THE EQUIVARIANT LS-CATEGORY OF POLAR ACTIONS

STEVEN HURDER AND DIRK TÖBEN

ABSTRACT. We will provide a lower bound for arbitrary proper actions in terms of the stratification by orbit types, and an upper bound for proper polar actions in terms of the equivariant LS-category of its generalized Weyl group. As an application we reprove a theorem of Singhof that determines the classical Lusternik-Schnirelmann category for $U(n)$ and $SU(n)$.

1. INTRODUCTION

The equivariant Lusternik-Schnirelmann category $\mathrm{cat}_G(M)$ of an action by a Lie group G on a manifold M (see Definition 2.1) was introduced by Marzantowicz in [Ma] for compact G, as a generalization the classical Lusternik-Schnirelmann category of a space [Ja, LS]. Theorem 3.2 of [Ma] shows that similar to the classical case, $\mathrm{cat}_G(M)$ is a lower bound for the number of critical orbits of a G-invariant C^1-function on M, and Theorem 1.13 proves that $\mathrm{cat}_G(M)$ is bounded below by the cuplength of a G-cohomology theory.

Colman studied the equivariant category $\mathrm{cat}_G(M)$ for G a finite group in [Co], and gave an upper bound in terms of the category of the connected components of the singular set for the action. Moreover, her work gives examples of finite group actions on compact surfaces for which $\mathrm{cat}_G(M)$ can be made arbitrarily large [Co], showing the necessity of working with the connected components. Note that for finite group actions, the singular set consists entirely of exceptional points.

Ayala, Lasheras and Quintero [ALQ] generalize the Marzantowicz results to proper group actions, although finite group actions were still their primary consideration.

In this paper, we will focus on the equivariant category of proper actions by higher dimensional Lie groups. In sections 2 and 3 we will introduce a refinement of the stratification by orbit types and provide a lower bound for arbitrary proper actions in terms of its bottom stratum. Section 4 defines the class of polar actions from the title, and section 5 introduces the Weyl group associated to a polar action. Section 6 contains our main result; for proper polar actions we will give an upper bound in terms of the equivariant category of its generalized Weyl group of a polar action, thereby reducing the computation to the discrete case. In section 7 we will use the previous results to calculate the equivariant category of $SU(n)$ and $U(n)$. We then

Date: February 20, 2007; May 1, 2018, corrected version of Theorem 2.6.
Key words and phrases. Lusternik-Schnirelmann category, equivariant category, polar actions, finite group actions, foliations.
SH was supported in part by NSF grant DMS-0406254.
DT was supported by the Schwerpunktprogramm SPP 1154 of the DFG.

observe that this also determines the classical Lusternik-Schnirelmann category of these spaces, which is a theorem by Singhof.

This work was begun while the second author was visiting the University of Illinois at Chicago for the academic year 2005-2006, and completed while the first author was a guest at the Institut Henri Poincaré in Paris. The authors would like to express their thanks for the hospitality shown by both host institutions.

2. Categorical bounds for proper actions

Let G be a topological group acting on a topological space M; in most of our cases this will be a Lie group acting smoothly on a manifold. A homotopy $H\colon U \times [0,1] \to M$ of an open G-invariant set $U \subset M$ is called *G-equivariant*, or just a *G-homotopy*, if $gH(x,t) = H(gx,t)$ for any $g \in G, x \in U$ and $t \in [0,1]$. We also write $x_t = H_t(x) = H(x,t)$. The set U is *G-categorical*, if there is a G-homotopy H with H_0 the identity, and H_1 maps U to a single orbit.

Definition 2.1. The *equivariant category* $\mathrm{cat}_G(M)$ is the least number of G-categorical open sets required to cover M. If there is no categorical cover of M, we set $\mathrm{cat}_G(M) = \infty$. If M is empty, set $\mathrm{cat}_G(M) = 0$.

Another common definition of LS category is the minimal number of open sets minus 1. The category of a point is 1 in our definition, while it is 0 is the other.

Remark 2.2. For the trivial group $G = \{e\}$ we recover the classical Lusternik-Schnirelmann category, denoted by $\mathrm{cat}(M)$.

Recall that an action is *proper* if for each compact subset $K \subset M$, $\{(g,x) \mid gx \in K\}$ is a compact subset of $G \times M$. For a proper action, the orbits Gx are closed and embedded submanifolds, hence the quotient M/G is a Hausdorff space. Then an immediate lower bound for $\mathrm{cat}_G(M)$ is $\mathrm{cat}(M/G) \leq \mathrm{cat}_G(M)$.

An important tool for studying the equivariant category is provided by the following well-known theorem [Bre, DuKa, Pa].

Theorem 2.3 (Tubular Neighborhood Theorem)**.** *Let G act properly on M. Then for any orbit Gx there is an invariant neighborhood U and a G-equivariant homotopy $H : U \times [0,1] \to M$ with H_0 the inclusion, $H_1(U) = Gx$, and $H_t(gx) = gx$ for all $t \in [0,1]$ and $g \in G$.*

We will give an outline of the proof in the case that the action is smooth, thereby introducing principles that will be useful later.

Proof. Choose a Riemannian metric on M for which the G-action is isometric. For $x \in M$, the orbit $Gx \subset M$ is a properly embedded smooth submanifold. Let $\nu(Gx) \to Gx$ be the normal bundle to Gx of M, and let $\nu^r(Gx) \subset \nu(Gx)$ denote the disk subbundle of vectors of length at most r. Then there exists $r > 0$ such that the geodesic exponential map $\exp : \nu^r(Gx) \to M$ is a diffeomorphism onto a tubular neighborhood U of the orbit Gx. Define the geodesic retraction $H : U \times [0,1] \to M$ onto Gx by $H_t(\exp(\vec{v})) = H(\exp(\vec{v}), t) = \exp(t\vec{v})$. It is easy to see that the homotopy H is G-equivariant. ◻

Now define the slice $S = H_1^{-1}(x)$, where $H : U \times [0,1] \to M$ is as in the proof. Let $G_x = \{g \in G \mid gx = x\}$ denote the stabilizer of x. Then there is a G-equivariant diffeomorphism

$$U \cong G \times_{G_x} S.$$

Example 2.4. Let G be a Lie group acting properly on an Hadamard manifold X, and assume the action is polar (see Definition 4.1). Let K be a maximal compact subgroup of G. One can show $K = G_x$ for some $x \in X$. In [Tö3] the second author proved that the normal exponential map $\exp^\perp : \nu(Gx) \to X$ of Gx is a G-equivariant diffeomorphism. In other words, X is a global tubular neighborhood of Gx. Therefore $\mathrm{cat}_G(X) = 1$.

The Tubular Neighborhood Theorem directly implies:

Corollary 2.5. *Let G be a compact Lie group acting on a compact manifold M. Then $\mathrm{cat}_G(M) < \infty$.*

The Tubular Neighborhood Theorem is generalized by the following well-known theorem.

Theorem 2.6 (Generalized Tubular Neighborhood Theorem)**.** *Let G act properly on M. Then any invariant submanifold A has an invariant neighborhood U for which A is a strong G-deformation retract, i.e. there is G-homotopy $H : U \times [0,1] \to M$ with $H_1(U) = A$ and $H_t|A$ is the inclusion of A into M for all $t \in [0,1]$.*

From now on we assume that G is a Lie group acting properly, isometrically on a complete Riemannian manifold M without boundary, such that M/G is connected.

An orbit Gx of maximal dimension is called *regular*, and each point $y \in Gx$ is said to be regular. Let r be the dimension of a regular orbit. The *cohomogeneity of the action* is defined to be the codimension q of a regular orbit. An orbit with dimension less than r is said to be *singular*. The set of regular (respectively, singular) points is denoted by R (respectively, S). The union of the regular orbits R forms an open dense connected subset of M.

A regular orbit Gx is said to have *non-trivial holonomy* if there exists $y \in Gx$ arbitrarily close to x such that the orbit Gy is a non-trivial covering of Gx; such an orbit is said to be *exceptional*. Let E denote the union of the exceptional orbits. The set $R_0 = R - E$ consists of regular orbits without holonomy, and equals the set of principle orbits as defined in section 3. The quotient space $B := R_0/G$ is a connected manifold of dimension q, and the quotient map $\rho\colon R_0 \to B$ is then a right G-fiber bundle.

