{a_0\, a_1 \ldots a_{k-1}}{b_0\, b_1 \ldots b_{k-1}} \cdot \mathrm{vol}\,(L_0) \;\leq\; \mathrm{vol}\,(L_k) \;\leq\; 2^{(2k-1)}\, \frac{a_0\, a_1 \ldots a_{k-1}}{b_0\, b_1 \ldots b_{k-1}} \cdot \mathrm{vol}\,(L_0) \tag{24}$$

Set $C(\pi_1(\mathfrak{L})) = \dfrac{a_0\, a_1 \ldots a_{k-1}}{b_0\, b_1 \ldots b_{k-1}}$. $\square$

This estimate is intuitively clear, as can be seen in the case when all of the covering maps have degree $a_\ell = b_\ell = 1$ and each $\mathfrak{L}_\ell = L_\ell$ or $\mathfrak{L}_\ell = L_{\ell+1}$. Then, each successive compact leaf $L_{\ell+1}$ is diffeomorphic to the previous leaf L_ℓ, and is obtained by perturbing L_ℓ by a distance of less than ϵ_0. The volume change for each of these small deformations by at most a factor of 2, based on the volume form estimate in the inequality (2). The choice of the scale factor 2 in the estimate (2) was made for simplicity, and could be replaced by an optimal constant, which would yield a more precise form version of (21).

Lemma 4.3 and Proposition 4.5 yield an estimate comparing the volume of the compact leaf at the end of a homotopy with the volume of the compact leaf at the beginning. The following technical lemma allows us to give a uniform estimate for all leaves in the homotopy in Proposition 4.7.

LEMMA 4.6 *Let $\mathcal{F}$ be a C^1 foliation of a compact manifold M. Let L, L', L_0 be compact leaves, and suppose ϵ is L_0–normal. Suppose that*

1. *$L' \subset \mathcal{N}(L_0, \epsilon)$ with $d_0 = \deg(\Pi\colon L' \to L_0) > 0$*
2. *$G\colon L \times [s', s''] \to \mathcal{N}(L_0, \epsilon)$ is a continuous foliated map such that $G_{s'}(L) = L'$*
3. *the map $G\colon L \to L'$ has degree $d_1 > 0$.*

Then for all $s' \leq t \leq s''$, the leaf $L_t = G_t(L)$ has volume bounded by

$$\mathrm{vol}\,(L_0)/2 \;\leq\; \mathrm{vol}\,(L_t) \;\leq\; 2\, d_0 d_1 \mathrm{vol}\,(L_0) \tag{25}$$

Moreover, the degrees d_t of the covering maps $\Pi\colon L_t \to L_0$ are uniformly bounded, $1 \leq d_t \leq d_0 d_1$ and likewise $d_0 d_1$ is an upper bound on the degree of the maps $G_t\colon L \to L_t$.

Proof: For all $s' \leq t \leq s''$, the leaf L_t is compact by the proof of Proposition 4.1. As $L_t \subset \mathcal{N}(L_0, \epsilon)$, the restriction of the normal projection $\Pi\colon \mathcal{N}(L_0, \epsilon) \to L_0$ induces a covering map $\Pi\colon L_t \to L_0$.

The composition $\Pi \circ G_{s'}\colon L \to L_0$ has degree $d_0 d_1$ by assumption. As $\Pi \circ G_t\colon L \to L''$ is continuously defined for all $s' \leq t \leq s''$, the degree d_t of $\Pi \circ G_t$ must equal $d_0 d_1$ also. It follows that the degrees of both factor maps Π and G_t must divide $d_0 d_1$. Thus, $d_t = \deg(\Pi\colon L_t \to L_0)$ divides $d_0 d_1$ and hence $1 \leq d_t \leq d_0 d_1$. Similarly, $1 \leq \deg(G_t) \leq d_0 d_1$.

For all $s' \leq t \leq s''$ we have that $\mathrm{vol}'(L_t) = d_t\, \mathrm{vol}\,(L'')$, where $\mathrm{vol}'(L_t)$ denotes the volume of L_t for the covering metric. As in the proof of Proposition 4.5, the estimate (2) on the leafwise volume forms implies the inequalities (25). $\square$

We can now state and prove the main result of this section.

PROPOSITION 4.7 *Let $\mathcal{F}$ be a C^1 foliation of a compact manifold M. If $H\colon L \times [0,1] \to M$ is a leafwise homotopy with L a compact leaf, then for all $0 \leq t \leq 1$,*

$$\mathrm{vol}\,(L_t) \leq 4^k \cdot \mathrm{vol}\,(L) \tag{26}$$

where k is the length of an ϵ_0-chain obtained from H. Moreover, there is an integer $d_* > 0$ such that

$$1 \leq \deg(H_t\colon L_0 \to L_t) \leq d_* \tag{27}$$

Proof: By Lemma 4.3 and its proof, there is an ϵ_0-chain $L_0, \mathfrak{L}_0, L_1, \ldots, L_{k-1}, \mathfrak{L}_{k-1}, L_k$, constants $0 < \epsilon''_0, \epsilon''_1, \ldots, \epsilon''_k \leq \epsilon_0$ and times $0 = t_0 < s_0 < t_1 < s_1 < t_2 < \cdots < t_{k-1} < s_{k-1} < t_k = 1$ so that for each $0 \leq \ell \leq k$

- $L_\ell = H_{t_\ell}(L)$, $\mathfrak{L}_\ell = H_{s_\ell}(L)$
- ϵ''_ℓ is L_ℓ–normal
- $\mathfrak{L}_\ell \subset \mathcal{N}(L_\ell, \epsilon''_\ell) \cap \mathcal{N}(L_{\ell+1}, \epsilon''_{\ell+1})$
- $H_t(L) \subset \mathcal{N}(L_\ell, \epsilon''_\ell)$ for all $s_{\ell-1} \leq t \leq s_\ell$

where we set $s_{-1} = 0$ and $s_k = 1$ for notational convenience.

Recall that $\Pi\colon \mathcal{N}(L_\ell, \epsilon''_\ell) \to L_\ell$ is the normal projection. The composition

$$\Pi \circ H_t\colon L \to \mathcal{N}(L_\ell, \epsilon''_\ell) \to L_\ell \ , \ t_\ell \leq t \leq s_\ell$$

is a homotopy from $\Pi \circ H_{t_\ell} = H_{t_\ell}$ to $\Pi \circ H_{s_\ell}$. Thus, the maps $\Pi \circ H_{s_\ell}$ and H_{t_ℓ} have the same homological degree. Recall that a_ℓ is the covering degree of $\Pi\colon \mathfrak{L}_\ell \to L_\ell$, or equivalently the homological degree of Π, so it follows that $a_\ell \cdot \deg(H_{s_\ell}) = \deg(H_{t_\ell})$.

Similarly, using the homotopy

$$\Pi \circ H_t\colon L \to \mathcal{N}(L_{\ell+1}, \epsilon''_{\ell+1}) \to L_{\ell+1} \ , \ s_\ell \leq t \leq t_{\ell+1}$$

we have that $b_\ell \cdot \deg(H_{s_\ell}) = \deg(H_{t_{\ell+1}})$.

Thus, we obtain

$$\frac{b_\ell}{a_\ell} \ = \ \frac{\deg(H_{t_{\ell+1}})}{\deg(H_{t_\ell})} \tag{28}$$

Given $0 \leq s \leq 1$, let ν be the least index such that $s \leq s_\nu$, so that $s \in \mathcal{I}_\nu = (t_\nu - \xi_{t_\nu}, t_\nu + \xi_{t_\nu})$. Then either $t_\nu \leq s \leq s_\nu$ or $s_{\nu-1} \leq s < t_\nu$.

Set $d_\nu = \dfrac{b_0\, b_1 \ldots b_{\nu-1}}{a_0\, a_1 \ldots a_{\nu-1}}$. Let $d_* = \max\{d_1, \ldots, d_{k+1}\}$.

