CLASSIFYING MATCHBOX MANIFOLDS

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Abstract. Matchbox manifolds are foliated spaces whose transversal spaces are totally disconnected. In this work, we show that the local dynamics of a certain class of minimal matchbox manifolds classify their total space, up to homeomorphism. A key point is the use Alexandroff's notion of a Y–like continuum, where Y is an aspherical closed manifold which satisfies the Borel Conjecture. In particular, we show that two equicontinuous $\mathbb{T}^n$–like matchbox manifolds of the same dimension, are homeomorphic if and only if their corresponding restricted pseudogroups are return equivalent. With an additional geometric assumption, our results apply to Y-like weak solenoids where Y satisfies these conditions. At the same time, we show that these results cannot be extended to include classes of matchbox manifolds fibering over a closed surface of genus 2 manifold which we call "*adic*-surfaces". These are 2–dimensional matchbox manifolds that have structure induced from classical 1-dimensional Vietoris solenoids. We also formulate conjectures about a generalized form of the Borel Conjecture for minimal matchbox manifolds.

1. Introduction

In this paper, we study the problem of when do the local dynamics and shape type of a matchbox manifold $\mathfrak{M}$ determine the homeomorphism type of $\mathfrak{M}$. For example, it is folklore [15, 32] that two connected compact abelian groups with the same shape (or even just isomorphic first Čech cohomology groups) are homeomorphic. In another direction, for minimal, 1–dimensional matchbox manifolds, Fokkink [23, Theorems 3.7,4.1], and Aarts and Oversteegen [2, Theorem 17] show that:

THEOREM 1.1. *Two orientable, minimal, 1–dimensional matchbox manifolds are homeomorphic if and only if they are return equivalent.*

Since any non–orientable minimal, matchbox manifold admits an orientable double cover, this demonstrates that the local dynamics effectively determines the global topology in dimension one.

The local dynamics of a minimal matchbox manifold $\mathfrak{M}$ is defined using the pseudogroup $\mathcal{G}_W$ of local holonomy maps for a local transversal W of $\mathfrak{M}$. In Section 4 we show that this notion of return equivalence is well-defined for minimal matchbox manifolds, and show that if $\mathfrak{M}_1, \mathfrak{M}_2$ are any two homeomorphic minimal matchbox manifolds, then for any local transversals $W_i \subset \mathfrak{M}_i$ we have that $\mathcal{G}_{W_1}$ is return equivalent to $\mathcal{G}_{W_2}$. It thus makes sense to ask to what extent two return equivalent minimal matchbox manifolds have the same topology for matchbox manifolds with leaves of dimension greater than one.

There are many difficulties in extending the topological classification for 1-dimensional matchbox manifolds to the cases with higher dimensional leaves. In Section 8, we show by way of examples, that such extensions are not always possible. Thus, one seeks sufficient conditions for which return equivalence implies topological conjugacy.

For example, the one-dimensional case uses implicitly the basic property of 1-dimensional flows, that every cover of a circle is again a circle. This leads to the introduction of shape theoretic properties of matchbox manifolds, which imposes some broad constraints on the topology of the leaves that appear necessary. In this work, we impose the following notion, introduced by Alexandroff in [3]:

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DEFINITION 1.2. *Let Y be a compact metric space. A metric space X is said to be Y–*like *if for every $\epsilon > 0$, there is a continuous surjection $f_\epsilon \colon X \to Y$ such that the fiber $f_\epsilon^{-1}(y)$ of each point $y \in Y$ has diameter less than ϵ.*

Recall that a CW-complex Y is *aspherical* if it is connected, and $\pi_n(Y)$ is trivial for all $n \geq 2$. Equivalently, Y is aspherical if it is connected and its universal covering space is contractible. Let $\mathcal{A}$ denote the collection of CW-complexes which are aspherical. Our first main result is an extension of a main result in [12].

THEOREM 1.3. *Suppose that $\mathfrak{M}$ is an equicontinuous Y-like matchbox manifold, where $Y \in \mathcal{A}$. Then $\mathfrak{M}$ admits a presentation as an inverse limit*

$$\mathfrak{M} \overset{\text{top}}{\approx} \varprojlim \{ q_{\ell+1} \colon B_{\ell+1} \to B_\ell \mid \ell \geq 0\} \tag{1}$$

where each $B_{\ell+1}$ is a closed manifold with $B_{\ell+1} \in \mathcal{A}$, and each bonding map q_ℓ is a finite covering.

We formulate our main results regarding the topological conjugacy of matchbox manifolds with leaves of arbitrary dimension $n \geq 1$. The first result is for the special case where $Y = \mathbb{T}^n$, which gives a direct generalization of the classification of 1-dimensional matchbox manifolds.

THEOREM 1.4. *Suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are equicontinuous $\mathbb{T}^n$-like matchbox manifolds. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent if and only if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic.*

As shown in Sections 6 and 7, an equicontinuous $\mathbb{T}^n$-like matchbox manifold $\mathfrak{M}$ is homeomorphic to an inverse limit of finite covering maps of the base $\mathbb{T}^n$. The possible homeomorphism types for such the inverse limit spaces known to be "unclassifiable", in the sense of descriptive set theory, as discussed in [31, 46, 47, 27]. Thus, it is not possible to give a classification for the family of matchbox manifolds obtained using the covering data in a presentation as the invariant. The notion of return equivalence provides an alternate approach to classification of these spaces.

In order to formulate a version of Theorem 1.4 for manifolds more general than $\mathbb{T}^n$, we use the *Borel Conjecture* for higher dimensional aspherical closed manifolds, which characterizes their homeomorphism types in terms of their fundamental groups. As discussed in Section 8, when combined with Definition 1.2, this yields a weak form of the self-covering property of the circle, for leaves of general matchbox manifolds. Recall that the *Borel Conjecture* is that if Y_1 and Y_2 are homotopy equivalent, aspherical closed manifolds, then a homotopy equivalence between Y_1 and Y_2 is homotopic to a homeomorphism between Y_1 and Y_2. The Borel Conjecture has been proven for many classes of aspherical manifolds, including:

- the torus $\mathbb{T}^n$ for all $n \geq 1$,
- all *infra-nilmanifolds*,
- closed Riemannian manifolds Y with negative sectional curvatures,
- closed Riemannian manifolds Y with non-positive sectional curvatures, dimension $n \neq 3, 4$,

where a compact connected manifold Y is an *infra-nilmanifold* if its universal cover $\widetilde{Y}$ is contractible, and the fundamental group of M has a nilpotent subgroup with finite index. The above list is not exhaustive. The history and current status of the Borel Conjecture is discussed in the surveys of Davis [16] and Lück [34]. We introduce the notion of a *strongly Borel* manifold.