Theorem 2.7. *Let $G \times M \to M$ be a proper smooth action of a Lie group G on a connected, complete Riemannian manifold M without boundary. If either S or E is non-empty, then*

$$\mathrm{cat}_G(M) \le \mathrm{cat}_G(S) + \mathrm{cat}_G(R) \le \mathrm{cat}_G(S) + \mathrm{cat}_G(E) + q \tag{1}$$

Otherwise, if all orbits are regular and there are no exceptional orbits, then

$$\mathrm{cat}_G(M) \le q + 1 \tag{2}$$

Proof. Let us first show that

$$\mathrm{cat}_G(M) \le \mathrm{cat}_G(S) + \mathrm{cat}_G(R) \tag{3}$$

Assume that S is not empty. Then S is a locally finite union of submanifolds. Following [CoHu] S is a strong deformation retraction of an invariant neighborhood U. We can therefore extend a G-categorical cover for S to a G-categorical cover of U with the same cardinality. Together with an equivariant cover for R, we obtain a G-categorical cover for M. This proves (3).

Assume that E is not empty. We use ideas from [CoHu, CoMa, HuTö] to show

$$\mathrm{cat}_G(R) \leq \mathrm{cat}_G(E) + q \tag{4}$$

Endow M with a G-invariant Riemannian metric, then the projection ρ becomes a Riemannian submersion for an appropriate metric on the open manifold B. Let $\mathcal{H} \subset TR_0$ denote the orthogonal bundle to the orbits of G, so that $\mathcal{H}$ is G-invariant, and is the horizontal distribution for ρ in the sense of Riemannian submersions.

Given an open set $U \subset B$, set $\widehat{U} = \rho^{-1}(U)$. Given a C^1-contraction $h : U \times I \to B$ to a point $b_0 \in B$, we define a G-equivariant lift $H : \widehat{U} \times I \to R_0$ by requiring that for $x \in \widehat{U}$,

$$H_0(x) = x \ , \quad d\rho\left(\frac{d}{dt}H_t(x)\right) = \frac{d}{dt}h_t(\rho(x)) \ , \quad \frac{d}{dt}H_t(x) \in \mathcal{H} \tag{5}$$

The differential conditions (5) mean that $H_t(x)$ is the horizontal curve over $h_t(\rho(x))$, and the G-invariance of $\mathcal{H}$ implies that H_t is a G-equivariant map for all t. As H_1 maps $\widehat{U}$ into the G orbit over b_0, $\widehat{U}$ is a G-categorical set in R_0.

Now note that B is a connected open manifold of dimension q, so there exists a categorical covering $\{U_1, \ldots, U_k\}$ with smooth homotopies, for some $k \leq q$. Their inverse images $\{\widehat{U}_1, \ldots, \widehat{U}_k\}$ form a G-categorical covering for R_0.

It remains to note that by the technique from [CoHu] already used above, we can extend a G-categorical cover for E to a G-categorical cover of some invariant neighborhood U of E with the same cardinality.

Finally, in the case where $M = R$ and there are no exceptional orbits, then M/G is a manifold. If M/G is compact, then it admits a categorical covering with at most $q + 1$ open sets. In the case where M/G is non-compact, then it admits a categorical covering with at most q open sets, so that (2) can be sharpened. □

3. Lower bound estimates

Our next aim is to give two lower bounds for $\mathrm{cat}_G(M)$ which are fundamental in applications. Note that each orbit Gx is a G-subspace.

Definition 3.1. A *G-path* from an orbit Gx to an orbit Gy is a G-equivariant map map $I : Gx \times [0, 1] \to M$ such that

(1) I_0 is the inclusion of Gx in M,
(2) $I_1(Gx) = Gy$.

If, in addition, every I_t is a diffeomorphism then we call I a *G-isotopy.*

Recall that given a G-invariant subset $X \subset M$, a G-homotopy is a continuous family of G-maps $H_t : X \times [0, 1] \to M$. For each $x \in X$ we then obtain a G-path $H_t : Gx \times [0, 1] \to M$ from Gx to Gy where $y = H_1(x)$.

We will now recall some well-known algebraic properties of proper actions [DuKa]. Given a closed subgroup $H \subset G$, we denote by (H) the conjugacy class of H in G. While for an orbit Gx the isotropy group G_y depends on the choice of $y \in Gx$, the conjugacy class (G_x) does not and is therefore an invariant of Gx. The class (G_x) is called the *orbit type* of Gx. There is a partial order on the set of orbit types of the G-space M: given isotropy groups $H, K \subset G$,

$$(H) \leq (K) \quad \text{if } gKg^{-1} \subset H \text{ for some } g \in G. \tag{6}$$

An orbit Gx and its orbit type (G_x) are called *principal* if Gx has a G-invariant open neighborhood that contains no orbit of greater orbit type. The union of principal orbit types is open and dense in M. If M is connected, then the space of principal orbits is connected, and hence there is exactly one principal orbit type; this orbit type is comparable to any other orbit type and it is the maximum with respect to the partial order in (6). The orbit Gx and its orbit type (G_x) are called *minimal* if (G_x) is a minimum with respect to the partial order in (6).

Lemma 3.2. *Let $I : Gx \times [0,1] \to M$ be a G-path in M, and write $x_t = I_t(x)$. Then for all $0 \leq t \leq 1$,*

$$G_x \subset G_{x_t} \quad \textit{and therefore} \quad (G_{x_t}) \leq (G_x), \ \textit{hence} \ \dim Gx_t \leq \dim Gx. \tag{7}$$

Proof. For $g \in G_x$ we have $gI(t,x) = I(t,gx) = I(t,x)$. □

The second property of (7) means that I_t respects the partial order of orbit types. It follows that a minimal orbit type is preserved under a G-path, i.e. $(G_{x_t}) = (G_x)$. We will generalize this principle in the next paragraphs.

For an isotropy group H, define the (H)-*orbit type submanifold*

$$M_{(H)} = \{x \in M \mid G_x \in (H)\} \tag{8}$$

which is the union of orbits of the same orbit type (H). One knows that each connected component of $M_{(H)}$ is a submanifold, possibly open, and the quotient map $M_{(H)} \to M_{(H)}/G$ is a fiber bundle when restricted to connected component of $M_{(H)}$. Also, define the invariant set

$$M_{\leq(H)} = \{x \in M \mid (G_x) \leq (H)\} = G \cdot \mathrm{Fix}(H) \tag{9}$$

which is closed by the Tubular Neighborhood Theorem, but in general need not be a submanifold.

Let $H \subset G$ be an isotropy subgroup, and suppose that $Gx \subset M_{\leq(H)}$ (respectively, $M_{\leq(H)} \cap U \neq \emptyset$). Then equation (7) implies

$$I_t(Gx) \subset M_{\leq(H)} \quad \text{respectively,} \quad H_t(M_{\leq(H)} \cap U) \subset M_{\leq(H)}. \tag{10}$$

Hence

$$\mathrm{cat}_G(M_{\leq(G_x)}) \leq \mathrm{cat}_G(M). \tag{11}$$

This proves again that each minimal orbit type is preserved under G-homotopy.

Note that a G-homotopy H_t preserves the connected components of $M_{\leq(H)}$. This motivates the introduction of the following: for $x \in M$ we define $\mathcal{M}_x = G \cdot (M_{(G_x)})_x$, the G-orbit of the connected component of $M_{(G_x)}$ containing x. Let us call a union of connected components of an invariant set of M a G-*component* if it is itself invariant. Then $G \cdot (M_{(G_x)})_x$ is the smallest G-component of $M_{(G_x)}$ containing x.

It is not difficult to show that $\mathcal{M}_x$ is the union of orbits that can be reached from Gx by a G-isotopy. The decomposition $\mathfrak{M}'$ of M into connected components of orbit type submanifolds $M_{(H)}$ is a Whitney stratification (see e.g. [DuKa]). The decomposition

$$\mathfrak{M} = \{\mathcal{M}_x \mid x \in M\}.$$

is a coarser Whitney stratification; the G-orbit of one element in $\mathfrak{M}'$ constitutes one element in $\mathfrak{M}$. Both stratifications induce the same Whitney stratification of M/G. The *incidence relations* on $\mathfrak{M}$ are defined by the partial order

$$\mathcal{M}_y \preceq \mathcal{M}_x :\Longleftrightarrow \mathcal{M}_y \subset \overline{\mathcal{M}}_x \tag{12}$$

which is related to the already introduced partial order on $\mathfrak{M}'$ by

$$\mathcal{M}_y \preceq \mathcal{M}_x \Longrightarrow (G_y) \leq (G_x), \text{i.e. } M_{(G_y)} \subset M_{\leq (G_x)}.$$

The following is a property of a stratification.