The map H_0 is the identity, so by applying (28) recursively, we have that $d_\nu = \deg(H_{t_\nu})$. In particular, d_ν is a positive integer, so $1/d_\nu \leq 1$. By the proof of Proposition 4.5, there is an estimate

$$\frac{1}{2^{(2\nu-1)}\, d_\nu} \cdot \mathrm{vol}\,(L) \ \leq \ \mathrm{vol}\,(L_\nu) \ \leq \ \frac{2^{(2\nu-1)}}{d_\nu} \cdot \mathrm{vol}\,(L) \tag{29}$$

Consider the case $t_\nu \leq s \leq s_\nu$. We then have $H\colon L \times [t_\nu, s_\nu] \to \mathcal{N}(L_\nu, \epsilon''_\ell)$ so we can apply Lemma 4.6 with $s' = t_\nu$, $s'' = s_\nu$, and $L = L$, $L_0 = L' = L_\nu$. The degree of the map H_{t_ν} is d_ν, while $L' \to L_0$ is the identity so has degree 1. Combine (25) and (29) to obtain the upper bound estimate

$$\mathrm{vol}\,(L_s) \ \leq \ 2\, d_\nu \cdot \mathrm{vol}\,(L_\nu) \ \ \leq \ 2\, d_\nu \cdot \frac{2^{(2\nu-1)}}{d_\nu} \cdot \mathrm{vol}\,(L) = 2^{2\nu} \cdot \mathrm{vol}\,(L) \leq 4^k \cdot \mathrm{vol}\,(L) \tag{30}$$

Note also that by Lemma 4.6 the degree $d_s = \deg(H_s\colon L \to L_s)$ divides d_ν, hence $1 \leq d_s \leq d_\nu \leq d_*$.

In the alternate case $s_{\nu-1} \leq s < t_\nu$ we proceed in exactly the same way. $\square$

5 Tame points in the bad set

The bad set X_1 consists of the points $y \in M$ where the leaf volume function $v(x)$ is not bounded in any open neighborhood of y. This set is closed, saturated and has no interior, but could otherwise be an arbitrarily pathological continua. A point $x_1 \in X_1$ is regular if the restriction of the leaf volume function $v: X_1 \to \mathbb{R}^+$ is is continuous at x_1. Equivalently, $x_1 \in X_1$ is a regular point if the holonomy of the restriction of $\mathcal{F}$ to X_1 is trivial in some relatively open neighborhood of $x_1 \in X_1$.

In this section we introduce the concept of a "tame point" in the bad set, where the local properties of the bad set have some additional regularity. We then prove the existence of tame points. Tame points are used in section 6 for studying the deformations of the bad set under foliated homotopy.

DEFINITION 5.1 *A regular point $x_1 \in X_1$ is* tame *if for any $\epsilon > 0$, there is a transverse C^1-path*

$$\gamma: [0,1] \to (\mathcal{N}(x_1, \epsilon) \cap G_e) \cup \{x_1\} \tag{31}$$

with $\gamma(t) \in G_e$ for $0 \leq t < 1$, $\gamma(1) = x_1$ and $v(\gamma(t)))$ tends uniformly to infinity as $t \to 1$.

Since the restricted path $\gamma: [0,1) \to G_e$ lies in the set of leaves without holonomy, it follows that there is a foliated isotopy $\Gamma: L_{\gamma(0)} \times [0,1) \to G_e$ such that $\Gamma_t(\gamma(0)) = \gamma(t)$. Thus, a tame point x is directly approachable by a family of moving leaves whose volumes tend uniformly to infinity.

In the examples constructed by Sullivan [30], it is easy to see that every regular point is a tame point. In general, though, Edwards, Millet, and Sullivan specifically point out their proof of the Moving Leaf Proposition 3.1 in [10] does not claim that a regular point is a tame point. The problem is due to the possibility that the complement of the bad set need not be locally connected in a neighborhood of a point in the bad set. In their proof, the moving leaf is defined by a curved that follows "an end of the good set" out to infinity, passing through points where the volume is tending to infinity along the way. This end of the good set is contained in arbitrarily small ϵ-neighborhoods of the bad set, but they do not control the behavior of the end. Thus, the existence of a tame point is asserting the existence of a "tame end" of the good set on which the volume function is unbounded, and which is defined by open neighborhoods of some point in the bad set.

Let $X^t \subset X_1$ denote the subset of tame points.

PROPOSITION 5.2 *Let $\mathcal{F}$ be a compact, C^1 foliation of a manifold M. Then the set of tame points X^t is dense in X_1.*

Proof: Let $x_1 \in X_1$ be a regular point, and L_1 the leaf through x_1. We will build up a detailed geometric description of the foliation in a sufficiently small neighborhood of L_1 and use this to prove there is a tame point arbitrarily close to x_1.

Choose a finite generating set $\{[\tau_1], \ldots, [\tau_k]\}$ for $\pi_1(L_1, x_1)$, where $[\tau_i]$ is represented by a closed path $\tau_i: [0,1] \to L_1$ with basepoint x_1. Let $\|\tau_i\|$ denote the Riemannian length of τ_i. Then set

$$\mathfrak{P} = 2 \max \{\mathrm{diam}\,(L_1), \|\tau_1\|, \ldots, \|\tau_k\|\} \tag{32}$$

Let $0 < \epsilon_1 \leq \epsilon_0$ be such that the normal projection $\Pi: \mathcal{N}(L_1, \epsilon_1) \to L_1$ is well-defined. Then set $\epsilon_2 = \Delta(\mathfrak{P}, \epsilon_1)$ where $\Delta(\mathfrak{P}, \epsilon_1)$ is defined in Definition 2.5.

By Corollary 2.6, the holonomy map $\mathbf{h}_i$ along the closed path τ_i is defined on the transverse disk $\mathcal{N}(x_1, \epsilon_2)$. That is, the transverse holonomy along τ_i is represented by a local homeomorphism into

$$\mathbf{h}_i: \mathcal{N}(x_1, \epsilon_2) \to \mathcal{N}(x_1, \epsilon_1) \tag{33}$$

Moreover, for any path $\sigma\colon[0,1] \to L_1$ with $\sigma(0) = x_1$ and $\|\sigma\| \leq \mathfrak{P}$ the transverse holonomy maps are defined for all $0 \leq t \leq 1$,

$$\mathbf{h}_\sigma\colon \mathcal{N}(x_1, \epsilon_2) \to \mathcal{N}(\sigma(t), \epsilon_1) \tag{34}$$

Choose $0 < \delta \leq \epsilon_2$ sufficiently small so that the induced holonomy on $X_1 \cap \mathcal{N}(x_1, 2\delta)$ is trivial. The induced holonomy of $\mathcal{F}$ is trivial on the closure

$$Z_1 = \overline{X_1 \cap \mathcal{N}(x_1, \delta)} = X_1 \cap \overline{\mathcal{N}(x_1, \delta)} \subset X_1 \cap \mathcal{N}(x_1, 2\delta)$$

so the saturation $Z_{\mathcal{F}}$ of Z_1 is a fibration over the closed set Z_1. We assume that δ is sufficiently small so that $Z_{\mathcal{F}} \subset \mathcal{N}(L_1, \epsilon_1)$.

The leaf volume function $v(y)$ is uniformly continuous and hence bounded on the compact set Z_1 as the induced holonomy is trivial. Hence, every point in Z_1 is a regular point of the bad set.

For each $z \in Z_1$, the normal projection $\Pi\colon \mathcal{N}(L_1, \epsilon_1) \to L_1$ restricts to the leaf $L_z \subset Z_{\mathcal{F}}$ to yield a covering projection $\pi^z\colon L_z \to L_1$, which must be a diffeomorphism as $\mathcal{F}|Z_1$ has no holonomy.

By estimate (3) the map π^z is also a quasi-isometry with expansion bound 2. Note that as $\mathcal{N}(x_1, 2\delta)$ is the normal neighborhood of x_1, by definition we have that $\pi^z(z) = x_1$ for $z \in Z_1$.

For $z \in Z_1$, the closed loop τ_i lifts via π^z to a closed loop $\tau_i^z\colon[0,1] \to L_z$. The homotopy classes of the lifts, $\{[\tau_1^z], \ldots, [\tau_k^z]\}$, give a generating set for $\pi_1(L_z, z)$, which by (3) and (32) have a uniform bound on their path lengths, $\|\tau_i^z\| \leq \mathfrak{P}$.

In fact, since $L_z \subset \mathcal{N}(L_1, \epsilon_1)$, for any path $\sigma\colon[0,1] \to L_1$ with $\sigma(0) = x_1$ there is a lift $\sigma^z\colon[0,1] \to L_z$ with $\sigma^z(0) = z$ and $\pi^z \circ \sigma^z(t) = \sigma(t)$ for all $0 \leq t \leq 1$.

For an arbitrary point $y_0 \in \mathcal{N}(x_1, \delta)$ and path $\sigma\colon[0,1] \to L_1$ with $\sigma(0) = x_1$ and $\|\sigma\| \leq \mathfrak{P}$, the existence of the transverse holonomy map in (34) means that there is a lift $\sigma^y\colon[0,1] \to L_y \cap \mathcal{N}(L_1, \epsilon_1)$ with $\sigma^y(0) = y_0$ and $\pi^y \circ \sigma^y(t) = \sigma(t)$ for all $0 \leq t \leq 1$.