DEFINITION 1.5. *A collection $\mathcal{A}_B$ of closed manifolds is called* Borel *if it satisfies the conditions*

1) *Each $Y \in \mathcal{A}_B$ is aspherical,*
2) *Any closed manifold X homotopy equivalent to some $Y \in \mathcal{A}_B$ is homeomorphic to Y, and*
3) *If $Y \in \mathcal{A}_B$, then any finite covering space of Y is also in $\mathcal{A}_B$.*

A closed manifold Y is strongly Borel *if the collection $\mathcal{A}_Y \equiv \langle Y \rangle$ of all finite covers of Y forms a Borel collection.*

Each class of manifolds in the above list is strongly Borel. Here is our second main result:

THEOREM 1.6. *Suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are equicontinuous, Y–like matchbox manifolds, where Y is a strongly Borel closed manifold. Assume that each of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have a leaf which is simply connected. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent if and only if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic.*

The requirement that there exists a simply connected leaf implies that the global holonomy maps associated to each of these foliations are injective maps, as shown in Proposition 5.9. This conclusion yields a connection between return equivalence for the foliations of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and the homotopy types of the approximating manifolds in a shape presentation. This requirement is not imposed for the case of $Y = \mathbb{T}^n$ in Theorem 1.4, due to the algebraic properties of its fundamental group.

The equicontinuous hypotheses is defined in Section 3, and is used to obtain towers of approximations in (1). Theorem 1.1 holds for more general matchbox manifolds. It remains an open question whether a more general form of Theorems 1.4 and 1.6 can be shown for classes of matchbox manifolds which are not equicontinuous. The last Section 9 of this paper formulates other generalizations of these results which we conjecture may be true.

In Section 8 we give some basic examples of equicontinuous matchbox manifolds which are not Y-like, for any CW-complex Y, and which are return equivalent but not homeomorphic. These examples show the strong relation between the Y-like hypothesis, and the property of a closed manifold Y that it has the *non-co-Hopfian Property*. This section also defines a class of examples, the *adic*-surfaces, which are not Y-like yet it is possible to give a form of classification result as an application of the ideas of this paper. In general, the examples of this section show that we cannot hope to generalize Theorem 1.4 to matchbox manifolds approximated by a sequence of arbitrary manifolds.

The rest of this paper is organized as follows. Sections 2 and 3 below collect together some definitions and results concerning matchbox manifolds and their dynamical properties that we use in the paper. Then in Section 4, we introduce the basic notion of return equivalence of matchbox manifolds.

Section 5 introduces the notion of *foliated Cantor bundles*, which play a fundamental role in the study of equicontinuous matchbox manifolds. Various results related to showing that these spaces are homeomorphism are developed, and Proposition 5.9 gives the main technical result required.

Section 6 recalls the properties of equicontinuous matchbox manifolds, and especially the notion of a *presentation* for such a space. Section 7 contains technical results concerning the *pro-homotopy groups* of equicontinuous matchbox manifolds. The proofs of Theorems 1.4 and 1.6 are given at the end of Section 7.

Finally, in Section 9 we offer several conjectures based on the results of this paper. In particular, we formulate an analogue of the Borel Conjecture for weak solenoids.

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2. Foliated spaces and matchbox manifolds

In this section we present the necessary background needed for our analysis of matchbox manifolds. More details can be found in the works [10, 12, 13, 14, 38]. Recall that a *continuum* is a compact connected metrizable space.

DEFINITION 2.1. *A* foliated space of dimension n *is a continuum $\mathfrak{M}$, such that there exists a compact separable metric space $\mathfrak{X}$, and for each $x \in \mathfrak{M}$ there is a compact subset $\mathfrak{T}_x \subset \mathfrak{X}$, an open subset $U_x \subset \mathfrak{M}$, and a homeomorphism defined on the closure $\varphi_x \colon \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ such that $\varphi_x(x) = (0, w_x)$ where $w_x \in int(\mathfrak{T}_x)$. Moreover, it is assumed that each φ_x admits an extension to a foliated homeomorphism $\widehat{\varphi}_x \colon \widehat{U}_x \to (-2,2)^n \times \mathfrak{T}_x$ where $\overline{U}_x \subset \widehat{U}_x$.*

The subspace $\mathfrak{T}_x$ of $\mathfrak{X}$ is the *local transverse model* at x.

Let $\pi_x : \overline{U}_x \to \mathfrak{T}_x$ denote the composition of φ_x with projection onto the second factor.

For $w \in \mathfrak{T}_x$ the set $\mathcal{P}_x(w) = \pi_x^{-1}(w) \subset \overline{U}_x$ is called a *plaque* for the coordinate chart φ_x. We adopt the notation, for $z \in \overline{U}_x$, that $\mathcal{P}_x(z) = \mathcal{P}_x(\pi_x(z))$, so that $z \in \mathcal{P}_x(z)$. Note that each plaque $\mathcal{P}_x(w)$ is given the topology so that the restriction $\varphi_x : \mathcal{P}_x(w) \to [-1,1]^n \times \{w\}$ is a homeomorphism. Then $int(\mathcal{P}_x(w)) = \varphi_x^{-1}((-1,1)^n \times \{w\})$.

Let $U_x = int(\overline{U}_x) = \varphi_x^{-1}((-1,1)^n \times int(\mathfrak{T}_x))$. Note that if $z \in U_x \cap U_y$, then $int(\mathcal{P}_x(z)) \cap int(\mathcal{P}_y(z))$ is an open subset of both $\mathcal{P}_x(z)$ and $\mathcal{P}_y(z)$. The collection of sets

$$\mathcal{V} = \{\varphi_x^{-1}(V \times \{w\}) \mid x \in \mathfrak{M} \ , \ w \in \mathfrak{T}_x \ , \ V \subset (-1,1)^n \text{ open}\}$$

forms the basis for the *fine topology* of $\mathfrak{M}$. The connected components of the fine topology are called leaves, and define the foliation $\mathcal{F}$ of $\mathfrak{M}$. For $x \in \mathfrak{M}$, let $L_x \subset \mathfrak{M}$ denote the leaf of $\mathcal{F}$ containing x.

Note that in Definition 2.1, the collection of transverse models $\{\mathfrak{T}_x \mid x \in \mathfrak{M}\}$ need not have union equal to $\mathfrak{X}$. This is similar to the situation for a smooth foliation of codimension q, where each foliation chart projects to an open subset of $\mathbb{R}^q$, but the collection of images need not cover $\mathbb{R}^q$.

DEFINITION 2.2. *A* smooth foliated space *is a foliated space $\mathfrak{M}$ as above, such that there exists a choice of local charts $\varphi_x : \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ such that for all $x, y \in \mathfrak{M}$ with $z \in U_x \cap U_y$, there exists an open set $z \in V_z \subset U_x \cap U_y$ such that $\mathcal{P}_x(z) \cap V_z$ and $\mathcal{P}_y(z) \cap V_z$ are connected open sets, and the composition*

$$\psi_{x,y;z} \equiv \varphi_y \circ \varphi_x^{-1} : \varphi_x(\mathcal{P}_x(z) \cap V_z) \to \varphi_y(\mathcal{P}_y(z) \cap V_z)$$

is a smooth map, where $\varphi_x(\mathcal{P}_x(z) \cap V_z) \subset \mathbb{R}^n \times \{w\} \cong \mathbb{R}^n$ and $\varphi_y(\mathcal{P}_y(z) \cap V_z) \subset \mathbb{R}^n \times \{w'\} \cong \mathbb{R}^n$. The leafwise transition maps $\psi_{x,y;z}$ are assumed to depend continuously on z in the C^∞-topology.