Lemma 3.3. $\mathcal{M}_y \prec \mathcal{M}_x \iff y \in \overline{\mathcal{M}}_x \backslash \mathcal{M}_x \iff \mathcal{M}_y \subset \overline{\mathcal{M}}_x \backslash \mathcal{M}_x.$

Here $\mathcal{M}_y \prec \mathcal{M}_x$ means $\mathcal{M}_y \preceq \mathcal{M}_x$, but $\mathcal{M}_y \neq \mathcal{M}_x$. The lemma shows that $\mathcal{M}_x \preceq \mathcal{M}_y$ and $\mathcal{M}_y \preceq \mathcal{M}_x$ implies $\mathcal{M}_x = \mathcal{M}_y$.

Definition 3.4. An orbit Gx is *locally minimal* if $\overline{\mathcal{M}}_x$ has a G-invariant open neighborhood U that contains no smaller orbit type. In this case $\mathcal{M}_x$ is called a *locally minimal stratum.*

The above notion will allow us to give a lower bound for $\mathrm{cat}_G(M)$ (see Theorem 3.7) and is illustrated by the examples in section 7. It is also surprisingly connected to the question whether the transverse saturated LS-category of a Riemannian foliation is finite or not (see [HuTö], which also gives a detailed discussion of the properties of locally minimal strata in section 6.) Obviously, a minimal orbit with respect to the orbit type relations is locally minimal. From Lemma 3.3 we derive the following characterization:

Proposition 3.5. *A stratum $\mathcal{M}_x$ is locally minimal if and only if it is minimal with respect to the incidence partial order, if and only if $\mathcal{M}_x$ is closed.*

Proof. By Lemma 3.3, $\mathcal{M}_x$ is closed if and only if it is minimal with respect to the incidence partial order. If $\mathcal{M}_x$ is closed then it is a submanifold and therefore locally minimal by the Generalized Tubular Neighborhood Theorem. Now assume it is locally minimal. Let $y \in \overline{\mathcal{M}}_x$. By the Tubular Neighborhood Theorem we have $(G_y) \leq (G_x)$ and, since $\mathcal{M}_x$ is locally minimal, $(G_y) = (G_x)$. Thus $y \in \overline{\mathcal{M}}_x$, so $\mathcal{M}_x$ is closed. □

Remark 3.6. For all $x \in M$, $\overline{\mathcal{M}}_x$ always contains a locally minimal stratum.

Let $\mathfrak{M}_0$ be the set of locally minimal strata. Let $\mathcal{M}_x \in \mathfrak{M}_0$ and V a invariant neighborhood in which there is no smaller orbit type than (G_x). Then $\mathcal{M}_x = M_{\leq (G_x)} \cap V$. For a G-homotopy $H : U \times [0,1] \to M$, by equation (10) we have

$$H_t(\mathcal{M}_x \cap U) \subset \mathcal{M}_x, \quad \text{hence} \quad \mathrm{cat}_G(\mathcal{M}_x) \leq \mathrm{cat}_G(M). \tag{13}$$

Theorem 3.7. *Let G be a Lie group acting properly on M. Then*

$$|\mathfrak{M}_0| \leq \sum_{\mathcal{M}_x \in \mathfrak{M}_0} \mathrm{cat}_G(\mathcal{M}_x) \leq \mathrm{cat}_G(M).$$

Proof. We show the second inequality; then the first follows directly. Let $\{U_j\}_{j\in J}$ be a covering by G-categorical sets with corresponding G-equivariant homotopies $H^j : U_j \times [0,1] \to M$. Let $y_j \in M$ be such that $H^j_1(U_j) = Gy_j$. For each $\mathcal{M}_x \in \mathfrak{M}_0$ let $J(\mathcal{M}_x) \subset J$ denote the subset of indices for which $U_j \cap \mathcal{M}_x$ is non-empty. By equation (13) the restriction of the corresponding homotopies to $\mathcal{M}_x$ gives a G-categorical covering of $\mathcal{M}_x$; in particular $\mathrm{cat}_G(\mathcal{M}_x) \leq |J(\mathcal{M}_x)|$. On the other hand, the image $H^j_1(U_j \cap \mathcal{M}_x)$ is contained in a single orbit Gy_j which must lie in $U_j \cap \mathcal{M}_x$ by (13). Hence, each open set U_j intersects at most one element of $\mathfrak{M}_0$, so the $J(\mathcal{M}_x), \mathcal{M}_x \in \mathfrak{M}_0$ are disjoint. This proves the statement. □

Remark 3.8. Marzantowicz gives in [Ma] an upper bound for $\mathrm{cat}_G(M)$ in terms of minimal orbit types. See Colman [Co] for a refinement in case of finite G.

Each component of the fixed point set of an action is a locally minimal set (each point of it is an orbit with minimal orbit type (G).) Thus we have:

Corollary 3.9. *Let G be a Lie group acting properly on M. The number of components of the fixed point set is a lower bound for* $\mathrm{cat}_G(M)$.

This corollary also justifies the refinement of orbit types by separating its orbit type submanifolds into its basic components, the G-components. Counting the number of minimal orbit type submanifolds would not provide a good lower bound, as the entire fixed point set is the sole orbit type submanifold $M_{(G)}$.

Example 3.10. Let us prove that

$$n + 1 \leq \mathrm{cat}_{SU(n+1)} SU(n + 1) \tag{14}$$

for the equivariant category of the action of $G = SU(n)$ on itself by conjugation. The center $Z(G)$ of a Lie group G is the fixed point set $\mathrm{Fix}_G(G)$ of its action on itself by conjugation. In this case we have $Z(SU(n+1)) = \{e^{\frac{2\pi i k}{n+1}} I_{n+1} \mid k = 0, \ldots, n\}$. By Corollary 3.9 we have $n + 1 \leq \mathrm{cat}_{SU(n+1)} SU(n + 1)$. This lower bound is optimal as we will see in Example 7.1. For the conjugation action of other Lie groups, say $SO(2n)$ for example, this estimate is not optimal and can be improved by the estimate in Theorem 3.7.

LS category theory has a close relationship with critical point theory for functions, and the next concept develops this parallel for G-category.

Definition 3.11. A *hierarchy* of (M, G) is a monotone function $f : \mathfrak{M}' \to \mathbb{N}_0$ with respect to the ordering (6). That is, $(G_x) \leq (G_y)$ implies $f(M_{(G_x)}) \leq f(M_{(G_y)})$.

A hierarchy $f : \mathfrak{M}' \to \mathbb{N}_0$ has a natural extension to a map $\bar{f} : M \to \mathbb{N}_0$ defined by $\bar{f}(x) := f(M_{(G_x)})$. Then f defines a partition $\mathfrak{M}'_f$ of M by

$$\mathfrak{M}'_f = f^{-1}(\mathbb{N}_0) = \{f^{-1}(n) \mid n \in \mathbb{N}_0\} \quad \text{and set} \quad \mathfrak{M}'_{f,n} := \bigcup_{i=0}^{n} f^{-1}(i) \tag{15}$$

We can think of a hierarchy f as a function on the directed graph of orbit types of (M, G) (see section 2.8 of [DuKa]) that respects the partial order between the vertices. Note that there is a directed vertex from any orbit type to a principal orbit type.

Example 3.12. We consider a few examples to illustrate hierarchy functions.

(1) The most common example is given by $f(M_{(G_x)}) := \dim Gx$. This function is well defined by Lemma 3.3, and $\mathfrak{M}'_f$ is the stratification of M by orbit dimension. Let $S^k = \mathfrak{M}'_{f,k}$.