The lifting property need not hold for paths longer than $\mathfrak{P}$, as there may be leaves of $\mathcal{F}$ which intercect the normal neighborhood $\mathcal{N}(L_1, \delta)$ but are not contained in $\mathcal{N}(L_1, \epsilon_1)$. However, when $L_y \subset \mathcal{N}(L_1, \delta)$ and there is given a path $\sigma\colon[0,1] \to L_1$ with $\sigma(0) = x_1$, then Corollary 2.6 implies that the lift $\sigma^{y'}\colon[0,1] \to L_{y'}$ can be defined for y' sufficiently close to y. From a geometric approach, $L_y \subset \mathcal{N}(L_1, \epsilon_1)$ is a compact subset of an open set, hence for $\epsilon_3 > 0$ sufficiently small, the open normal neighborhood $\mathcal{N}(L_y, \epsilon_3) \subset \mathcal{N}(L_1, \epsilon_1)$, and the leaves of the foliation restricted to $\mathcal{N}(L_y, \epsilon_3)$ satisfy an arbitrarily long path lifting property with respect to paths in L_y as they approach L_1.

For $y \in \mathcal{N}(x_1, \delta)$, let $B_T(y, \delta) \subset \mathcal{N}(x_1, \delta)$ denote the open ball of radius δ about y for the induced Riemannian metric on $\mathcal{N}(x_1, \delta)$. Note that, unless $\mathcal{N}(x_1, \delta)$ is a totally geodesic submanifold of M, $B_T(y, \delta)$ and $\mathcal{N}(y, \delta)$ are distinct submanifolds of M, though both are transverse to $\mathcal{F}$.

By the proof of the Moving Leaf Proposition, which is based on ideas of Montgomery [24] and Newman [25], there is an open connected component U of $\mathcal{N}(x_1, \delta) - Z_1$ on which the volume function $v(y)$ is unbounded on the open neighborhood $U \cap \mathcal{N}(x_1, \delta/2)$ of x_1. (See the details of the proof on page 23 of [10], especially Figure 3.)

Choose a point $y_0 \in U \cap \mathcal{N}(x_1, \delta/2)$ and let $x_* \in Z_1$ be a closest point to y_0 for the induced metric on $\mathcal{N}(x_1, \delta)$. That is, consider a sequence of balls $B_T(y_0, \delta) \subset \mathcal{N}(x_1, \delta) - Z_1$ expanding until there is a first contact with the frontier of U, then x_* is this first point of contact. Let δ_0 denote the distance from y_0 to x_* in this induced metric. Then $B_T(y_0, \delta_0) \subset U$ and $\overline{B_T(y_0, \delta_0)} \cap Z_1 \neq \emptyset$. Let $L_* = L_{x_*}$ be the leaf containing x_*. This is illustrated in Figure 1 below.

We claim that x_* is a tame point. As $\delta > 0$ was chosen to be arbitrarily small, and the regular points are dense in the bad set, the proof of Proposition 5.2 will then follow from this claim.

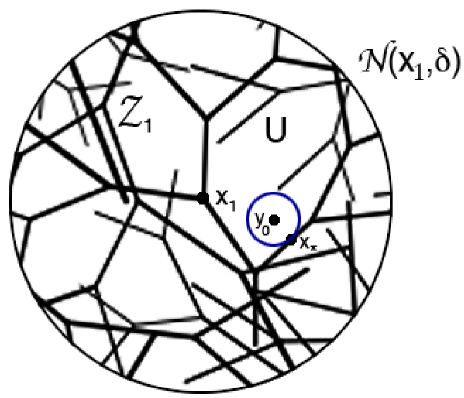

Figure 1: tame point in the bad set

As noted before, $B_T(y_0, \delta_0) \subset U$, and there is a geodesic path $\gamma: [0,1] \to \mathcal{N}(x_1, \delta)$ with length δ_0 such that $\gamma(0) = y_0$, $\gamma(1) = x_*$ and $\gamma[0,1) \subset B_T(y_0, \delta_0)$. The complement of X_1 is the good set, so the image $\gamma[0,1) \subset G$. The set of leaves with holonomy G_h in G is a union of codimension two (or higher) submanifolds by Proposition 3.2, so by a small C^1-perturbation of the path γ in U, we can assume that its image is disjoint from the set G_h. That is, $\gamma(t) \in L_t \subset G_e$ and $\gamma(1) \in L_*$.

We claim that the volumes of the leaves L_t tend uniformly to infinity. Our proof is by contradiction – we assume this is false, that is, there exists a constant $\mathfrak{M} > 0$ and a sequence $0 < t_1 < \cdots < t_n \cdots \to 1$ such that $x_n = \gamma(t_n) \to x_*$ and the volumes of the leaves $L_n = L_{x_n}$ are bounded above by $\mathfrak{M}$. The contradiction is then provided by the following "rigidity" result, which has an intuitive description as saying that for each $\mathfrak{M} > 0$ there is an $\epsilon > 0$ so that when $y \in U$ gets ϵ–close to the regular point $x_* \in X_1$, and if the volume of the leaf L_y is less than $\mathfrak{M}$, then L_y acts as a "seed" upon which all the leaves in the saturation of the open set U "crystalize", forcing all the leaves in U to have volumes bounded above by $4\mathfrak{M}$.

LEMMA 5.3 *For each $\mathfrak{M} > 0$ there is an $\epsilon_* > 0$ so that if there exists $y \in U \cap G_e$ such that $d(y, x_*) < \epsilon_*$ and $\mathrm{vol}\,(L_y) \leq \mathfrak{M}$, then for all $y' \in U$, the leaf $L' = L_{y'}$ has the volume bound* $\mathrm{vol}\,(L_{y'}) \leq 4\,\mathfrak{M}$.

Proof: By Proposition 2.3 there is $\mathfrak{R}$ such that if $L \subset M$ satisfies $\mathrm{vol}\,(L) \leq \mathfrak{M}$ then $\mathrm{diam}\,(L) \leq \mathfrak{R}$.

Recall that $L_* = L_{x_*}$ and set $\pi^* = \pi^{x_*}: L_* \to L_1$. For each closed path τ_i let $\tau_i^*: [0,1] \to L_*$ be the lift with basepoint x_*. Their homotopy classes $\{[\tau_1^*], \ldots, [\tau_k^*]\}$ form a generating set for $\pi_1(L_*, x_*)$. Note that the path length $\|\tau_i^*\| \leq \mathfrak{P}$. The holonomy along τ_i^* will be denoted by $\mathbf{h}_i^*$.

Let $0 < \epsilon_3 \leq \epsilon_2$ be such that $\mathcal{N}(L_*, \epsilon_3) \subset \mathcal{N}(L_1, \epsilon_2)$. Set $\epsilon_* = \Delta(\mathfrak{P}, \epsilon_3)$. Then the holonomy $\mathbf{h}_i^*$ along τ_i^* is represented by a local homeomorphism into

$$\mathbf{h}_i^*: \mathcal{N}(x_*, \epsilon_*) \to \mathcal{N}(x_*, \epsilon_3) \subset \mathcal{N}(x_1, \epsilon_2) \tag{35}$$

and each map in (35) extends to map

$$\mathbf{h}_i^*: \mathcal{N}(x_*, \epsilon_2) \to \mathcal{N}(x_*, \epsilon_1) \tag{36}$$

Then for $y \in U \cap B_M(x_*, \epsilon_*)$ with $\mathrm{vol}\,(L_y) \leq \mathfrak{M}$, by Corollary 2.6 we have that $L_y \subset \mathcal{N}(L_*, \epsilon_3)$. In particular, as $\mathcal{N}(x_*, \epsilon_3) \subset \mathcal{N}(x_*, \epsilon_2)$, this implies that

$$\mathfrak{F}_y = L_y \cap \mathcal{N}(x_*, \epsilon_2) = L_y \cap \mathcal{N}(x_*, \epsilon_3) \tag{37}$$

and their holonomy maps satisfy

$$h_i^*(\mathfrak{F}_y) \subset L_y \cap \mathcal{N}(x_*, \epsilon_2) = \mathfrak{F}_y$$

Thus, the finite set of points $\mathfrak{F}_y$ is permuted by the action of a set of generators for $\pi_1(L_*, x_*)$. It follows that the holonomy of all compositions of the generators are defined when restricted to the set $\mathfrak{F}_y$. That is, for any $w \in \pi_1(L_*, x_*)$ the holonomy h_w^* along w contains the finite set $\mathfrak{F}_y$ in its domain. Let $\mathcal{H}_* \subset \pi_1(L_*, x_*)$ denote the normal subgroup of finite index consisting of all words whose holonomy fixes every point in $\mathfrak{F}_y$. Let $\{w_1, \ldots, w_N\}$ be a set of generators for $\mathcal{H}_*$.