A map $f : \mathfrak{M} \to \mathbb{R}$ is said to be *smooth* if for each flow box $\varphi_x : \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ and $w \in \mathfrak{T}_x$ the composition $y \mapsto f \circ \varphi_x^{-1}(y,w)$ is a smooth function of $y \in (-1,1)^n$, and depends continuously on w in the C^∞-topology on maps of the plaque coordinates y. As noted in [38] and [10, Chapter 11], this allows one to define smooth partitions of unity, vector bundles, and tensors for smooth foliated spaces. In particular, one can define leafwise Riemannian metrics. We recall a standard result, whose proof for foliated spaces can be found in [10, Theorem 11.4.3].

THEOREM 2.3. *Let $\mathfrak{M}$ be a smooth foliated space. Then there exists a leafwise Riemannian metric for $\mathcal{F}$, such that for each $x \in \mathfrak{M}$, L_x inherits the structure of a complete Riemannian manifold with bounded geometry, and the Riemannian geometry depends continuously on x .* □

Bounded geometry implies, for example, that for each $x \in \mathfrak{M}$, there is a leafwise exponential map $\exp_x^{\mathcal{F}} : T_x\mathcal{F} \to L_x$ which is a surjection, and the composition $\exp_x^{\mathcal{F}} : T_x\mathcal{F} \to L_x \subset \mathfrak{M}$ depends continuously on x in the compact-open topology on maps.

DEFINITION 2.4. *A* matchbox manifold *is a continuum with the structure of a smooth foliated space $\mathfrak{M}$, such that for each $x \in \mathfrak{M}$, the transverse model space $\mathfrak{T}_x \subset \mathfrak{X}$ is totally disconnected, and for each $x \in \mathfrak{M}$, $\mathfrak{T}_x \subset \mathfrak{X}$ is a clopen (closed and open) subset.*

The maximal path-connected components of $\mathfrak{M}$ define the leaves of a foliation $\mathcal{F}$ of $\mathfrak{M}$. All matchbox manifolds are assumed to be smooth, with a given leafwise Riemannian metric, and with a fixed choice of metric $d_{\mathfrak{M}}$ on $\mathfrak{M}$. A matchbox manifold $\mathfrak{M}$ is *minimal* if every leaf of $\mathcal{F}$ is dense.

We next formulate the definition of a *regular covering* of $\mathfrak{M}$.

For $x \in \mathfrak{M}$ and $\epsilon > 0$, let $D_{\mathfrak{M}}(x, \epsilon) = \{y \in \mathfrak{M} \mid d_{\mathfrak{M}}(x,y) \leq \epsilon\}$ be the closed ϵ-ball about x in $\mathfrak{M}$, and $B_{\mathfrak{M}}(x, \epsilon) = \{y \in \mathfrak{M} \mid d_{\mathfrak{M}}(x,y) < \epsilon\}$ the open ϵ-ball about x.

Similarly, for $w \in \mathfrak{X}$ and $\epsilon > 0$, let $D_{\mathfrak{X}}(w, \epsilon) = \{w' \in \mathfrak{X} \mid d_{\mathfrak{X}}(w,w') \leq \epsilon\}$ be the closed ϵ-ball about w in $\mathfrak{X}$, and $B_{\mathfrak{X}}(w, \epsilon) = \{w' \in \mathfrak{X} \mid d_{\mathfrak{X}}(w,w') < \epsilon\}$ the open ϵ-ball about w.

Each leaf $L \subset \mathfrak{M}$ has a complete path-length metric, induced from the leafwise Riemannian metric:

$$d_{\mathcal{F}}(x,y) = \inf\left\{\|\gamma\| \mid \gamma\colon [0,1] \to L \text{ is piecewise C}^1 \ ,\ \gamma(0) = x \ ,\ \gamma(1) = y \ ,\ \gamma(t) \in L \quad \forall\, 0 \le t \le 1\right\}$$

where $\|\gamma\|$ denotes the path-length of the piecewise C^1-curve $\gamma(t)$. If $x, y \in \mathfrak{M}$ are not on the same leaf, then set $d_{\mathcal{F}}(x,y) = \infty$.

For each $x \in \mathfrak{M}$ and $r > 0$, let $D_{\mathcal{F}}(x,r) = \{y \in L_x \mid d_{\mathcal{F}}(x,y) \le r\}$.

The leafwise Riemannian metric $d_{\mathcal{F}}$ is continuous with respect to the metric $d_{\mathfrak{M}}$ on $\mathfrak{M}$, but otherwise the two metrics have no relation. The metric $d_{\mathfrak{M}}$ is used to define the metric topology on $\mathfrak{M}$, while the metric $d_{\mathcal{F}}$ depends on an independent choice of the Riemannian metric on leaves.

For each $x \in \mathfrak{M}$, the Gauss Lemma implies that there exists $\lambda_x > 0$ such that $D_{\mathcal{F}}(x, \lambda_x)$ is a *strongly convex* subset for the metric $d_{\mathcal{F}}$. That is, for any pair of points $y, y' \in D_{\mathcal{F}}(x, \lambda_x)$ there is a unique shortest geodesic segment in L_x joining y and y' and contained in $D_{\mathcal{F}}(x, \lambda_x)$. Then for all $0 < \lambda < \lambda_x$ the disk $D_{\mathcal{F}}(x, \lambda)$ is also strongly convex. As $\mathfrak{M}$ is compact and the leafwise metrics have uniformly bounded geometry, we obtain:

LEMMA 2.5. *There exists $\lambda_{\mathcal{F}} > 0$ such that for all $x \in \mathfrak{M}$, $D_{\mathcal{F}}(x, \lambda_{\mathcal{F}})$ is strongly convex.*

It follows from standard considerations (see [12, 13]) that a matchbox manifold admits a covering by foliation charts which satisfies additional regularity conditions.