(2) We can refine the last example. The action of G on $S_k := S^k \backslash S^{k-1}$ gives a foliation by orbits of dimension k. For $x \in S_k$ its orbit $Gx \subset S_k$ may have normal holonomy, as a leaf of the induced foliation. This holonomy group must be finite. Moreover, if G is compact then there are at most finitely many orbits with holonomy, and we can define n_k as the maximal cardinality of the holonomy groups of leaves in S_k. For G non-compact, we assume that each n_k is finite. Let $S_{k,i}$ for $i = 1, \ldots, n_k$ be the union of orbits with holonomy group of cardinality i. Now define

$$\bar{f}(x) = i - 1 + \sum_{j=1}^{k} n_j \quad \text{if} \quad x \in S_{k,i} \tag{16}$$

This map only depends on the orbit type and therefore defines a hierarchy, with S'_f being the refined stratification of M. This refines the stratification of M by dimension, to a stratification by dimension and holonomy cardinality. The stratification S'_f has been introduced in the study of the singular Riemannian foliations defined by the leaf closures in Riemannian foliations (see e.g. [Ha1, Ha2, Mo].)

(3) The following hierarchy is defined if there is only one principal orbit type (H) (e.g. if M/G is connected) and there are only a finite number of orbit types (e.g. if G is compact). Let (G_x) be an arbitrary orbit type. A chain

$$\mathcal{C}: \quad (G_x) = (H_1) \leq \cdots \leq (H_n) = (H)$$

of orbit types from (G_x) to (H) is said to have length n. Let $l(G_x)$ be the maximal length of chains from (G_x) to (H) and $L = \max\{l(G_x) \mid x \in M\}$. We define $\bar{f}(x) := L - l(G_x)$.

(4) Alternatively we can define $\bar{f}(x)$ as the maximal length of chains from some minimal orbit type to (G_x). This yields the stratification by holonomy groups used in the paper [CoHu] on the study of transverse LS-category for compact Hausdorff foliations.

A hierarchy function provides a lower bound estimate on the G-category:

Proposition 3.13. *Let G be a Lie group acting properly on M. Let $f : \mathfrak{M}' \to \mathbb{N}_0$ be a hierarchy function, with induced map $\bar{f} : M \to \mathbb{N}_0$. Then $\bar{f}$ is lower semicontinuous; that is,*

$$\liminf_{y \to x} \bar{f}(y) \geq \bar{f}(x).$$

Hence the strata $\mathfrak{M}'_{f,n}$ are preserved under G-homotopy, and thus

$$\operatorname{cat}_G(\mathfrak{M}'_{f,n}) \leq \operatorname{cat}_G(\mathfrak{M}'_{f,n+1}) \leq \operatorname{cat}_G(M) \tag{17}$$

Proof. Let x_n be a sequence in M converging to x. By the Tubular Neighborhood Theorem we have $(G_x) \leq (G_{x_n})$ for large n. Since f is monotone, $\bar{f}(x) \leq \bar{f}(x_n)$

for large n. Thus $\bar{f}$ is lower semicontinuous. We now show that $\mathfrak{M}'_{f,n}$ is preserved under G-homotopy. Let $H : U \times [0,1] \to M$ be a G-homotopy with $x \in U$. Then $(G_{x_t}) \leq (G_x)$, so $\bar{f}(x_t) \leq \bar{f}(x)$, i.e. $x_t \in \mathfrak{M}'_{f,f(x)}$. □

We apply this result to the hierarchy f of example (3.12.2) to obtain:

Corollary 3.14. *Let G be a Lie group acting properly on M. Then we have*

$$\operatorname{cat}_G(S^0) \leq \operatorname{cat}_G(S_{k,i}) \leq \operatorname{cat}_G(S_{l,j}) \leq \operatorname{cat}_G(S^l) \leq \operatorname{cat}_G(S) \leq \operatorname{cat}_G(M), \tag{18}$$

for $k \leq l$ and if $k = l$ for $i \geq j$. Here, S denotes the singular stratum.

4. Polar actions

In this section we will review the definition and properties of a polar action. In section 6, we will give an upper bound for the category of a polar action in terms of the action of its Weyl group.

Definition 4.1. Let G Lie group acting smoothly by isometries on a complete Riemannian manifold M. A *section* for the G-action is an isometrically immersed complete submanifold $i : \Sigma \to M$ which meets every orbit and always orthogonally. Then the dimension of Σ is equal to the cohomogeneity of the action, which was denoted by q. Note that for any $g \in G$, the map $g \circ i \colon g\Sigma \to M$ is again a section. A *polar action* is a G-action with a section. If Σ is a flat submanifold, then the action is called *hyperpolar*.

Remark 4.2. The set of regular points in a section is open and dense in it. A section is always a totally geodesic submanifold (see [Sz]).

Remark 4.3. The immersion i might not be injective. If injectivity fails in a regular point of the action, we can write $i = j \circ \rho$ where $\rho : \Sigma \to \Sigma'$ is a covering map and $j : \Sigma' \to M$ is a section that is injective in regular points. We will always assume that i is reduced in this sense. On the other hand, injectivity can still fail at singular points of the action.

The geometry of polar actions has been extensively studied [BeTa, Da, Ko1, Ko2, Kos, PaTe, PoTh, Sz, Th1, Th2]. Let us consider a few examples.

Example 4.4. The following examples are all hyperpolar actions.

(1) Isometric cohomogeneity one actions. The sections are the normal geodesics of a regular orbit. These have been classified in special cases, although remains an open problem to classify all such actions [Ko1, BeTa].

(2) A compact Lie group G with bi-invariant metric acting on itself by conjugation. The maximal tori are the sections.

(3) Let N be a symmetric space. The identity component of the isometry group, $G = I(N)_0$, acts transitively on N. We can write $N = G/K$, where $K = G_p$ for some point $p \in N$, and (G, K) is called a *symmetric pair*. Then the isotropy action

$$K \times G/K \to G/K \ ; \quad (k, gK) \mapsto kgK$$

and its linearization $K \times (T_{[K]}G/K) \to T_{[K]}G/K$ at the the tangent space to the point [K], are hyperpolar. The sections are the maximal flat submanifolds through $[K]$, and their tangent spaces in $[K]$, respectively. These are called *s-representations.*

(4) Let (G, K_1) and (G, K_2) be two symmetric spaces of the above form. Then the left action of K_1 on G/K_2 and its linearization are hyperpolar. These actions are called *Hermann actions.* They generalize examples (2) and (3).

Remark 4.5. Dadok has classified all linear representations that are polar in [Da]: they are orbit equivalent to the linearized actions of example (3), the s-representations. Kollross has classified in [Ko1] all hyperpolar actions on irreducible, simply-connected symmetric spaces of compact type: they are of type (1) and of type (4). Polar actions that are not hyperpolar on symmetric spaces of compact type have been found only on compact rank one symmetric spaces; for a classification, see [PoTh]. For a survey on these objects as well as on polar actions, see [Th1] and [Th2].

Theorem 4.6 (Slice Theorem for polar actions [PaTe])**.** *Let G a Lie group with a proper polar action on M. Then the slice representation of G_x on $\nu_x(Gx)$ is hyperpolar with sections of the form $T_x\Sigma$, where Σ is a section through x.*

Note that any polar action acts transitively on the set of sections. Indeed, let Gx be an arbitrary regular orbit. By definition any two sections Σ_1 and Σ_2 meet Gx say in x_1 respectively x_2. Let $g \in G$ such that with $gx_1 = x_2$. Then $g_*\nu_{x_1}Gx = \nu_{x_2}Gx$. Since $\nu_{x_i}Gx = T_{x_i}\Sigma_i$ and Σ_i is totally geodesic, we have $g(\Sigma_1) = \Sigma_2$. We have the following corollary.

Corollary 4.7. *The isotropy group G_x acts transitively on the sections through x.*

Let us also note that Dadok's classification of hyperpolar actions implies:

Corollary 4.8. *The slice representation of a proper polar action is an s-representation.*

5. Weyl group and G-equivariant blow-up

We recall the definition of the Weyl group for a polar action, which generalizes the Weyl group of a classical Lie group G, acting on itself via the adjoint map.