Let $z \in \mathfrak{F}_y$. For each $w \in \mathcal{H}_*$, the holonomy $\mathbf{h}_w^*$ map is defined at z and so must be defined on some open neighborhood $z \in V_z^w \subset U$ of z, which may depend on w. As $y \in U \cap G_e$ the leaf $L_y \subset G_e$ is without holonomy, so the restriction of $\mathbf{h}_w^*$ to the open set V_z^w must fix an open neighborhood of $z \in U_z^w \subset V_z^W$. Thus, the fix-point set of $\mathbf{h}_w^*$ contains an open neighborhood of $\mathfrak{F}_y$ and in particular, contains an open neighborhood $y \subset U_y^w \subset U \cap B_T(x_*, \epsilon_*)$.

Each holonomy map $\mathbf{h}_i : \mathcal{N}(x_1, \epsilon_2) \to \mathcal{N}(x_1, \epsilon_1)$ fixes the set Z_1 and its restriction to the open set $B_T(x_*, \epsilon_3)$ equals the restriction of $\mathbf{h}_i^*$ to $\mathcal{N}(x_*, \epsilon_3)$. Thus, $\mathbf{h}_w$ is defined on and fixes the open set $U_y^w \subset B_T(x_*, \epsilon_*)$. We will show that $U \subset \mathrm{Fix}(\mathbf{h}_w)$, hence for $y' \in U$ the leaf $L_{y'}$ is a finite covering of L_1 and isotopic to L_y, hence $\mathrm{vol}\,(L_{y'}) \leq 4\,\mathfrak{M}$.

Let m_ℓ denote the word length of w_ℓ with respect to the generating set $\{[\tau_1], \ldots, [\tau_k]\}$, and set $m_* = \max\{m_1, \ldots, m_N\}$. Then for all $z \in \mathcal{N}(x_1, \epsilon_1)$ and for each $1 \leq \ell \leq N$ the closed path representing w in L_1 can be lifted to a path τ_w^z in the leaf L_z with length at most $2m_*\mathfrak{P}$.

Fix a choice of $w = w_\ell \in \mathcal{H}_*$. Choose $0 < \delta_* \leq \Delta(2m_*\mathfrak{P}, \epsilon_*) \leq \epsilon_*$ such that $B_T(y, \delta_*) \subset U_y^w$.

Given any point $y' \in U$ there is a continuous path $\sigma : [0,1] \to U \cap G_e$ such that $\sigma(0) = y$ and $\sigma(1) = y'$. Choose a sequence of points $0 = t_0 < t_1 < \cdots < t_\lambda = 1$ such that for $y_i = \sigma(t_i)$ we have $\sigma([t_i, t_{i+1}]) \subset B_T(y_i, \delta_*)$. See Figure 2 below.

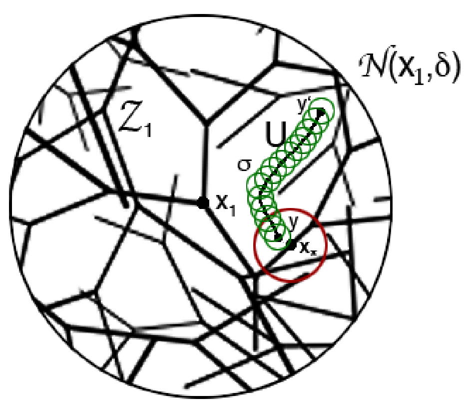

Figure 2: path chain in the good set

We prove by induction on the index i that $\sigma([0,1]) \subset \mathrm{Fix}(\mathbf{h}_w)$. For $i = 0$, $y_0 = y$ and by assumption, the disk $B_T(y, \delta_*) \subset U_y^w \subset \mathrm{Fix}(\mathbf{h}_w)$ so $\sigma([0, t_1]) \subset \mathrm{Fix}(\mathbf{h}_w)$.

Now assume $\sigma([0,t_n]) \subset \mathrm{Fix}(\mathbf{h}_w)$. We show that $\sigma([t_n,t_{n+1}]) \subset \mathrm{Fix}(\mathbf{h}_w)$.

Note that $y_n = \sigma(t_n) \in \mathrm{Fix}(\mathbf{h}_w)$.

The leaf L_{x_n} through y_n is a covering of L_1 such that the word w is represented by a closed loop of length at most $2m_*\mathfrak{P}$. The holonomy map for w at y_n fixes y_n so is defined by a map

$$\mathbf{h}_w^{y_n}\colon \mathcal{N}(y_n,\delta_*) \to \mathcal{N}(y_n,\epsilon_*)$$

As the points of $U \cap G_e$ determine leaves without holonomy, the set of fixed-points for $\mathbf{h}_w^{y_n}$ is an open subset of $\mathcal{N}(y_n,\delta_*) \cap U \cap G_e$. The set of fixed-points is also always a (relatively) closed subset, hence $\mathrm{Fix}(\mathbf{h}_w^{y_n})$ contains the connected component of $\mathcal{N}(y_n,\delta_*) \cap U \cap G_e$ which contains the point y_n. By assumption we have that $\sigma([t_n,t_{n+1}]) \subset \mathcal{N}(y_n,\delta_*) \cap U \cap G_e$, hence

$$\sigma([t_n,t_{n+1}]) \subset \mathrm{Fix}(\mathbf{h}_w^{y_n}) \subset \mathrm{Fix}(\mathbf{h}_w) \tag{38}$$

This concludes the proof of Lemma 5.3, and thus the proof of Proposition 5.2. $\square$.

6 Proof of Main Theorem

PROPOSITION 6.1 *Let $\mathcal{F}$ be a compact C^1-foliation of a compact manifold M. If $V \subset M$ is a saturated open set which contains a tame point, then V cannot be transversely categorical.*

Proof of Proposition 6.1: We assume that V is a categorical saturated open neighborhood of a tame point $x_1 \in X_1$ and show that this yields a contradiction.

Let $H\colon V \times [0,1] \to M$ be a leafwise homotopy with $H_0 = Id$, and $H_1(V) \subset L_*$ for some leaf L_*.

As x_1 is a tame point, there is a continuous path $\gamma\colon [0,1] \to V$ such that $\gamma(1) = x_1$, $\gamma(t) \in G_e$ for $0 \le t < 1$, and the volume $v(\gamma(t))$ of the leaf L_t containing the point $\gamma(t)$ satisfies $\lim_{t\to 1} v(\gamma(t)) = \infty$.

Define a map $\phi\colon [0,1] \times [0,1] \to M$ by setting $\phi_s(t) = \phi(s,t) = H_s(\gamma(t))$. The key to obtaining a contradiction is to analyze the behavior of the leaf volume function $v(\phi(s,t))$.

Set $x_t = \gamma(t)$. Then $x_t \in G_e$ for $0 < t \le 1$, while $x_1 \in X_1$ is the tame point.

Set $x_{s,t} = \phi(s,t)$, so that $x_t = x_{0,t}$. Let $L_{s,t}$ be the compact leaf through the point $x_{s,t}$.

We note that:

- For all $0 \le t \le 1$, $L_t = L_{0,t}$ and $L_{1,t} = L_*$.
- $L_{s,t}$ is the image of the leaf L_t by the map H_s.
- $L_{s,1}$ is the image of the leaf L_1 by the map H_s.
- For $t = 0$, the path $s \mapsto x_{s,0}$ is the trace of the initial point x_0 under the homotopy H_s.
- For $t = 1$, the path $s \mapsto x_{s,1}$ is the trace of the tame point x_1 under the homotopy H_s.
- For $s = 0$, the volume function $v(x_{0,t}) = v(x_t)$ is unbounded as $t \to 1$.
- For $s = 1$, $t \mapsto x_{1,t}$ is a continuous path in the leaf L_*, hence $v(x_{1,t})$ is constant for $0 \le t \le 1$.

As remarked after Proposition 6.1, the restricted path $\gamma\colon [0,1) \to G_e$ lies in the set of leaves without holonomy, hence there is a foliated isotopy $\Gamma\colon L_0 \times [0,1) \to G_e$ such that $\Gamma_t(x_0) = x_t$. Note that each map $\Gamma_t\colon L_0 \to L_t$ is a homeomorphism, hence has homological degree 1.

For $t = 0$ and each $0 \leq s \leq 1$, the map $H_s\colon L_0 = L_{0,0} \to L_{s,0}$ is surjective by Proposition 4.7. Let $d_{s,0}$ denote its homological degree. The path of leaves $s \mapsto L_{s,0}$ starting at L_0 has an upper bound $\mathfrak{P}_0$ on their volumes by Proposition 4.7, and moreover, there is an upper bound $d_0 = \sup\{d_{s,0} \mid 0 \leq s \leq 1\}$.