PROPOSITION 2.6. [12] *For a smooth foliated space $\mathfrak{M}$, given $\epsilon_{\mathfrak{M}} > 0$, there exist $\lambda_{\mathcal{F}} > 0$ and a choice of local charts $\varphi_x\colon \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ with the following properties:*

1. *For each $x \in \mathfrak{M}$, $U_x \equiv int(\overline{U}_x) = \varphi_x^{-1}\left((-1,1)^n \times B_{\mathfrak{X}}(w_x, \epsilon_x)\right)$, where $\epsilon_x > 0$.*
2. *Locality: for all $x \in \mathfrak{M}$, each $\overline{U}_x \subset B_{\mathfrak{M}}(x, \epsilon_{\mathfrak{M}})$.*
3. *Local convexity: for all $x \in \mathfrak{M}$ the plaques of φ_x are leafwise strongly convex subsets with diameter less than $\lambda_{\mathcal{F}}/2$. That is, there is a unique shortest geodesic segment joining any two points in a plaque, and the entire geodesic segment is contained in the plaque.*

By a standard argument, there exists a finite collection $\{x_1, \ldots, x_\nu\} \subset \mathfrak{M}$ where $\varphi_{x_i}(x_i) = (0, w_{x_i})$ for $w_{x_i} \in \mathfrak{X}$, and regular foliation charts $\varphi_{x_i}\colon \overline{U}_{x_i} \to [-1,1]^n \times \mathfrak{T}_{x_i}$ satisfying the conditions of Proposition 2.6, which form an open covering of $\mathfrak{M}$. Relabel the various maps and spaces accordingly, so that $\overline{U}_i = \overline{U}_{x_i}$ and $\varphi_i = \varphi_{x_i}$ for example, with transverse spaces $\mathfrak{T}_i = \mathfrak{T}_{x_i}$ and projection maps $\pi_i = \pi_{x_i}\colon \overline{U}_i \to \mathfrak{X}_i$. Then the projection $\pi_i(U_i \cap U_j) = \mathfrak{T}_{i,j} \subset \mathfrak{T}_i$ is a clopen subset for all $1 \le i,j \le \nu$.

Moreover, without loss of generality, we can impose a uniform size restriction on the plaques of each chart. Without loss of generality, we can assume there exists $0 < \delta^{\mathcal{F}}_{\mathcal{U}} < \lambda_{\mathcal{F}}/4$ so that for all $1 \le i \le \nu$ and $\omega \in \mathfrak{T}_i$ with plaque "center point" $x_\omega = \tau_i(\omega) \stackrel{\text{def}}{=} \varphi_i^{-1}(0,\omega)$, then the plaque $\mathcal{P}_i(\omega)$ for φ_i through x_ω satisfies the uniform estimate of diameters:

$$D_{\mathcal{F}}(x_\omega, \delta^{\mathcal{F}}_{\mathcal{U}}/2) \ \subset\ \mathcal{P}_i(\omega) \ \subset\ D_{\mathcal{F}}(x_\omega, \delta^{\mathcal{F}}_{\mathcal{U}}). \tag{2}$$

For each $1 \le i \le \nu$ the set $\mathcal{T}_i = \varphi_i^{-1}(0, \mathfrak{T}_i)$ is a compact transversal to $\mathcal{F}$. Again, without loss of generality, we can assume that the transversals $\{\mathcal{T}_1, \ldots, \mathcal{T}_\nu\}$ are pairwise disjoint, so there exists a constant $0 < \epsilon_1 < \delta^{\mathcal{F}}_{\mathcal{U}}$ such that

$$d_{\mathcal{F}}(x,y) \ge \epsilon_1 \quad \text{for } x \ne y \ , x \in \mathcal{T}_i \ ,\ y \in \mathcal{T}_j \ ,\ 1 \le i,j \le \nu. \tag{3}$$

In particular, this implies that the centers of disjoint plaques on the same leaf are separated by distance at least ϵ_1.

We assume in the following that a regular foliated covering of $\mathfrak{M}$ as in Proposition 2.6 has been chosen. Let $\mathcal{U} = \{U_1, \ldots, U_\nu\}$ denote the corresponding open covering of $\mathfrak{M}$. We can assume that the spaces $\mathfrak{T}_i$ form a *disjoint clopen covering* of $\mathfrak{X}$, so that $\mathfrak{X} = \mathfrak{T}_1 \,\dot{\cup} \cdots \dot{\cup}\, \mathfrak{T}_\nu$.

A *regular covering* of $\mathfrak{M}$ is a finite covering $\{\varphi_i\colon U_i \to (-1,1)^n \times \mathfrak{T}_i \mid 1 \le i \le \nu\}$ by foliation charts which satisfies these conditions.

A map $f\colon \mathfrak{M} \to \mathfrak{M}'$ between foliated spaces is said to be a *foliated map* if the image of each leaf of $\mathcal{F}$ is contained in a leaf of $\mathcal{F}'$. If $\mathfrak{M}'$ is a matchbox manifold, then each leaf of $\mathcal{F}$ is path connected, so its image is path connected, hence must be contained in a leaf of $\mathcal{F}'$. Thus we have:

LEMMA 2.7. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be matchbox manifolds, and $h\colon \mathfrak{M}' \to \mathfrak{M}$ a continuous map. Then h maps the leaves of $\mathcal{F}'$ to leaves of $\mathcal{F}$. In particular, any homeomorphism $h\colon \mathfrak{M} \to \mathfrak{M}'$ of matchbox manifolds is a foliated map.* □

A *leafwise path* is a continuous map $\gamma\colon [0,1] \to \mathfrak{M}$ such that there is a leaf L of $\mathcal{F}$ for which $\gamma(t) \in L$ for all $0 \leq t \leq 1$. If $\mathfrak{M}$ is a matchbox manifold, and $\gamma\colon [0,1] \to \mathfrak{M}$ is continuous, then γ is a leafwise path by Lemma 2.7. In the following, we will assume that all paths are piecewise differentiable.

3. Holonomy

The holonomy pseudogroup of a smooth foliated manifold $(M, \mathcal{F})$ generalizes the induced dynamical systems associated to a section of a flow. The holonomy pseudogroup for a matchbox manifold $(\mathfrak{M}, \mathcal{F})$ is defined analogously to the smooth case.

A pair of indices (i, j), $1 \leq i, j \leq \nu$, is said to be *admissible* if the *open* coordinate charts satisfy $U_i \cap U_j \neq \emptyset$. For (i, j) admissible, define clopen subsets $\mathfrak{D}_{i,j} = \pi_i(U_i \cap U_j) \subset \mathfrak{T}_i \subset \mathfrak{X}$. The convexity of foliation charts imply that plaques are either disjoint, or have connected intersection. This implies that there is a well-defined homeomorphism $h_{j,i}\colon \mathfrak{D}_{i,j} \to \mathfrak{D}_{j,i}$ with domain $D(h_{j,i}) = \mathfrak{D}_{i,j}$ and range $R(h_{j,i}) = \mathfrak{D}_{j,i}$.

The maps $\mathcal{G}_{\mathcal{F}}^{(1)} = \{h_{j,i} \mid (i,j) \text{ admissible}\}$ are the transverse change of coordinates defined by the foliation charts. By definition they satisfy $h_{i,i} = Id$, $h_{i,j}^{-1} = h_{j,i}$, and if $U_i \cap U_j \cap U_k \neq \emptyset$ then $h_{k,j} \circ h_{j,i} = h_{k,i}$ on their common domain of definition. The *holonomy pseudogroup* $\mathcal{G}_{\mathcal{F}}$ of $\mathcal{F}$ is the topological pseudogroup modeled on $\mathfrak{X}$ generated by the elements of $\mathcal{G}_{\mathcal{F}}^{(1)}$. The elements of $\mathcal{G}_{\mathcal{F}}$ have a standard description in terms of the "holonomy along paths", which we next describe.