Definition 5.1. Let G a Lie group acting smoothly by isometries on a complete Riemannian manifold M, and assume the action is polar with section $i : \Sigma \to M$. Let

$$N := N_G(\Sigma) \quad = \quad \{g \in G \mid g(i(\Sigma)) = i(\Sigma)\} \tag{19}$$

$$Z := Z_G(\Sigma) \quad = \quad \{g \in G \mid gi(x) = i(x) \text{ for any } x \in \Sigma\} \tag{20}$$

Then the Weyl group is

$$W = N_G(\Sigma)/Z_G(\Sigma) \tag{21}$$

The action of N descends to an action on Σ for which $i : \Sigma \to M$ is N-equivariant. The generalized Weyl group W therefore acts effectively and isometrically on Σ. The orbits of G and W are related in the following way:

$$Gi(x) \cap i(\Sigma) = i(Wx) \tag{22}$$

or in a more suggestive form by $Gx \cap \Sigma = Wx$ for any $x \in \Sigma$. Note that the normalizer $N_G(\Sigma)$ of the section Σ need not be discrete; in example (4.4.2) the section Σ is a maximal torus T, and $T = Z_G(T) \subset N_G(\Sigma)$.

Proposition 5.2. *The action of the Weyl group on Σ is properly discontinous.*

Proof. Since G acts properly on M, the orbits are closed and embedded. In particular, every G-orbit intersects Σ discretely. By (22) the W-orbits must also be discrete. For an isometric action, this is equivalent to the action being properly discontinuous, i.e. for any compact subset K of Σ, the set of $w \in W$ with $w(K) \cap K \neq \emptyset$ is finite. □

We next introduce the "blow-up" of a polar action. This blow-up is different from the inductive normal projectivization of strata beginning with the lowest dimensional (see e.g. [DuKa]). The blow-up we use here has been introduced in the context of singular Riemannian foliations admitting sections in [Bou] and studied further in [Tö1] (see also [Tö2], [Tö3].)

Recall that for $x \in M$ a regular point, the orbit Gx has maximal dimension, hence there exists exactly one section Σ_x containing x: since the intersection of Σ_x with Gx is orthogonal and Σ_x is totally geodesic, we have $\Sigma_x = \exp^{\perp}(\nu_x(Gx))$.

For a singular point $y \in M$, there is a family of sections running through this point. The blow-up of a singular point y is obtained using the space of all sections through the singular point.

Let $G_q(TM)$ denote the Grassmann bundle of q-planes in TM. The fiber $G_q(T_xM)$ over $x \in M$ is the Grassmann manifold of q-planes in T_xM. Recall r is the dimension of a regular orbit, so that $G_q(TM) \cong Fr(TM)/(O(r)\times O(q))$ where $Fr(TM) \to M$ denotes the orthogonal frame bundle of TM with the right action by $O(r+q)$.

Given a section $i : \Sigma \to M$, let $\tau_x\Sigma = i_*(T_x\Sigma)$ denote the tangent space $T_x\Sigma$ considered as a subspace of T_xM via the map i_* and hence as an element of the Grassmannian $G_q(T_x\Sigma)$.

Define the (Grassmann) *blow-up* of (M, G) by

$$\widehat{M} := \{\tau_x\Sigma = i_*(T_x\Sigma) \mid i : \Sigma \to M \text{ is a section}, x \in \Sigma\} \subset G_q(TM) \tag{23}$$

Let $\widehat{\pi} : \widehat{M} \to M$ be the restriction of the canonical projection $\pi : G_q(TM) \to M$ to $\widehat{M}$; so $\widehat{\pi}(\tau_x\Sigma)) = i(x)$. The action of G on $\widehat{M}$ defined by

$$g_*\tau_x\Sigma = (g \circ i)_*(T_x\Sigma)$$

is proper and the projection $\widehat{\pi}$ is G-equivariant with respect to this action. The blow-up $\widehat{M}$ can be endowed with a differentiable structure such that its inclusion $\iota : \widehat{M} \hookrightarrow G_q(TM)$ is an immersion (see section 3.2 of [Tö1].)

Note that given a section $i : \Sigma \to M$ there is a tautological lift to a map $\tau : \Sigma \to \widehat{M}$, where for $x \in \Sigma$ we set $\tau(x) = \tau_x\Sigma$.

Proposition 5.3. *Let G be a Lie group with a proper polar action on M, $i : \Sigma \to M$ a section, and $N = N_G(\Sigma)$ the normalizer of Σ. Then there is a G-equivariant diffeomorphism $\widehat{M} \cong G \times_N \Sigma$.*

Proof. The action of the normalizer group N on $G \times \Sigma$ is defined by $h \cdot (g, x) = (gh^{-1}, hx)$. The space $G \times_W \Sigma$ is the quotient by this action, which is just the quotient space defined by the equivalence relation $(gh, x) \sim (g, hx)$ for $h \in N$.

Consider the map $\tilde{\Phi}(g, x) = g_* \tau_x \Sigma$. Then

$$\tilde{\Phi}(gh, x) = (gh)_* \tau_x \Sigma = g_*(h_* \tau_x \Sigma) = g_* \tau_{hx} \Sigma = \tilde{\Phi}(g, hx)$$

where we use that $h\Sigma = \Sigma$ as $h \in N_G(\Sigma)$. Thus, there is a well-defined map

$$\begin{aligned} \Phi\colon G \times_N \Sigma &\rightarrow \widehat{M} \\ \Phi([(g, x)]) &= g_* \tau_x \Sigma \end{aligned} \tag{24}$$

The Lie group G acts on $G \times_N \Sigma$ by $g \cdot [(h, x)] = [(gh, x)]$.

First, let us show that Φ is onto. Every orbit of G in M intersects the image of Σ, so it suffices to show that given $x \in \Sigma$, the action of the isotropy group G_x is transitive on sections through x. This follows from Corollary 4.7.

We claim that Φ is injective. Assume $[(g, x)]$ and $[(h, y)]$ have the same image, where $x, y \in \Sigma$. Without loss we can assume that $h = e$. Then $g_* \tau_x \Sigma = \tau_y \Sigma$ which implies $gx = y$ and $g(\Sigma) = \Sigma$, i.e. $g \in N$. Therefore $[(g, x)] = [(e, y)]$. □

Example 5.4. Let G be a connected Lie group and $K \subset G$ a compact subgroup. Assume the center $Z(G)$ is not empty, and $\Gamma \subset Z(G)$ is a finite subgroup which acts effectively, isometrically on a compact manifold N. Now assume that the quotient space $M = G/K \times_\Gamma N$ is a manifold, where for each $h \in \Gamma$ and $y \in N$, we identify $(gKh, y) \sim (gK, hy)$. Let $[(gH, y)] \in M$ denote the equivalence class of (gK, y). Then $i : N \to M$, $i(y) = [(eK, y)]$, is a section and $N_G(\Sigma) = K\Gamma$, $Z_G(\Sigma) = K$, so $W = \Gamma/(\Gamma \cap K)$. The diffeomorphism of Proposition 5.3 is the tautology $G \times_{K\Gamma} N \cong G/K \times_\Gamma N$.

This class of examples are standard models in the theory of compact Hausdorff foliations [Mi], where the quotient manifold $B = G\backslash M \cong W\backslash N$ is an orbifold. Conversely, given an orbifold B of dimension q, there is an associated manifold M with a locally-free action of $O(q)$, so that $B \cong O(q)\backslash M$. However, for such a group action, there need not exists a section Σ.

Example 5.5. (Polar coordinates on $\mathbb{R}^3$) Let $M = \mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere, and let $G = SO(2)$ act via rotations in the plane $(x, y, 0)$. Let Σ be the embedding $i\colon \mathbb{S}^1 \to \mathbb{S}^2$ given by $i(w) = (\sin(\theta), 0, \cos(\theta))$ for $w = (\sin(\theta), \cos(\theta) \in \mathbb{S}^1$. Then $N_G(\Sigma) = \{\pm 1\} \subset SO(2)$, $Z_G(\Sigma) = \{1\}$, $W = \mathbb{Z}/2\mathbb{Z}$ and Proposition 5.3 yields

$$G \times \Sigma = \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \widehat{M} = \mathbb{S}^1 \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{S}^1 \longrightarrow M = \mathbb{S}^2$$

which is just the standard blow-down map of the 2-torus to obtain the 2-sphere.

6. Category for polar actions

We can now give an upper estimate for the equivariant category of a proper polar action in terms of the proper action of the Weyl group on a section Σ. The category of the Weyl group action is often easier to compute, as the action is discrete.