For $t = 1$ and each $0 \leq s \leq 1$, the map $H_s\colon L_1 = L_{0,1} \to L_{s,1}$ is surjective by Proposition 4.7. Let $d_{s,1}$ denote its homological degree. The path of leaves $s \mapsto L_{s,1}$ starting at L_1 has an upper bound $\mathfrak{P}_1$ on their volumes by Proposition 4.7, and moreover, there is an upper bound $d_1 = \sup\{d_{s,1} \mid 0 \leq s \leq 1\}$.

Set $\mathfrak{P} = \max\{\mathfrak{P}_0, \mathfrak{P}_1\}$.

For the values $0 \leq t < 1$, we use the isotopy Γ_t extend the map $\phi(s,t)$ to a continuous 2–parameter family of maps $\Phi\colon [0,1] \times [0,1) \times L_0 \to M$ by setting $\Phi_{s,t}(y) = H_s(\Gamma_t(y))$ for $y \in L_0$.

Observe that $\Phi_{1,t}\colon L_0 \to L_*$, $0 \leq t < 1$, is a homotopic family of maps, hence its homological degree is constant. Thus, for all $0 \leq t < 1$ we have

$$\begin{aligned}
\deg(H_1\colon L_0 \to L_*) &= \deg(\Phi_{1,0}\colon L_0 \to L_{1,0}) \\
&= \deg(\Phi_{1,t}\colon L_0 \to L_{1,t}) \\
&= \deg(\Gamma_t\colon L_0 \to L_t) \cdot \deg(H_1\colon L_t \to L_{1,t}) \\
&= \deg(H_1\colon L_t \to L_{1,t})
\end{aligned}$$

It follows that

$$\deg(H_1\colon L_t \to L_{1,t}) \leq d_0 \ , \ \forall\ 0 \leq t < 1 \tag{39}$$

Let $\mathfrak{K} = R(2\, d_0\, d_1\, \mathfrak{P})$ be the maximum diameter of a leaf with volume at most $2\, d_0\, d_1\, \mathfrak{P}$.

For each $0 \leq s \leq 1$, let $0 < \epsilon'_s \leq \epsilon_0$ be such that the normal projection $\Pi\colon \mathcal{N}(L_{s,1}, \epsilon'_s) \to L_{s,1}$ is well-defined. Set $\delta'_s = \Delta(\mathfrak{K}, \epsilon'_s)$.

By the choice of $\mathfrak{K}$ and δ'_s, if L is a compact leaf such that $\mathrm{vol}\,(L) \leq 2\, d_0\, d_1\, \mathfrak{P}$ and $L \cap \mathcal{N}(L_{s,1}, \delta'_s) \neq \emptyset$, then Proposition 2.9 implies that $L \subset \mathcal{N}(L_{s,1}, \epsilon'_s)$, the restriction $\Pi\colon L \to L_{s,1}$ is a covering map, and moreover by estimate (2) we have the estimate

$$\mathrm{vol}\,(L) \leq 2 \deg(\Pi\colon L \to L_{s,1}) \cdot \mathrm{vol}\,(L_{s,1}) \leq 2 \deg(\Pi\colon L \to L_{s,1}) \cdot \mathfrak{P} \tag{40}$$

For each s, $\mathcal{N}(L_{s,1}, \delta'_s)$ is an open neighborhood of $L_{s,1}$, so there exists $\lambda_s > 0$ such that

$$\phi([s - \lambda_s, s + \lambda_s] \times [1 - \lambda_s, 1]) \subset \mathcal{N}(L_{s,1}, \delta'_s) \tag{41}$$

Choose a sequence $0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = 1$ of points such that for $\lambda_n = \lambda_{s_n}$ the collection of open intervals $\{\mathcal{I}_n = (s_n - \lambda_n, s_n + \lambda_n) \mid n = 0, 1, \ldots, N\}$ is an open covering of $[0,1]$. Moreover, we can assume that the covering is minimal; that is, for each n we can choose a point

$$\xi_n \in (s_{n-1}, s_{n-1} + \lambda_{n-1}) \cap (s_n - \lambda_n, s_n) \ , n = 1, 2, \ldots, N$$

The closed intervals $\{[0, \xi_1], [\xi_1, \xi_2], \ldots, [\xi_{N-1}, \xi_N], [\xi_N, 1]\}$ form a closed cover $[0,1]$.

Set $\delta''_n = \delta'_{s_n}$ and $\epsilon''_n = \epsilon'_{s_n}$ for $0 \leq n \leq N$.

Set $\lambda_* = \min\{\lambda_n \mid n = 0, 1, \ldots, N\} > 0$. We claim that

$$0 \leq s \leq 1 \ \ \& \ \ 1 - \lambda_* \leq t < 1 \implies \mathrm{vol}\,(L_{s,t}) \leq 2\, d_0\, d_1\, \mathfrak{P} \tag{42}$$

For $s = 0$ this contradicts the assumption that $\mathrm{vol}\,(L_{0,t}) = v(\phi_0(t)) \to \infty$.

The idea of the proof of (42) is that we are given two paths of compact leaves: The first, for $t = 1$, is the trace of the homotopy H_s applied to L_1, which yields the path $s \mapsto L_{s,1} = H_s(L_1)$ whose volumes are uniformly bounded by $\mathfrak{P}_1 \leq \mathfrak{P}$. The second, for $t = \mu$ with $1 - \lambda_* \leq \mu < 1$, is the trace of the homotopy H_s applied to L_μ, which yields the path $s \mapsto L_{s,\mu}$ whose volumes we want to show are uniformly bounded. What we show (essentially) is that each leaf $L_{s,\mu}$ is a covering of the corresponding leaf $L_{s,1}$ and that the degrees of these coverings are uniformly bounded by $d_0\, d_1$. The proof of this is by downward induction on the index $0 \leq n \leq N$, and we formulate the covering property for the discrete set of leaves corresponding to the values $\{s_n \mid n = 0, 1, \ldots, N\}$. The proof uses the techniques for studying a homotopy of compact leaves introduced in section 4.

It is important to recall the usual caution with the study of compact foliations: a path leaves $t \mapsto L_t$ with unbounded volumes cannot limit on any normal neighborhood of a compact leaf in the bad set. Thus, the paths $s \mapsto L_{s,\mu}$ must become more chaotic as $\mu \to 1$. The key point of the proof below is that the behavior of the path of leaves $\mu \mapsto L_{1,\mu}$ is controlled at $s = 1$, and we then use an inductive process to control the the limiting behavior as $\mu \to 1$, for all $s < 1$.

Let μ denote a fixed number such that $1 - \lambda_* \leq \mu < 1$. For $n = N$, by (41) we have that

$$\phi([1 - \xi_N, 1] \times [\mu, 1]) \subset \phi([1 - \lambda_N, 1] \times [1 - \lambda_N, 1]) \subset \mathcal{N}(L_{1,1}, \delta''_N)$$

and thus for each $1 - \xi_N \leq s \leq 1$ the point $\phi(s, \mu) \in \mathcal{N}(L_{1,1}, \delta''_N)$.

Note that $L_{1,\mu} = L_{1,1} = L_*$, thus for $s < 1$ sufficiently close to 1 we have $H_s(L_\mu) \subset \mathcal{N}(L_{1,1}, \epsilon''_N)$.

Let $r_N < 1$ be the infimum of r such that $r \leq s \leq 1$ implies $L_{s,\mu} \subset \mathcal{N}(L_{1,1}, \epsilon''_N)$. The above remark implies $r_N < 1$. We claim that $r_N < \xi_N$.

Assume, to the contrary, that $r_N \geq \xi_N$. Let $r_N < r < 1$. Then for $r \leq s \leq 1$, $L_{s,\mu} \subset \mathcal{N}(L_{1,1}, \epsilon''_N)$ and so the normal projection $\Pi\colon L_{s,\mu} \to L_{1,1}$ is well-defined and a covering map. The restriction

$$H\colon L_\mu \times [r, 1] \to \mathcal{N}(L_{1,1}, \epsilon''_N)$$

yields a homotopy between $H_r\colon L_\mu \to L_{r,\mu}$ and $H_1\colon L_\mu \to L_{1,\mu} = L_{1,1}$. Thus,

$$\deg(\Pi \circ H_r\colon L_\mu \to L_{r,\mu} \to L_{1,1}) = \deg(\Pi \circ H_1\colon L_\mu \to L_{1,\mu} \to L_{1,1}) = \deg(H_1\colon L_\mu \to L_{1,1})$$

as $\Pi\colon L_{1,\mu} \to L_{1,1}$ is the identity. The upper bound (39) implies $\deg(H_1\colon L_\mu \to L_{1,1}) \leq d_0$ hence the covering degree of $\Pi\colon L_{r,\mu} \to L_{1,1}$ is bounded above by d_0, as it is an integer which divides $\deg(H_1\colon L_\mu \to L_{1,1})$. By (2) it follows that

$$\mathrm{vol}\,(L_{r,\mu}) \leq 2\, d_0 \cdot \mathrm{vol}\,(L_{1,1}) \leq 2\, d_0 \cdot \mathfrak{P} \tag{43}$$

The leaf volume function is lower semi-continuous, hence we also have that

$$\mathrm{vol}\,(L_{r_N,\mu}) \;\leq\; \lim_{r \to r_N+} \mathrm{vol}\,(L_{r,\mu}) \;\leq\; 2\, d_0 \cdot \mathfrak{P}$$

Thus, the estimate (43) holds for all $r_N \leq r \leq 1$ and $1 - \lambda_* \leq \mu < 1$.