A sequence $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ is *admissible*, if each pair $(i_{\ell-1}, i_\ell)$ is admissible for $1 \leq \ell \leq \alpha$, and the composition

$$h_{\mathcal{I}} = h_{i_\alpha, i_{\alpha-1}} \circ \cdots \circ h_{i_1, i_0} \tag{4}$$

has non-empty domain. The domain $\mathfrak{D}_{\mathcal{I}}$ of $h_{\mathcal{I}}$ is the *maximal clopen subset* of $\mathfrak{D}_{i_0} \subset \mathfrak{T}_{i_0}$ for which the compositions are defined.

Given any open subset $U \subset \mathfrak{D}_{\mathcal{I}}$ we obtain a new element $h_{\mathcal{I}}|U \in \mathcal{G}_{\mathcal{F}}$ by restriction. Introduce

$$\mathcal{G}_{\mathcal{F}}^* = \{h_{\mathcal{I}}|U \mid \mathcal{I} \text{ admissible } \& \ U \subset \mathfrak{D}_{\mathcal{I}}\} \subset \mathcal{G}_{\mathcal{F}} \ . \tag{5}$$

For $g \in \mathcal{G}_{\mathcal{F}}^*$ denote its domain by $\mathfrak{D}(g)$ then its range is the clopen set $\mathfrak{R}(g) = g(\mathfrak{D}(g)) \subset \mathfrak{X}$.

The orbit of a point $w \in \mathfrak{X}$ by the action of the pseudogroup $\mathcal{G}_{\mathcal{F}}$ is denoted by

$$\mathcal{O}(w) = \{g(w) \mid g \in \mathcal{G}_{\mathcal{F}}^* \ , \ w \in \mathfrak{D}(g)\} \subset \mathfrak{T}_* \ . \tag{6}$$

Given an admissible sequence $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ and any $0 \leq \ell \leq \alpha$, the truncated sequence $\mathcal{I}_\ell = (i_0, i_1, \ldots, i_\ell)$ is again admissible, and we introduce the holonomy map defined by the composition of the first ℓ generators appearing in $h_{\mathcal{I}}$,

$$h_{\mathcal{I}_\ell} = h_{i_\ell, i_{\ell-1}} \circ \cdots \circ h_{i_1, i_0} \ . \tag{7}$$

Given $\xi \in D(h_{\mathcal{I}})$ we adopt the notation $\xi_\ell = h_{\mathcal{I}_\ell}(\xi) \in \mathfrak{T}_{i_\ell}$. So $\xi_0 = \xi$ and $h_{\mathcal{I}}(\xi) = \xi_\alpha$.

Given $\xi \in D(h_{\mathcal{I}})$, let $x = x_0 = \tau_{i_0}(\xi_0) \in L_x$. Introduce the *plaque chain*

$$\mathcal{P}_{\mathcal{I}}(\xi) = \{\mathcal{P}_{i_0}(\xi_0), \mathcal{P}_{i_1}(\xi_1), \ldots, \mathcal{P}_{i_\alpha}(\xi_\alpha)\} \ . \tag{8}$$

Intuitively, a plaque chain $\mathcal{P}_{\mathcal{I}}(\xi)$ is a sequence of successively overlapping convex "tiles" in L_0 starting at $x_0 = \tau_{i_0}(\xi_0)$, ending at $x_\alpha = \tau_{i_\alpha}(\xi_\alpha)$, and with each $\mathcal{P}_{i_\ell}(\xi_\ell)$ "centered" on the point $x_\ell = \tau_{i_\ell}(\xi_\ell)$.

Recall that $\mathcal{P}_{i_\ell}(x_\ell) = \mathcal{P}_{i_\ell}(\xi_\ell)$, so we also adopt the notation $\mathcal{P}_{\mathcal{I}}(x) \equiv \mathcal{P}_{\mathcal{I}}(\xi)$.

We next associate an admissible sequence $\mathcal{I}$ to a leafwise path γ, and thus obtain the holonomy map $h_\gamma = h_{\mathcal{I}}$ defined by γ.

Let γ be a leafwise path, and $\mathcal{I}$ be an admissible sequence. For $w \in D(h_{\mathcal{I}})$, we say that $(\mathcal{I}, w)$ *covers* γ, if the domain of γ admits a partition $0 = s_0 < s_1 < \cdots < s_\alpha = 1$ such that the plaque chain $\mathcal{P}_{\mathcal{I}}(w_0) = \{\mathcal{P}_{i_0}(w_0), \mathcal{P}_{i_1}(w_1), \ldots, \mathcal{P}_{i_\alpha}(w_\alpha)\}$ satisfies

$$\gamma([s_\ell, s_{\ell+1}]) \subset int(\mathcal{P}_{i_\ell}(w_\ell)) \ , \ 0 \leq \ell < \alpha, \quad \& \quad \gamma(1) \in int(\mathcal{P}_{i_\alpha}(w_\alpha)). \tag{9}$$

It follows that $h_{\mathcal{I}}$ is well-defined, with $w_0 = \pi_{i_0}(\gamma(0)) \in D(h_{\mathcal{I}})$. The map $h_{\mathcal{I}}$ is said to define the holonomy of $\mathcal{F}$ along the path γ, and satisfies $h_{\mathcal{I}}(w_0) = \pi_{i_\alpha}(\gamma(1)) \in \mathfrak{T}_{i_\alpha}$.

Given two admissible sequences, $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ and $\mathcal{J} = (j_0, j_1, \ldots, j_\beta)$, such that both $(\mathcal{I}, w_0)$ and $(\mathcal{J}, v_0)$ cover the leafwise path $\gamma \colon [0,1] \to \mathfrak{M}$, then

$$\gamma(0) \in int(\mathcal{P}_{i_0}(w_0)) \cap int(\mathcal{P}_{j_0}(v_0)) \quad , \quad \gamma(1) \in int(\mathcal{P}_{i_\alpha}(w_\alpha)) \cap int(\mathcal{P}_{j_\beta}(v_\beta))$$

Thus both (i_0, j_0) and (i_α, j_β) are admissible, and $v_0 = h_{j_0,i_0}(w_0)$, $w_\alpha = h_{i_\alpha,j_\beta}(v_\beta)$.

The proof of the following standard observation can be found in [12].