Theorem 6.1. *Let G be a Lie group with a proper polar action on M, $i : \Sigma \to M$ a section, and $W = N_G(\Sigma)/Z_G(\Sigma)$ the generalized Weyl group acting on Σ. Then*

$$\mathrm{cat}_G(M) \leq \mathrm{cat}_W(\Sigma) \tag{25}$$

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a W-categorical covering of Σ, with W-equivariant homotopies $H_i : U_i \times [0,1] \to \Sigma$. Let $V_i = G \cdot U_i$ be the orbit saturation of U_i for the G-action on M. Then $\widehat{V}_i := \widehat{\pi}^{-1}(V_i) \subset \widehat{M}$ is the orbit saturation of the G-action on the blow-up $\widehat{M}$. The strategy of the proof is to show that the sets $\widehat{V}_i$ are G-categorical in $\widehat{M}$, and hence the V_i form a G-categorical cover for M.

Note that U_i is W-invariant by assumption, so invariant under the induced action of $N = N_G(\Sigma)$. Using the identification $\widehat{M} \cong G \times_N \Sigma$, each $\widehat{V}_i$ thus has the alternate description

$$\widehat{V}_i = \bigcup_{[g]\in G/N} g_*\tau(U_i), \tag{26}$$

Note that if $g_*\tau(U_i) \cap h_*\tau(U_i) \neq \emptyset$ then $h^{-1}g \in N$, or $g = hk$ for some $k \in N$, as $U_i \subset \Sigma$ always contains regular points for the G-action. Thus the above union is disjoint.

Define $(\widehat{H}_i)_t | g_*\tau(U_i) = (g_* \circ \tau) \circ H_t \circ (g_* \circ \tau)^{-1}$. We claim this yields a well-defined map on $\widehat{V}_i$. Recall that the homotopy H_i is assumed to be W-equivariant, and that $g_* \circ \tau = \tau \circ g$ for all $g \in G$.

Let $y, y' \in U_i$ with $g_*\tau(y) = h_*\tau(y')$ so that $g = hk$ for some $k \in N$. Then $y = g^{-1}hy' = k^{-1}y'$ and we calculate

$$\begin{aligned}
(\widehat{H}_i)_t(g_*\tau(y)) &= (g_* \circ \tau) \circ H_t \circ (g_* \circ \tau)^{-1}(g_*\tau(y)) \\
&= (g_* \circ \tau) \circ H_t(y) \\
&= (g_* \circ \tau) \circ H_t(k^{-1}y') \\
&= (g_* \circ \tau) \circ k^{-1} H_t(y') \\
&= (g_* k_*^{-1} \circ \tau) \circ H_t(y') \\
&= (h_* \circ \tau) \circ H_t \circ (h_* \circ \tau)^{-1}(h_*\tau(y')) \\
&= (\widehat{H}_i)_t(h_*\tau(y'))
\end{aligned}$$

Thus, $(\widehat{H}_i)_t : \widehat{V}_i \to \widehat{M}$ is well-defined, and G-equivariant by construction.

We next show that each $\widehat{V}_i$ is G-categorical in $\widehat{M}$; that is, the image $(\widehat{H}_i)_1(\widehat{V}_i)$ is contained in a G-orbit in $\widehat{M}$. By the definitions, the following diagram commutes

$$\begin{array}{ccccc}
 & g_*\tau U_i & \xrightarrow{(\widehat{H}_i)_t} & g_*\tau\Sigma & \\
g_* \circ \tau & \uparrow & & \uparrow & g_* \circ \tau \\
 & U_i & \xrightarrow{(H_i)_t} & \Sigma &
\end{array}$$

For $t = 1$, the map $(H_i)_1 : U_i \to Wy_i$ for some $y_i \in \Sigma$. Thus, the image $(\widehat{H}_i)_1(\widehat{V}_i)$ is contained in the union of the images of the sets $g_* \circ \tau(Wy_i)$ for $g \in G$, which are all contained in the G-orbit $G \cdot \tau(y_i)$. We have thus shown that $\mathrm{cat}_G(\widehat{M}) \leq \mathrm{cat}_W(\Sigma)$.

Now we want to define a homotopy $H_i' : V_i \times [0,1] \to M$ satisfying

$$\widehat{\pi} \circ (\widehat{H}_i)_t = (H_i')_t \circ \widehat{\pi}. \tag{27}$$

The above formula determines H_i' if it is well-defined, because $\widehat{\pi} : \widehat{V}_i \to V_i$ is surjective. Also the G-equivariance of $(H_i')_t$ follows immediately using $g \circ \widehat{\pi} = \widehat{\pi} \circ g_*$.

For existence, it suffices to show that $(\widehat{H}_i)_t$ respects $\widehat{\pi}$-fibers; that is,

$$\sigma, \sigma' \in \widehat{V}_i \ , \ \widehat{\pi}(\sigma) = \widehat{\pi}(\sigma') \implies \widehat{\pi} \circ (\widehat{H}_i)_t(\sigma) = \widehat{\pi} \circ (\widehat{H}_i)_t(\sigma'). \tag{28}$$

Let $\sigma, \sigma' \in \widehat{V}_i$ satisfy $\widehat{\pi}(\sigma) = \widehat{\pi}(\sigma')$. Then $\sigma = g_* \tau_x \Sigma$ for some $g \in G$ and $x \in U_i$. Moreover, $\widehat{\pi}(\sigma) = gx$ and so σ' is a section through gx. By Corollary 4.7 $G_{\widehat{\pi}(\sigma)}$ acts transitively on the set of sections through $\widehat{\pi}(\sigma)$ which is $\widehat{\pi}^{-1}(\widehat{\pi}(\sigma))$, so $\widehat{\pi}^{-1}(\widehat{\pi}(\sigma)) = (G_{\widehat{\pi}(\sigma)})_* \sigma$. Hence there exists $h \in G_{gx}$ such that

$$\sigma' = h_* \sigma = h_* g_* \tau_x \Sigma = (hg)_* \tau_x \Sigma.$$

Set $x_t = (H_i)_t(x)$. We show in Proposition 6.2 below that $G_x \subset G_{x_t}$ for all $0 \le t \le 1$, hence

$$G_{gx} = g G_x g^{-1} \subset g G_{x_t} g^{-1} = G_{g x_t}$$

and so

$$\widehat{\pi} \circ (\widehat{H}_i)_t(\sigma') = \widehat{\pi} \circ (\widehat{H}_i)_t((hg)_* \tau_x \Sigma) = hg x_t = g x_t = \widehat{\pi} \circ (\widehat{H}_i)_t(\sigma)$$

as was to be shown. □

It remains to show that the isotropy groups are stable under the homotopies $(H_i)_t$. We use the same notation as above.

Proposition 6.2. *Given $U \subset \Sigma$, let $H : U \times [0,1] \to \Sigma$ be a W-equivariant homotopy. For $x \in U$ we write $x_t = H(t,x)$. Then $G_x \subset G_{x_t}$ for any $t \in [0,1]$.*

Proof. Clearly $W_x \subset W_{x_t}$. This means $x_t \in (\operatorname{Fix} W_x)_x$, which is the connected component of $\operatorname{Fix} W_x$ containing x.

Recall that the action of $W = N_G(\Sigma)/Z_G(\Sigma)$ on Σ is the quotient of the action of $N_G(\Sigma) \subset G$ on $\Sigma \subset M$. We are going to show

$$(\operatorname{Fix} W_x)_x \subset (\operatorname{Fix} G_x)_x \cap \Sigma.$$

This implies $x_t \in \operatorname{Fix} G_x$, so $G_x \subset G_{x_t}$.