As we assumed that $r_N \geq \xi_N \geq \lambda_N$ we have that $\phi(r_N, \mu) \in \mathcal{N}(L_{1,1}, \delta''_N)$ hence $L_{r_N,\mu} \subset \mathcal{N}(L_{1,1}, \epsilon''_N)$. By the continuity of H_s at $s = r_N$ there is $r < r_N$ such that $r < s \leq r_N$ implies $L_{s,\mu} \subset \mathcal{N}(L_{1,1}, \epsilon''_N)$. This contradicts the choice of r_N as the infimum of such r, hence we must have that $r_N < \xi_N$. This proves the first statement of the inductive hypothesis for $n = N$, which is that the estimate (43) holds for all $\xi_N \leq r \leq 1$ and $1 - \lambda_* \leq \mu < 1$.

The second statement statement of the inductive hypothesis involves the "internal geometry" of the foliation near the path of leaves $\{s \mapsto L_{s,1}\}$. We prove the equality between the ratios of covering degrees for a pair of leaves in adjacent normal neighborhoods of a covering of the path $\{L_{s,1}\}$.

For $\xi_N \leq s \leq 1$, we have $\phi(s,1) \in \mathcal{N}(L_{1,1}, \delta''_N)$ and $\mathrm{vol}\,(L_{s,1}) \leq \mathfrak{P}$ hence $L_{s,1} \subset \mathcal{N}(L_{1,1}, \epsilon''_{N-1})$ and the normal projection restricts to a covering map $\Pi\colon L_{s,1} \to L_{1,1}$. Therefore, both $L_{\xi_N,\mu}$ and $L_{\xi_N,1}$ are coverings of $L_{1,1}$, and their homological degrees are denoted by

$$\begin{aligned} \alpha_N^\mu &= \deg(\Pi\colon L_{\xi_N,\mu} \to L_{1,1}) \\ a_N &= \deg(\Pi\colon L_{\xi_N,1} \to L_{1,1}) \end{aligned}$$

The leaves $L_{\xi_N,\mu}$ and $L_{\xi_N,1}$ are also coverings of $L_{s_{N-1},1}$, so have covering degrees with respect to $L_{s_{N-1},1}$. We next show that the degrees of the maps $t \mapsto \Phi_{\xi_N,t}$ have a well-defined limit, which will be used to prove the equality of the ratios of the covering degrees with respect to $L_{1,1}$ and $L_{s_{N-1},1}$. We formulate this in a generality that allows us to quote it again for the general inductive step.

For $0 \leq s \leq 1$, the path $t \mapsto \phi(s,t)$ has limit $x_{s,1} = H_s(x_1)$. Assume that we have proven the volume bound (43) holds for a given s and all $1 - \lambda_* \leq t < 1$. We then have that

$$H_s(L_t) = L_{s,t} \subset \mathcal{N}(L_{s,1}, \epsilon'_s) \tag{44}$$

and thus there is a well-defined limit

$$\deg\left(\Phi_{s,1}\colon L_0 \to L_{s,1}\right) \equiv \lim_{t \to 1} \left\{\deg\left(\Pi \circ \Phi_{s,t}\colon L_0 \to \mathcal{N}(L_{s,1}, \epsilon'_s) \to L_{s,1}\right)\right\}$$

The terminology $\deg\left(\Phi_{s,1}\colon L_0 \to L_{s,1}\right)$ is a small abuse of notation, as given $y \in L_0$ there is no assurance that $t \mapsto \Phi_{s,t}(y)$ has a limit at $t = 1$; it is only given that the image is trapped in the open neighborhood $\mathcal{N}(L_{s,1}, \epsilon'_s)$. For $1 - \lambda_s \leq t < 1$, define

$$\Xi(s,t) = \frac{\deg\left(\Phi_{s,1}\colon L_0 \to L_{s,1}\right)}{\deg\left(\Phi_{s,t}\colon L_0 \to L_{s,t}\right)} \tag{45}$$

We now apply this discussion in the case $s = \xi_N$ where we have the volume bound (43). It follows from our choices that, for $1 - \lambda_* \leq t < 1$ and noting that $s_N = 1$,

$$H_{\xi_N}(L_t) = L_{\xi_N,t} \subset \mathcal{N}(L_{s_N,1}, \epsilon''_N) \cap \mathcal{N}(L_{s_{N-1},1}, \epsilon''_{N-1}) \tag{46}$$

Thus, for $1 - \lambda_* \leq \mu \leq t < 1$ the maps $\Pi \circ \Phi_{\xi_N,\mu} \sim \Pi \circ \Phi_{\xi_N,t}\colon L_0 \to \mathcal{N}(L_{1,1}, \epsilon''_N)$ are homotopic, hence

$$\deg(\Pi \circ \Phi_{\xi_N,\mu}) = \deg(\Pi \circ \Phi_{\xi_N,t}) \tag{47}$$

For t sufficiently close to 1 the map $\Pi \circ \Phi_{\xi_N,t}$ on the left-hand-side of (47) factors

$$\Pi \circ \iota \circ \Pi \circ \Phi_{\xi_N,t}\colon L_0 \to \mathcal{N}(L_{\xi_N,1}, \epsilon'_{\xi_{N-1}}) \to L_{\xi_N,1} \subset \mathcal{N}(L_{1,1}, \epsilon''_N) \to L_{1,1}$$

while the map $\Pi \circ \Phi_{\xi_N,\mu}$ on left-hand-side of (47) factors

$$\Pi \circ \Phi_{\xi_N,\mu}\colon L_0 \to L_{\xi_N,\mu} \to L_{1,1}$$

Identifying the degrees of these maps in our terminology, we obtain from (47) that

$$\deg(\Phi_{\xi_N,\mu}\colon L_0 \to L_{\xi_N,\mu}) \cdot \alpha_N^\mu = \deg(\Pi \circ \Phi_{\xi_N,\mu}) = \deg(\Pi \circ \Phi_{\xi_N,t}) = \deg(\Phi_{\xi_N,1}\colon L_0 \to L_{\xi_N,1}) \cdot a_N$$

and so

$$\alpha_N^\mu = \Xi(\xi_N, \mu) \cdot a_N \tag{48}$$

This completes the proof of the first stage of the induction.

The general inductive hypotheses involves two statements. Given $0 \leq n \leq N$, we first assume that:

$$\text{for all } \xi_n \leq s \leq 1 \ \& \ 1 - \lambda_* \leq t \leq 1, \ \ \mathrm{vol}\,(L_{s,t}) \leq 2\, d_0\, d_1 \cdot \mathfrak{P} \tag{49}$$

Given (49) is valid for n, then for $n \leq \ell \leq N$ and $1 - \lambda_* \leq \mu \leq 1$ define the integers $a_\ell, b_\ell, \alpha_\ell^\mu, \beta_\ell^\mu$.

$$\begin{array}{lclcccl}
L_{\xi_\ell,1} & \subset & \mathcal{N}(L_{s_\ell,1}, \epsilon''_\ell) & , & a_\ell & = & \deg\,(\Pi\colon L_{\xi_\ell,1} \to L_{s_\ell,1}) \\
L_{\xi_\ell,1} & \subset & \mathcal{N}(L_{s_{\ell-1},1}, \epsilon''_{\ell-1}) & , & b_\ell & = & \deg\big(\Pi\colon L_{\xi_\ell,1} \to L_{s_{\ell-1},1}\big) \\
L_{\xi_\ell,\mu} & \subset & \mathcal{N}(L_{s_\ell,1}, \epsilon''_\ell) & , & \alpha_\ell^\mu & = & \deg\,(\Pi\colon L_{\xi_\ell,\mu} \to L_{s_\ell,1}) \\
L_{\xi_\ell,\mu} & \subset & \mathcal{N}(L_{s_{\ell-1},1}, \epsilon''_{\ell-1}) & , & \beta_\ell^\mu & = & \deg\big(\Pi\colon L_{\xi_\ell,\mu} \to L_{s_{\ell-1},1}\big)
\end{array}$$

For notational convenience, set $b_{N+1} = \beta_{N+1}^\mu = 1$ and $a_0 = \alpha_0^\mu = 1$.