PROPOSITION 3.1. [12] *The maps $h_{\mathcal{I}}$ and $h_{i_\alpha,j_\beta} \circ h_{\mathcal{J}} \circ h_{j_0,i_0}$ agree on their common domains.*

Let $U, U', V, V' \subset \mathfrak{X}$ be open subsets with $w \in U \cap U'$. Given homeomorphisms $h \colon U \to V$ and $h' \colon U' \to V'$ with $h(w) = h'(w)$, then h and h' have the same *germ at* w, and write $h \sim_w h'$, if there exists an open neighborhood $w \in W \subset U \cap U'$ such that $h|W = h'|W$. Note that $\sim_w$ defines an equivalence relation.

DEFINITION 3.2. *The* germ of h at w *is the equivalence class $[h]_w$ under the relation $\sim_w$. The map $h \colon U \to V$ is called a* representative *of $[h]_w$. The point w is called the source of $[h]_w$ and denoted $s([h]_w)$, while $w' = h(w)$ is called the range of $[h]_w$ and denoted $r([h]_w)$.*

Given a leafwise path γ and plaque chain $\mathcal{P}_{\mathcal{I}}(w_0)$ chosen as above, we let $h_\gamma \in \mathcal{G}^*_{\mathcal{F}}$ denote a representative of the germ $[h_{\mathcal{I}}]_{w_0}$. Then Proposition 3.1 yields:

COROLLARY 3.3. *Let γ be a leafwise path as above, and $(\mathcal{I}, w_0)$ and $(\mathcal{J}, v_0)$ be two admissible sequences which cover γ. Then $h_{\mathcal{I}}$ $h_{i_\alpha,j_\beta} \circ h_{\mathcal{J}} \circ h_{j_0,i_0}$ determine the same germinal holonomy maps, $[h_{\mathcal{I}}]_{w_0} = [h_{i_\alpha,j_\beta} \circ h_{\mathcal{J}} \circ h_{j_0,i_0}]_{w_0}$. In particular, the germ of h_γ is well-defined for the path γ.*

Two leafwise paths $\gamma, \gamma' \colon [0,1] \to \mathfrak{M}$ are homotopic if there exists a family of leafwise paths $\gamma_s \colon [0,1] \to \mathfrak{M}$ with $\gamma_0 = \gamma$ and $\gamma_1 = \gamma'$. We are most interested in the special case when $\gamma(0) = \gamma'(0) = x$ and $\gamma(1) = \gamma'(1) = y$. Then γ and γ' are *endpoint-homotopic* if they are homotopic with $\gamma_s(0) = x$ for all $0 \leq s \leq 1$, and similarly $\gamma_s(1) = y$ for all $0 \leq s \leq 1$. Thus, the family of curves $\{\gamma_s(t) \mid 0 \leq s \leq 1\}$ are all contained in a common leaf L_x and we have:

LEMMA 3.4. [12] *Let $\gamma, \gamma' \colon [0,1] \to \mathfrak{M}$ be endpoint-homotopic leafwise paths. Then the holonomy maps h_γ and $h_{\gamma'}$ admit representatives which agree on some clopen subset $U \subset \mathfrak{T}_*$. In particular, they determine the same germinal holonomy maps, $[h_{\mathcal{I}}]_{w_0} = [h_{i_\alpha,j_\beta} \circ h_{\mathcal{J}} \circ h_{j_0,i_0}]_{w_0}$.*

We next consider some properties of the pseudogroup $\mathcal{G}_{\mathcal{F}}$. First, let $W \subset \mathfrak{T}$ be an open subset, and define the restriction to W of $\mathcal{G}^*_{\mathcal{F}}$ by:

$$\mathcal{G}_W = \{g \in \mathcal{G}^*_{\mathcal{F}} \mid \mathfrak{D}(g) \subset W \ , \ \mathfrak{R}(g) \subset W\} \, . \tag{10}$$

Introduce the filtrations of $\mathcal{G}^*_{\mathcal{F}}$ by word length. For $\alpha \geq 1$, let $\mathcal{G}^{(\alpha)}_{\mathcal{F}}$ be the collection of holonomy homeomorphisms $h_{\mathcal{I}}|U \in \mathcal{G}^*_{\mathcal{F}}$ determined by admissible paths $\mathcal{I} = (i_0, \ldots, i_k)$ such that $k \leq \alpha$ and $U \subset \mathfrak{D}(h_{\mathcal{I}})$ is open. Then for each $g \in \mathcal{G}^*_{\mathcal{F}}$ there is some α such that $g \in \mathcal{G}^{(\alpha)}_{\mathcal{F}}$. Let $\|g\|$ denote the least such α, which is called the *word length* of g. Note that $\mathcal{G}^{(1)}_{\mathcal{F}}$ generates $\mathcal{G}^*_{\mathcal{F}}$.

We note the following finiteness result, whose proof is given in [14, Section 4]:

LEMMA 3.5. *Let $W \subset \mathfrak{X}$ be an open subset. Then there exists an integer α_W such that $\mathfrak{X}$ is covered by the collection $\{h_{\mathcal{I}}(W) \mid h_{\mathcal{I}} \in \mathcal{G}^{(\alpha_W)}_{\mathcal{F}}\}$.*

Finally, we recall the definition of an equicontinuous pseudogroup.

DEFINITION 3.6. *The action of the pseudogroup $\mathcal{G}_{\mathcal{F}}$ on $\mathfrak{X}$ is* equicontinuous *if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $g \in \mathcal{G}_{\mathcal{F}}^*$, if $w, w' \in D(g)$ and $d_{\mathfrak{X}}(w, w') < \delta$, then $d_{\mathfrak{X}}(g(w), g(w')) < \epsilon$. Thus, $\mathcal{G}_{\mathcal{F}}^*$ is equicontinuous as a family of local group actions.*

Further dynamical properties of the pseudogroup $\mathcal{G}_{\mathcal{F}}$ for a matchbox manifold are discussed in the papers [12, 13, 14, 28].

4. Return equivalence

For an open subset $W \subset \mathfrak{T}_*$ the induced pseudogroup $\mathcal{G}_W$ is used to represent the local dynamics of a matchbox manifold $\mathfrak{M}$. We first introduce the key concept of *return equivalence* between two such pseudogroups, and then study the properties of the equivalence relation obtained. Return equivalence is the analog for matchbox manifolds of the notion of *Morita equivalence* for foliation groupoids, which is discussed by Haefliger in [26].

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be matchbox manifolds with transversals $\mathfrak{T}_*^1$ and $\mathfrak{T}_*^2$, respectively. Given clopen subsets $U_1 \subset \mathfrak{T}_*^1$ and $U_2 \subset \mathfrak{T}_*^2$ we say that the restricted pseudogroups $\mathcal{G}_{U_1}$ and $\mathcal{G}_{U_2}$ are *isomorphic* if there exists a homeomorphism $\phi \colon U_1 \to U_2$ such that the induced map $\Phi \colon \mathcal{G}_{U_1} \to \mathcal{G}_{U_2}$ is an isomorphism. That is, for all $g \in \mathcal{G}_{U_1}$ the map $\Phi(g) = \phi \circ g \circ \phi^{-1}$ defines an element of $\mathcal{G}_{U_2}$. Conversely, for all $h \in \mathcal{G}_{U_2}$ the map $\Phi^{-1}(h) = \phi^{-1} \circ h \circ \phi$ defines an element of $\mathcal{G}_{U_1}$.