We first want to show $\operatorname{Fix}(dW_x|T_x\Sigma) = \operatorname{Fix}(dG_x|V) \cap T_x\Sigma$, where $V = \nu_x(Gx)$. We remark that the right hand side is equal to $\operatorname{Fix}(dG_x|V)$ as $\operatorname{Fix} dG_x|V \subset T_x\Sigma$; this follows from the fact that G_x and its identity component G_x^0 act transitively on the set of sections through x. The linear action of G_x^0 on $\nu_x(Gx)$ is hyperpolar (and therefore orbit-equivalent to an s-representation) for which $T_x\Sigma$ is a section. Let W^0 be the associated Weyl group acting on $T_x\Sigma$; it is generated by the reflections through the singular hyperplanes in $T_x\Sigma$ through the origin. It is known that $W^0 = N_{G_x^0}(\Sigma)/Z_{G_x^0}(\Sigma)$. Let $F_1 = \operatorname{Fix}(W^0)$ and F_2 be its orthogonal complement in $T_x\Sigma$. This decomposes $T_x\Sigma$ into two W^0-invariant subspaces. Let $\pi : V \to T_x\Sigma$ be the orthogonal projection and let $D = \ker \pi$. As the image of an orbit $dG_x^0 v, v \in V$ under π is the convex hull of $W^0\pi(v)$ by Kostant's convexity theorem ([Kos], see also [PaTe] Theorem 8.6.2 and 8.6.4), $F_2 \oplus D$ is a $dG_x^0|V$-invariant subspace and therefore also F_1. The action of dG_x^0 on F_1 is trivial (for $v \in F_1$, the orbit $dG_x^0 v$ lies on the sphere of radius $\|v\|$ and on the other hand on $\pi^{-1}(v)$, since again by convexity $\pi(dG_x^0 v) = W^0 v = \{v\}$; the intersection of both submanifolds in V is exactly $\{v\}$), so $F_1 \subset \text{Fix}(dG_x^0|V) \cap T_x\Sigma$. The converse is obviously true. Thus $\text{Fix}(W^0) = \text{Fix}(dG_x^0|V) \cap T_x\Sigma$.

Now we want to show $(\text{Fix}\, W_x)_x = (\text{Fix}\, G_x)_x \cap \Sigma$. The natural inclusion

$$\phi : N_G(\Sigma)_x / N_{G_x^0}(\Sigma) \to G_x/G_x^0$$

is an isomorphism. We prove surjectivity. Let $[g] \in G_x/G_x^0$. Since G_x^0 acts transitively on the set of sections through x there is an $h \in G_x^0$ with $(gh)_*(T_x\Sigma) = T_x\Sigma$. Thus $gh \in N_G(\Sigma)_x$ and this proves surjectivity. We now prove injectivity. Assume $\phi([n_1]) = \phi([n_2])$ for $n_i \in N_G(\Sigma)_x$. Thus $n_1 h = n_2$ for some $h \in G_x^0$. It follows $h_* T_x\Sigma = T_x\Sigma$. Therefore $h \in N_{G_x^0}(\Sigma)$ and we have proven injectivity.

Thus $G_x/G_x^0 = \{n_i G_x^0\}_{i\in I}$ for a countable index set I and $n_i \in N_G(\Sigma)_x$. Together with $\text{Fix}(W^0) = \text{Fix}(dG_x^0|V)$ this implies $\text{Fix}(dW_x|T_x\Sigma) = \text{Fix}(dG_x|V) \cap T_x\Sigma = \text{Fix}(dG_x|T_xM) \cap T_x\Sigma$ and therefore by exponentiating we obtain $(\text{Fix}\, W_x)_x = (\text{Fix}\, G_x)_x \cap \Sigma$. □

7. Examples and applications

We provide a selection of examples of polar actions which show that the upper and lower bounds for the G-category provided in Theorem 6.1, and Theorem 3.7 and Corollary 3.14, can be very effective. In particular, we deduce the calculation of the the classical Lusternik-Schnirelmann category for $U(n)$ and $SU(n)$ due to Singhof [Si] mentioned in the Introduction.

7.1. **The LS-category and the equivariant category of $SU(n)$ and $U(n)$.** We will first prove

$$\text{cat}_{SU(n+1)} SU(n+1) = n+1$$

for the equivariant category of the action of $G = SU(n)$ on itself by conjugation. We have already seen in example 3.10 that $n+1$ is a lower bound.

We now want show with the help of Theorem 6.1 that $n+1$ is also an upper bound. The maximal torus of this action is

$$\mathbb{T}^n = \{\lambda_1 \oplus \cdots \oplus \lambda_{n+1} \mid \lambda_i \in S^1 \subset \mathbb{C}, \lambda_1 \cdots \lambda_{n+1} = 1\}.$$

The Weyl group $W_{SU(n+1)}$ is the group of permutations of the coordinates of $\mathbb{T}^n$

$$\sigma : \lambda_1 \oplus \cdots \oplus \lambda_{n+1} \mapsto \lambda_{\sigma(1)} \oplus \cdots \oplus \lambda_{\sigma(n+1)}.$$

Let $z_k = e^{\frac{2\pi i k}{n+1}} I_{n+1}, k = 0, \ldots, n$ be the set of central elements of $SU(n+1)$. We give a W-categorical covering U_k of Σ so that each U_k contracts radially to z_k.

Let $\varphi : \mathbb{R}^{n+1} \to \mathbb{T}^{n+1}, x = (x_0, \ldots, x_n) \mapsto (e^{2\pi i x_0}, \ldots, e^{2\pi i x_n})$ be the canonical covering map. The preimage of $\mathbb{T}^n \subset \mathbb{T}^{n+1}$ under φ intersected with the fundamental domain $[0,1]^{n+1}$ is exactly $\{x \in [0,1]^{n+1} \mid \sum_{i=1}^n x_i = k, k \in \{1, 2, \ldots, n\}\}$.

Define the n-simplex $\Delta_k = \{x \in [0,1]^{n+1} \mid \sum_{i=1}^n x_i = k\}$ for $k = 1, \ldots, n$. We observe that φ restricted to the interior $\text{int}(\Delta_k)$ of Δ_k in $\{x \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i = k\}$ is a diffeomorphism onto its image in $\mathbb{T}^n$. Clearly $\text{int}(\Delta_k)$ is invariant under permutation of coordinates and its radial contraction to $(\frac{k}{n+1}, \ldots, \frac{k}{n+1})$ is equivariant with respect to the permutation group. The conjugation of this homotopy with φ

is W-equivariant and contracts to z_k. An extension of this homotopy to a neighborhood of $\varphi(\Delta_k)$ would finish the proof but this is not possible. The injectivity of φ on the entire Δ_k fails exactly in its vertices which are mapped to z_0.

Now let V'_k be a small open neighborhood of Δ_k in $\{x \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i = k\}$ invariant under permutation of coordinates minus an $\epsilon/2$ ball around the vertices of Δ_k for small $\epsilon > 0$ such that V'_k is still star-shaped with respect to $(\frac{k}{n+1}, \dots, \frac{k}{n+1})$. Then, for appropriate choices, φ is injective and therefore an isometry of V'_k to V_k.

The $\{V_k\}, k = 1, \dots, n$ cover $\mathbb{T}^n$, together with the ball V_0 around z_0 of radius ϵ. Each V_k is W-categorical and radially contracts to z_k via the W-equivariant homotopy h_k.

Now the proof of Theorem 6.1 shows that h_k can be extended to a $SU(n+1)$-homotopy H_k of $SU(n+1) \cdot V_k \subset SU(n+1)$ to $SU(n+1) \cdot z_k = z_k$. This gives us a $SU(n+1)$-categorical covering of $SU(n+1)$ of cardinality $n+1$. Thus $\mathrm{cat}_{SU(n+1)} SU(n+1) = n+1$.

We can now quickly reprove the following theorem by Singhof [Si].

Theorem 7.1 (Singhof)**.** *The LS-categories of the unitary and the special unitary groups are* $\mathrm{cat}(SU(n)) = n$ *and* $\mathrm{cat}(U(n)) = n+1$.

Since the $SU(n+1)$-homotopy H_i from above contracts to a point $z_k = SU(n+1) \cdot z_k$, the open set $SU(n+1) \cdot V_k$ is also categorical in the classical sense of Lusternik and Schnirelmann. Therefore our equivariant cover also provides a LS-categorical cover of $SU(n+1)$. So $\mathrm{cat}(SU(n+1)) \leq n+1$. On the other hand $n+1 \leq \mathrm{cat}(SU(n+1))$ by the general formula $\mathrm{cuplength}(M) + 1 \leq \mathrm{cat}(M)$ and $\mathrm{cuplength}(SU(n+1)) = n$. Now $\mathrm{cat}(U(n)) = n+1$ follows from $U(n) \cong S^1 \times SU(n)$, from the formula $\mathrm{cat}(M \times N) + 1 \leq \mathrm{cat}(M) + \mathrm{cat}(N)$ and $\mathrm{cuplength}(U(n)) = n$.