The second statement of the inductive hypotheses is that for all $n \leq \ell \leq N$ and $1 - \lambda_* \leq \mu \leq 1$,

$$\frac{\alpha_\ell^\mu}{a_\ell} = \Xi(\xi_\ell, \mu) = \frac{\beta_\ell^\mu}{b_\ell} \tag{50}$$

We proceed by downward induction on n, and must show that (49) and (50) are true for $n-1$.

The choice of $\lambda_s > 0$ so that (41) holds implies that

$$\phi([s_{n-1} - \lambda_{n-1}, s_{n-1} + \lambda_{n-1}] \times [1 - \lambda_*, 1]) \subset \mathcal{N}(L_{s_{n-1},1}, \delta''_{n-1})$$

and hence $\phi(s,t) \in \mathcal{N}(L_{s_{n-1},1}, \delta''_{n-1})$ for all $\xi_{n-1} \leq s \leq \xi_n$ and $1 - \lambda_* \leq t < 1$.

For $s = \xi_n$ the hypothesis (49) implies that for all $1 - \lambda_* \leq t < 1$,

$$\mathrm{vol}\,(L_{\xi_n,t}) \leq 2\, d_0\, d_1 \cdot \mathfrak{P} \ \text{ and hence } \ L_{\xi_n,t} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1}) \tag{51}$$

Thus, the restriction $\Pi\colon L_{\xi_n,t} \to L_{s_{n-1},1}$ is a covering map. The key to the proof of the inductive step is to obtain a uniform estimate for the homological degree of this covering map.

LEMMA 6.2 *For all* $1 - \lambda_* \leq t < 1$, $\beta_n^t \cdot \deg\,(H_{\xi_n}\colon L_{0,t} \to L_{\xi_n,t}) \ \leq \ d_0\, d_1$.

Proof: Consider the diagram

$$\begin{tikzcd}
L_{0,t} \arrow[rr, "H_{\xi_n}"] & & L_{\xi_n,t} \arrow[dl, "\beta_n^t"'] \arrow[d, dotted, "\Xi(n,t)"] \arrow[dr, "\alpha_n^t"] & & L_{\xi_{n+1},t} \arrow[dl, "\beta_{n+1}^t"'] \arrow[d, dotted, "\Xi(n+1,t)"] & \cdots & L_{\xi_N,t} \arrow[d, dotted, "\Xi(N,t)"] \arrow[dr, "\alpha_N^t"] & \\
L_{0,1} \arrow[r, "H_{s_{n-1}}"'] & L_{s_{n-1},1} & L_{\xi_n,1} \arrow[l, "b_n"] \arrow[r, "a_n"'] & L_{s_n,1} & L_{\xi_{n+1},1} \arrow[l, "b_{n+1}"] & \cdots & L_{\xi_N,1} \arrow[r, "a_N"'] & L_{1,1}
\end{tikzcd}$$

where the integer next to a covering map indicates its homological degree.

The maps $H_{\xi_n}\colon L_{0,1} \to L_{\xi_n,1}$ and $H_{s_{n-1}}\colon L_{0,1} \to L_{s_{n-1},1}$ are homotopic through maps into $\mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$, hence

$$\deg\big(H_{s_{n-1}}\colon L_{0,1} \to L_{s_{n-1},1}\big) = b_n \cdot \deg\,(H_{\xi_n}\colon L_{0,1} \to L_{\xi_n,1}) \tag{52}$$

As $\deg\big(H_{s_{n-1}}\colon L_{0,1} \to L_{s_{n-1},1}\big) = d_{s,1} \le d_1$ and the degrees of the maps are positive integers, it follows that $1 \le b_n \le d_1$.

For $n \le \ell < N$ and $1-\lambda_* \le t < 1$, the maps $H_{\xi_\ell}\colon L_{0,t} \to L_{\xi_\ell,t}$ and $H_{\xi_{\ell+1}}\colon L_{0,t} \to L_{\xi_{\ell+1},t}$ are homotopic through maps into $\mathcal{N}(L_{s_\ell,1}, \epsilon''_\ell)$, hence

$$\alpha^t_\ell \cdot \deg\left(H_{\xi_\ell}\colon L_{0,t} \to L_{\xi_\ell,t}\right) = \beta^t_{\ell+1} \cdot \deg\big(H_{\xi_{\ell+1}}\colon L_{0,t} \to L_{\xi_{\ell+1},t}\big) \tag{53}$$

Likewise, for $n \le \ell < N$, the maps $H_{\xi_\ell}\colon L_{0,1} \to L_{\xi_\ell,1}$ and $H_{\xi_{\ell+1}}\colon L_{0,1} \to L_{\xi_{\ell+1},1}$ are homotopic through maps into $\mathcal{N}(L_{s_\ell,1}, \epsilon''_\ell)$, hence

$$a_\ell \cdot \deg\left(H_{\xi_\ell}\colon L_{0,1} \to L_{\xi_\ell,1}\right) = b_{\ell+1} \cdot \deg\big(H_{\xi_{\ell+1}}\colon L_{0,1} \to L_{\xi_{\ell+1},1}\big) \tag{54}$$

It follows from equation (53) that

$$\begin{aligned}
\deg\left(H_1\colon L_{0,t} \to L_{1,t}\right) &= \frac{\alpha^t_N}{\beta^t_{N+1}} \cdot \deg\left(H_{\xi_N}\colon L_{0,t} \to L_{\xi_N,t}\right) \\
&= \frac{\alpha^t_{N-1}\alpha^t_N}{\beta^t_N\beta^t_{N+1}} \cdot \deg\big(H_{\xi_{N-1}}\colon L_{0,t} \to L_{\xi_{N-1},t}\big) \\
&\;\vdots \\
&= \frac{\alpha^t_n \cdots \alpha^t_{N-1}\alpha^t_N}{\beta^t_{n+1}\cdots\beta^t_N\beta^t_{N+1}} \cdot \deg\left(H_{\xi_n}\colon L_{0,t} \to L_{\xi_n,t}\right) \\
&= \frac{\alpha^t_n \cdots \alpha^t_N}{\beta^t_n\cdots\beta^t_N} \cdot \beta^t_n \cdot \deg\left(H_{\xi_n}\colon L_{0,t} \to L_{\xi_n,t}\right)
\end{aligned}$$

so that by the inductive hypothesis (50) we have

$$\begin{aligned}
\beta^t_n \cdot \deg\left(H_{\xi_n}\colon L_{0,t} \to L_{\xi_n,t}\right) &= \frac{\beta^t_n \cdots \beta^t_N}{\alpha^t_n \cdots \alpha^t_N} \cdot \deg\left(H_1\colon L_{0,t} \to L_{1,t}\right) &(55)\\
&= \frac{b_n \cdots b_N}{a_n \cdots a_N} \cdot \deg\left(H_1\colon L_{0,t} \to L_{1,t}\right) &(56)
\end{aligned}$$

Using (54) we obtain

$$\deg\left(H_1\colon L_{0,1} \to L_{1,1}\right) = \frac{a_n \cdots a_N}{b_n \cdots b_N} \cdot \deg\big(H_{s_{n-1}}\colon L_{0,1} \to L_{s_{n-1},1}\big) \tag{57}$$

so that

$$\frac{b_n \cdots b_N}{a_n \cdots a_N} = \frac{\deg\big(H_{s_{n-1}}\colon L_{0,1} \to L_{s_{n-1},1}\big)}{\deg\left(H_1\colon L_{0,1} \to L_{1,1}\right)} \le d_1 \tag{58}$$

and hence combining (39), (56) and (58) we obtain

$$\beta^t_n \cdot \deg\left(H_{\xi_n}\colon L_{0,t} \to L_{\xi_n,t}\right) \le d_1 \cdot \deg\left(H_1\colon L_{0,t} \to L_{1,t}\right) \le d_0\, d_1 \tag{59}$$

This completes the proof of Lemma 6.2. □

Fix $1-\lambda_* \le \mu < 1$. Let $r_{n-1} \le \xi_n$ be the infimum of r satisfying $r \le \xi_n$ such that $r \le s \le \xi_n$ implies that $L_{s,\mu} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$. As $L_{\xi_n,\mu} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$, the continuity of H_s at $s = \xi_n$ implies that $r_{n-1} < \xi_n$. We claim that $r_{n-1} < \xi_{n-1}$.