DEFINITION 4.1. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be minimal matchbox manifolds, with transversals $\mathfrak{T}_*^1$ and $\mathfrak{T}_*^2$, respectively. Given clopen subsets $W_i \subset \mathfrak{T}_i$ for $i = 1, 2$, we say that the restricted pseudogroups $\mathcal{G}_{W_i}$ are* return equivalent *if there are non-empty clopen sets $U_i \subset W_i$ and homeomorphism $\phi \colon U_1 \to U_2$ such that the induced map $\Phi \colon \mathcal{G}_{U_1} \to \mathcal{G}_{U_2}$ is an isomorphism.*

The properties of this definition are given in the following sequence of results, but we first make a general remark. Recall that if $\mathfrak{M}_i$ is minimal, then every leaf of $\mathcal{F}_i$ intersects the local section $\tau_i(W_i)$ for any open set $W_i \subset \mathfrak{T}_*^i$. As seen in the proof of Lemma 4.5 below, this property is used to show that return equivalence satisfies the transitive axiom of an equivalence relation. In contrast, recall that a matchbox manifold $\mathfrak{M}$ is *transitive* if it contains a leaf L with $\overline{L} = \mathfrak{M}$. Definition 4.1 does not define a transitive equivalence relation for transitive spaces, as can be seen for particular transitive matchbox manifolds and suitably chosen clopen subsets.

LEMMA 4.2. *Let $\mathfrak{M}$ be a minimal matchbox manifold with transversal $\mathfrak{T}_*$. Let $W, W' \subset \mathfrak{T}_*$ be non–empty clopen subsets, then $\mathcal{G}_W$ and $\mathcal{G}_{W'}$ are return equivalent.*

Proof. Let $w \in W$ with $x = \tau(w)$. Let $w' \in W'$ be a point such that $y = \tau(w') \cap L_x$ which exists as L_x is dense in $\mathfrak{M}$. Choose a path γ with $\gamma(0) = x$ and $\gamma(1) = y$, and let $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ define a plaque chain which covers γ, with $W \subset \mathfrak{T}_{i_0}$ and $W' \subset \mathfrak{T}_{i_\alpha}$. Observe that $\alpha \leq \alpha_W$. Let $g = h_{\mathcal{I}}$ denote the holonomy transformation defined by the admissible sequence $\mathcal{I}$, with domain a clopen set $\mathfrak{D}(g) \subset \mathfrak{T}_{i_0}$. Chose a clopen set U with $w \in U \subset W \cap \mathfrak{D}(g)$ and $V = h_g(U) \subset W' \cap \mathfrak{T}_{i_\alpha}$. Then the restriction $\phi = h_g | U \colon U \to V$ is a homeomorphism which satisfies the conditions above, so induces an isomorphism of pseudogroups, $\Phi \colon \mathcal{G}_U \to \mathcal{G}_V$. Thus, $\mathcal{G}_W$ and $\mathcal{G}_{W'}$ are return equivalent. □

COROLLARY 4.3. *Let $\mathfrak{M}$ be a minimal matchbox manifold with transversal $\mathfrak{T}_*$. Let $W \subset \mathfrak{T}_*$ be a non–empty clopen subset, then $\mathcal{G}_{\mathcal{F}}$ and $\mathcal{G}_W$ are return equivalent.*

LEMMA 4.4. *Let $\mathfrak{M}$ be a minimal matchbox manifold, and suppose we are given regular coverings $\{\varphi_i \colon U_i \to (-1,1)^n \times \mathfrak{T}_i \mid 1 \leq i \leq \nu\}$ and $\{\varphi'_j \colon U'_j \to (-1,1)^n \times \mathfrak{T}'_j \mid 1 \leq j \leq \nu'\}$ of $\mathfrak{M}$, with transversals $\mathfrak{T}_*$ and $\mathfrak{T}'_*$ respectively. Let $W \subset \mathfrak{T}_*$ and $W' \subset \mathfrak{T}'_*$ be non–empty clopen subsets. Let $\mathcal{G}_W$ denote the restricted pseudogroup on W for the first covering, and $\mathcal{G}_{W'}$ the restricted pseudogroup for the second covering. Then $\mathcal{G}_W$ and $\mathcal{G}_{W'}$ are return equivalent.*

Proof. Let $w \in W$ with $x = \tau(w)$. Let $w' \in W'$ be a point such that $y = \tau'(w') \cap L_x$ which exists as L_x is dense in $\mathfrak{M}$. Choose a path γ with $\gamma(0) = x$ and $\gamma(1) = y$, and let $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$

define a plaque chain which covers γ, with $W \subset \mathfrak{T}_{i_0}$. Let $g = h_{\mathcal{I}}$ be the holonomy map defined by the admissible sequence $\mathcal{I}$, with domain a clopen set $\mathfrak{D}(g) \subset \mathfrak{T}_{i_0}$. Then there exists $i_{\alpha'}$ such that $y \in \mathcal{T}'_{j_{\alpha'}} = \tau'(\mathfrak{T}'_{j_{\alpha'}})$. Thus, $y \in U_{i_\alpha} \cap U'_{j_{\alpha'}}$. Also, we have that $W' \subset \mathfrak{T}'_{j_{\alpha'}}$, and define $W'' = \pi_{i_\alpha}(\pi^{-1}_{j_{\alpha'}}(W') \subset \mathfrak{T}_{i_\alpha}$. Chose a clopen set U with $w \in U \subset W \cap \mathfrak{D}(g)$ and $V' = h_g(U) \subset W'$.

The composition $\phi = \pi'_{j_{\alpha'}} \circ \tau_{i_\alpha} \circ h_g|U \colon U \to V \subset \mathfrak{T}'_{j_{\alpha'}}$ is a homeomorphism which induces an isomorphism of restricted pseudogroups, $\Phi \colon \mathcal{G}_U \to \mathcal{G}_{V'}$. Thus, $\mathcal{G}_W$ and $\mathcal{G}_{W'}$ are return equivalent. □

LEMMA 4.5. *Let $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$ be minimal matchbox manifolds with regular coverings defining transversals $\mathfrak{T}^1_*, \mathfrak{T}^2_*, \mathfrak{T}^3_*$ respectively. Suppose there exists non-empty clopen subsets $W_1 \subset \mathfrak{T}^1_*$ and $W_2 \subset \mathfrak{T}^2_*$ such that the restricted pseudogroups $\mathcal{G}_{W_1}$ and $\mathcal{G}_{W_2}$ are return equivalent, and that there exists non-empty clopen subsets $W'_2 \subset \mathfrak{T}^2_*$ and $W_3 \subset \mathfrak{T}^3_*$ such that the restricted pseudogroups $\mathcal{G}_{W'_2}$ and $\mathcal{G}_{W_3}$ are return equivalent. Then $\mathcal{G}_{W_1}$ and $\mathcal{G}_{W_3}$ are return equivalent.*

Proof. By definition, there exists non–empty clopen sets $U_i \subset W_i$ $(i = 1, 2)$ and a homeomorphism $\phi_1 \colon U_1 \to U_2$, which induces a pseudogroup isomorphism $\Phi_1 \colon \mathcal{G}_{U_1} \to \mathcal{G}_{U_2}$.