7.2. **The LS-category $\mathbb{CP}^n$.** We want to give an alternative computation of the LS-category of $\mathbb{CP}^n$ with the help of Theorem 6.1 as in the previous example. Consider the action of $\mathbb{T}^n = \{c = (c_0, \dots, c_n) \in \mathbb{T}^{n+1} \mid c_0 \cdots c_n = 1\}$ on an element $z = [z_0 : \dots : z_n] \in \mathbb{CP}^n$ by $c \cdot z := [c_0 z_0 : \dots c_n z_n]$. This action is polar. The natural embedding of $\mathbb{RP}^n$ into $\mathbb{CP}^n$ is a section. The elements of the Weyl group W are the even sign changes in the homogeneous coordinates. The fixed points of W are the $n+1$ points $e_i = [0 : \dots : 0 : 1 : 0 : \dots : 0]$, where the 1 is at position i, for $i = 0, \dots, n$. These are also $\mathbb{T}^n$-fixed points. For each i we want to define a W-equivariant homotopy that contracts to e_i.

Define $\varphi_i : \mathbb{R}^n \to \mathbb{RP}^n; x = (x_1 \dots x_n) \mapsto [x_1 : \dots : 1 : \dots : x_n]$. Let $U_i \subset \mathbb{RP}^n = \Sigma$ denote the image of φ_i. The radial contraction to the origin in $\mathbb{R}^n$ gives us via φ_i a W-homotopy $h^i : U_i \times [0,1] \to \mathbb{RP}^n; ([z], t) \mapsto [t\frac{z_0}{z_i} : \dots : 1 : \dots : t\frac{z_n}{z_i}]$ contracting to e_i. The collection $\{U_0, \dots, U_n\}$ form a W-categorical cover of the section $\mathbb{RP}^n$. (Of course, the open sets U_i are just the usual covering of $\mathbb{RP}^n$ by Grassmann cells.)

Now set $V_i = \mathbb{T}^n \cdot U_i$ By the proof of Theorem 6.1, we can extend the h_i to $\mathbb{T}^n$-homotopies $H_i : V_i \times [0,1] \to \mathbb{CP}^n$. These contract to the $\mathbb{T}^n$-fixed points e_i. Together with the lower bound from the cuplength we now have $\mathrm{cat}(\mathbb{CP}^n) = n+1$.

Note that together with the lower bound from Corollary 3.9, the arguments also show $\mathrm{cat}_{\mathbb{T}^n}(\mathbb{CP}^n) = n+1$.

7.3. **The LS-category $\mathbb{HP}^n$.** We consider the polar action of $G = Sp(1) \cdots Sp(1)$ (n factors) on $\mathbb{HP}^n$. It has the same section, namely $\mathbb{RP}^n$, and Weyl group W as the previous action. The W-fixed points are also G-fixed points. This gives us as before $\mathrm{cat}(\mathbb{HP}^n) = n + 1$.

7.4. **The LS-category $\mathbb{OP}^2$.** We consider the polar action of $G = Spin(8)$ on $\mathbb{OP}^n$ with section $\mathbb{OP}^2$ with Weyl group as before for $n = 2$. Then $\mathrm{cat}(\mathbb{OP}^2) = 3$.

References

[ALQ] R. Ayala, F.F. Lasheras, A. Quintero, *The equivariant category of proper G-spaces*, Rocky Mountain Journal of Mathematics, vol. **31**, nr. 4 (2001), 1111–1132.

[BeTa] J. Berndt, H. Tamaru, *Homogeneous codimension one foliations on noncompact symmetric spaces*, J. Differential Geom., vol. **63**, nr. 1 (2003), 1–40.

[Bor] K. Borsuk, **Theory of retracts**, Polish Academy Publishers, Warsaw 1967.

[Bou] H. Boualem, *Feuilletages riemanniens singuliers transversalement integrables*, Comp. Math. **95** (1995), 101-125.

[Bre] G. Bredon, **Introduction to compact transformation groups**, Pure and Applied Mathematics, Vol. 46, Academic Press, New York, 1972.

[Co] H. Colman, *Equivariant LS-category for finite group actions*, in **Lusternik-Schnirelmann category and related topics**, Contemp. Math. Vol. 316, 35-40, AMS, Providence 2002.

[CoHu] H. Colman, S. Hurder, *LS-category of compact Hausdorff foliations*, Trans. Amer. Math. Soc. **356** 2004, 1463-1487.

[CoMa] H. Colman and E. Macias, *Transverse Lusternik–Schnirelmann category of foliated manifolds*, Topology Vol. **40** (2) (2001), 419-430.

[Da] J. Dadok, *Polar Coordinates induced by Actions of Compact Lie Groups*, Trans. Amer. Math. Soc. **288** (1985), 125-137.

[DuKa] J.J. Duistermaat, J.A. Kolk, **Lie Groups**, Universitext, Springer, 1999.

[Ha1] A. Haefliger, *Pseudogroups of local isometries*, In **Differential geometry (Santiago de Compostela, 1984)**, Res. Notes in Math., Vol. 131:174–197, Pitman, Boston, MS, 1985.

[Ha2] A. Haefliger, *Leaf closures in Riemannian foliations*, In **A Fête of Topology**, Academic Press, Boston, MA, 1988, 3–32.

[HuTö] S. Hurder, D. Töben, *Riemannian foliations with finite transverse LS category*, Trans. Amer. Math. Soc., to appear.

[Ja] I.M. James, *Lusternik-Schnirelmann Category*, **Handbook of Algebraic Topology**, (ed. I. M. James, Elsevier Science, Amsterdam) pages 1293–1310, 1995.

[Ko1] A. Kollross, *A Classification of Hyperpolar and Cohomogeneity One Actions*, Trans. Amer. Math. Soc. **354**, no. 2 (2002), 571-612.

[Ko2] A. Kollross, *Homogeneous spaces with polar isotropy*, Manuscripta Math. **110**, no. 4 (2003), 487–503.

[Kos] B. Kostant, *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. Sci. Ec. Norm. Sup. **6** (1973), 413-455.

[LS] L. Lusternik and L. Schnirelmann, **Méthodes topologiques dans les Problèmes Variationnels**. Hermann, Paris, 1934.

[Ma] W. Marzantowicz, *A G-Lusternik Schnirelman category of space with an action of a compact Lie group*, Topology **28**, no.4 (1989), 403-412.

[Mi] K. Millett, *Compact foliations*, In **Foliations: Dijon 1974, Lect. Notes in Math. volume 484**, pages 277–287, Springer–Verlag, New York and Berlin, 1975.

[Mo] P. Molino, **Riemannian foliations**, Birkhäuser, Boston 1988.

[Pa] R.S. Palais, *On the existence of slices for actions of noncompact Lie groups*, Ann. of Math. **73** (1961), 295-323.

[PaTe] R.S. Palais, C.L.Terng, **Critical Point Theory and Submanifold Geometry**, Geometry Lecture Notes in Math., Vol. 1353, Springer, Berlin 1988.

[PoTh] F. Podestà, G. Thorbergsson, *Polar Actions on Rank One Symmetric Spaces*, J. Diff. Geom. **53**, no. 1 (1999), 131-175.

[Si] W. Singhof, *On the Lusternik-Schnirelmann category of Lie groups*, Math. Z. **145** (1975), 111-116.

[Sz] J. Szenthe, *Orthogonally Transversal Submanifolds and the Generalizations of the Weyl Group*, Period. Math. Hungar. **15**, no.4 (1984), 281-299.

[Th1] G. Thorbergsson, *A Survey on Isoparametric Submanifolds and their Generalizations*, in **Handbook of Differential Geometry**, Vol. I, 963-995, North Holland, Amsterdam, 2000.

[Th2] G. Thorbergsson, *Transformation Groups and Submanifold Geometry*, Rend. Mat. Appl. (7) **25**, no. 1 (2005), 1-16.

[Tö1] D. Töben, *Parallel focal structure and singular Riemannian foliations*, Trans. Amer. Math. Soc. **358** (2006), 1677-1704.

[Tö2] D. Töben, *The generalized Weyl group of a singular Riemannian foliation*, In *FOLIATIONS 2005*, 399–409, World Scientific, 2006.

[Tö3] D. Töben, *Singular Riemannian foliations on nonpositively curved manifolds*, Math. Z. **255** (2007), 427–436.

Steven Hurder, Department of Mathematics, University of Illinois at Chicago, USA

Email address: hurder@uic.edu

Dirk Töben, Mathematisches Institut, Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany

Email address: dtoeben@math.uni-koeln.de