Assume, to the contrary, that $r_{n-1} \geq \xi_{n-1}$. Let $r_{n-1} < r < \xi_n$, then for $r \leq s \leq \xi_n$, $L_{s,\mu} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$ and so the normal projection $\Pi\colon L_{s,\mu} \to L_{s_{n-1},1}$ is well-defined and a covering map. The restriction

$$H\colon L_\mu \times [r, \xi_n] \to \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$$

yields a homotopy between $H_r\colon L_\mu \to L_{r,\mu}$ and $H_{\xi_n}\colon L_\mu \to L_{\xi_n,\mu}$. Thus,

$$\deg(\Pi \circ H_r\colon L_\mu \to L_{r,\mu} \to L_{s_{n-1},1}) = \deg(\Pi \circ H_{\xi_n}\colon L_\mu \to L_{\xi_n,\mu} \to L_{\xi_{n-1},1})$$

It follows from the estimate (59) that

$$\deg(\Pi\colon L_{r,\mu} \to L_{s_{n-1},1}) \leq \deg(\Pi \circ H_r\colon L_\mu \to L_{r,\mu} \to L_{s_{n-1},1}) \leq d_0\, d_1 \tag{60}$$

hence

$$\mathrm{vol}\,(L_{r,\mu}) \leq 2\, d_0\, d_1 \cdot \mathrm{vol}\,(L_{s_{n-1},1}) \leq 2\, d_0\, d_1 \cdot \mathfrak{P} \tag{61}$$

The leaf volume function is lower semi-continuous, hence we also have that

$$\mathrm{vol}\,(L_{r_{n-1},\mu}) \;\leq\; \lim_{r \to r_{n-1}+} \mathrm{vol}\,(L_{r,\mu}) \;\leq\; 2\, d_0\, d_1 \cdot \mathfrak{P} \tag{62}$$

Thus, the estimate (61) holds for all $r_{n-1} \leq r \leq 1$ and $1 - \lambda_* \leq \mu < 1$.

As we assumed that $r_{n-1} \geq \xi_{n-1} \geq s_{n-1} - \lambda_{n-1}$ we have that $\phi(r_{n-1}, \mu) \in \mathcal{N}(L_{s_{n-1},1}, \delta''_{n-1})$ hence $L_{r_{n-1},\mu} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$. By the continuity of H_s at $s = r_{n-1}$ there is $r < r_{n-1}$ such that $r < s \leq r_{n-1}$ implies $L_{s,\mu} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$. This contradicts the choice of r_{n-1} as the infimum of such r, hence we must have that $r_{n-1} < \xi_{n-1}$. This proves the first statement of the inductive hypothesis for $n - 1$.

The second inductive statement (50) follows exactly as before.

Thus, we conclude by downward induction that (42) holds for all $1 - \lambda_* \leq t < 1$ and all $0 \leq s \leq 1$.

This yields a contradiction, and completes the proof of Proposition 6.1. $\square$

References

[1] C. Camacho and A. Lins Neto, **Geometric Theory of Foliations**, Translated from the Portuguese by Sue E. Goodman, Progress in Mathematics, Birkhäuser Boston, MA, 1985.

[2] A. Candel and L. Conlon, **Foliations I**, Amer. Math. Soc., Providence, RI, 2000.

[3] H. Colman, *Categoría LS en foliaciones*, **Publicaciones del Departamento de Topología y Geometría**, no. 93, Universidade de Santiago de Compostele, 1998.

[4] H. Colman, *LS-categories for foliated manifolds*, In **Foliations: Geometry and Dynamics (Warsaw, 2000)** World Scientific Publishing Co. Inc., River Edge, N.J., 2002:17–28.

[5] H. Colman, *Equivariant LS-category for finite group actions*, In **Lusternik-Schnirelmann category and related topics (South Hadley, MA, 2001)**, Contemp. Math. Vol. 316, Amer. Math. Soc., Providence, R.I., 2002, 35–40.

[6] H. Colman, *Transverse Lusternik–Schnirelmann category of Riemannian foliations*, **Topology Appl.**, 141:187–196, 2004.

[7] H. Colman and S. Hurder, *LS-category of compact Hausdorff foliations*, **Trans. Amer. Math. Soc.**, 356:1463–1487, 2004.

[8] H. Colman and E. Macias, *Transverse Lusternik–Schnirelmann category of foliated manifolds*, **Topology** Vol. **40** (2) (2000), 419-430.

[9] M. do Carmo, **Riemannian Geometry**, 2^{nd} edition, Birkhäuser, Boston, 1992.

[10] R. Edwards, K. Millett, and D. Sullivan, *Foliations with all leaves compact*, **Topology**, 16:13–32, 1977.

[11] D. B. A. Epstein, *Periodic flows on 3-manifolds*, **Annals of Math.**, 95:68–82, 1972.

[12] D. B. A. Epstein, *Foliations with all leaves compact*, **Ann. Inst. Fourier (Grenoble).**, 26:265–282, 1976.

[13] D. B. A. Epstein and E. Vogt, *A counterexample to the periodic orbit conjecture in codimension* 3, **Ann. of Math. (2)**, 100:539–552, 1978.

[14] C. Godbillon, **Feuilletages: Etudes géométriques I, II**, Publ. IRMA Strasbourg (1985-86), Progress in Math., Vol. 98, Birkhäuser, Boston, Mass., 1991.

[15] A. Haefliger. *Some remarks on foliations with minimal leaves*, **J. Differential Geom.**, 15:269–284, 1981.

[16] G. Hector and U. Hirsch, **Introduction to the Geometry of Foliations, Parts A,B**, Vieweg, Braunschweig, 1981.

[17] H. Holmann, *Seifertsche Faserräume*, **Math. Ann.**, 157:138–166, 1964.

[18] S. Hurder, *Category and compact leaves*, **preprint**.

[19] I.M. James, *On category, in the sense of Lusternik-Schnirelmann*, **Topology** 17:331–348, 1978.

[20] I.M. James, *Lusternik-Schnirelmann Category*, **Chapter 27, Handbook of Algebraic Topology**,1995,1293–1310.

[21] R. Langevin and P. Walczak, *Transverse Lusternik-Schnirelmann category and non-proper leaves*, In **Foliations: Geometry and Dynamics (Warsaw, 2000)**, World Scientific Publishing Co. Inc., River Edge, N.J., 2002:351–354.

[22] K. Millett, *Compact foliations*, In **Differential topology and geometry (Proc. Colloq., Dijon, 1974)**, Lect. Notes in Math. Vol. 484, 277–287,1975. Springer–Verlag, New York and Berlin.

[23] K. Millett, *Generic properties of proper foliations*, **Fund. Math.** 128:131–138, 1987.

[24] D. Montgomery, *Pointwise periodic homeomorphisms*, **Amer. Journal Math.**, 59:118–120, 1937.

[25] M.H.A. Newman, A theorem on periodic transformations of spaces, **Quart. Journal Math.** 2:1–9, 1931.

[26] J. Plante, *Foliations with measure preserving holonomy*, **Annals of Mathematics**, 102:327–361, 1975.

[27] G. Reeb, *Sur certaines propiétés topologiques des variétés feuilletés*, **Act. Sci. Ind.** 1183:91–154, 1952.

[28] H. Rummler, *Quelques notions simples en géométrie riemannienne et leurs applications aux feuilletages compacts*, **Comment. Math. Helv.** 54:224–239, 1979.

[29] I. Satake, *On a generalization of the notion of manifold*, **Proc. Nat. Acad. Sci. U.S.A.** , 42:359–363, 1956.

[30] D. Sullivan, *A counterexample to the periodic orbit conjecture*, **Publ. Math. Inst. Hautes Etudes Sci.**, 46:5–14, 1976.

[31] D. Sullivan, *A homological characterization of foliations consisting of minimal surfaces*, **Comment. Math. Helv.**, 54:218–223, 1979.

[32] I. Tamura, **Topology of Foliations: An Introduction**, 1976. Translations of Mathematical Monographs 97, Amer. Math. Soc., Providence, RI, 1992.

[33] E. Vogt, *Foliations of codimension 2 with all leaves compact*, **Manuscripta Math.**, 18:187–212, 1976.

[34] E. Vogt, *Foliations of codimension 2 on closed 3, 4 and 5-manifolds*, **Math. Zeit.**, 157:201–223, 1977.

[35] E. Vogt, *A periodic flow with infinite Epstein hierarchy*, **Manuscripta Math.**, 22:403–412, 1977.

Steven Hurder
Department of Mathematics (m/c 249)
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL 60607-7045 USA
Email: hurder@uic.edu
Web: http://www.math.uic.edu/∼hurder/

Paweł G. Walczak
Wydział Matematyki
Uniwersytet Łódzki
Banacha 22, 90-238
Łódź, Poland
Email: pawelwal@math.uni.lodz.pl
Web: http://www.math.uni.lodz.pl/∼pawelwal/