Similarly, there exists non–empty clopen subsets, $V_2 \subset W'_2 \subset \mathfrak{T}^2_*$ and $V_3 \subset W_3 \subset \mathfrak{T}^3_*$ and a homeomorphism $\phi_2 \colon V_2 \to V_3$, which induces a pseudogroup isomorphism $\Phi_2 \colon \mathcal{G}_{V_2} \to \mathcal{G}_{V_3}$.

Choose $w_2 \in W_2$ and set $y = \tau^2(w_2) \in \mathcal{T}^2_*$, then by minimality of $\mathfrak{M}_2$ the leaf L_y containing y intersects the transverse set $\tau^2(W'_2)$ in a point y'. Choose a path γ with $\gamma(0) = y$ and $\gamma(1) = y'$. Then the holonomy for $\mathcal{F}_2$ along γ defines a homeomorphism $h_\gamma \colon X \to X'$ for clopen sets satisfying $y \in X \subset U_2 \subset W_2$ and $y' \in X' \subset W'_2$.

Set $Y = \phi_1^{-1}(X)$ and $Z = \phi_2(X')$. Then the composition $\phi_3 = \phi_2 \circ h_\gamma \circ \phi_2 \colon Y \to Z$ is a homeomorphism between the clopen subsets $Y \subset W_1$ and $Z \subset W_3$ which induces an isomorphism of pseudogroups, $\Phi_3 \colon \mathcal{G}_Y \to \mathcal{G}_Z$. Thus, $\mathcal{G}_{W_1}$ and $\mathcal{G}_{W_3}$ are return equivalent, which was to be shown. □

PROPOSITION 4.6. Return equivalence *is an equivalence relation on the class of restricted pseudogroups obtained from minimal matchbox manifolds.*

Proof. It is immediate that return equivalence is reflexive and symmetric relation. That return equivalence is transitive follows from Lemmas 4.2, 4.4 and 4.5. □

DEFINITION 4.7. *Two minimal matchbox manifolds $\mathfrak{M}_i$ for $i = 1, 2$, are* return equivalent *if there exists regular coverings of $\mathfrak{M}_i$ and non–empty, clopen transversals W_i for each covering so that the restricted pseudogroups $\mathcal{G}_{W_i}$ for $i = 1, 2$ are return equivalent.*

We conclude this section by showing that homeomorphism implies return equivalence.

THEOREM 4.8. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be minimal matchbox manifolds. Suppose that there exists a homeomorphism $h \colon \mathfrak{M}_1 \to \mathfrak{M}_2$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent.*

Proof. First note that the homeomorphism h is a foliated map by Lemma 2.7. This implies that h is a homeomorphism between the leaves of $\mathfrak{M}_1$ and the leaves of $\mathfrak{M}_2$, and thus the leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same dimensions. However, we do not assume that h is smooth when restricted to leaves.

Choose a regular covering $\mathcal{U} = \{\varphi_i \colon U_i \to (-1, 1)^n \times \mathfrak{T}_i \mid 1 \leq i \leq \nu\}$ for $\mathfrak{M}_1$ with transversal $\mathfrak{T}_*$.

Also, choose a regular covering $\{\varphi'_j \colon U'_j \to (-1, 1)^n \times \mathfrak{T}'_j \mid 1 \leq j \leq \nu'\}$ of $\mathfrak{M}_2$, with transversal $\mathfrak{T}'_*$.

Consider the open covering of $\mathfrak{M}_1$ by the inverse images $\mathcal{V} = \{V_j = h^{-1}(U'_j) \mid 1 \leq j \leq \nu'\}$. Let $\epsilon_V > 0$ be a Lebesgue number for this covering.

Then choose a regular covering $\mathcal{U}'' = \{\varphi''_l \colon U''_k \to (-1, 1)^n \times \mathfrak{T}_k \mid 1 \leq k \leq \nu''\}$ for $\mathfrak{M}_1$ with transversal $\mathfrak{T}''_*$ as in Proposition 2.6, with constant $\epsilon_{\mathfrak{M}} < \epsilon_{\mathcal{V}}$ so that each chart each $\overline{U}''_k \subset B_{\mathfrak{M}}(z_k, \epsilon_{\mathcal{V}})$ where z_k is the "center point" for V_k. It follows that for each $1 \leq k \leq \nu''$ there exists $1 \leq \ell_k \leq \nu'$ with $\overline{U}''_k \subset V_{\ell_k}$, and thus $h(\overline{U}''_k) \subset U'_{\ell_k}$.

Choose a clopen set $X \subset \mathfrak{T}_1$ and a clopen set $Y \subset \mathfrak{T}_1''$. Then by Lemma 4.4, the restricted pseudogroups $\mathcal{G}_X$ and $\mathcal{G}_Y$ are return equivalent. That is, there exists clopen subsets $X' \subset X$ and $Y' \subset Y$ and a homeomorphism $\phi_1 \colon X' \to Y'$ which induces an isomorphism $\Phi_1 \colon \mathcal{G}_{X'} \to \mathcal{G}_{Y'}$.

Then the composition $\phi = \pi'_{\ell_1} \circ h \circ \tau_1 \circ \phi_1 \colon X' \to Z' \subset \mathfrak{T}'_{\ell_1}$ is well-defined, and is a homeomorphism onto the clopen subset Z', and induces an isomorphism $\Phi \colon \mathcal{G}_{X'} \to \mathcal{G}_{Z'}$. Set $Z = \mathfrak{T}'_{\ell_1}$, then it follows that $\mathcal{G}_X$ and $\mathcal{G}_Z$ are return equivalent, and so $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent. □

5. Foliated bundles

A matchbox manifold $\mathfrak{M}$ has the structure of a *foliated bundle* if there is a closed connected manifold B of dimension $n \geq 1$, and a fibration map $\pi \colon \mathfrak{M} \to B$ such that for each leaf $L \subset \mathfrak{M}$, the restriction $\pi \colon L \to B$ is a covering map. For each $b \in B$, the fiber $\mathfrak{F}_b = \pi^{-1}(b)$ is then a totally disconnected compact space. If $\mathfrak{F}_b$ is a Cantor space, then we say that $\mathfrak{M}$ is a *foliated Cantor bundle*.

The standard texts on foliations, such as [10] and [9], discuss the suspension construction for foliated manifolds, which adapts to the context of foliated spaces without difficulties. Also, the seminal work by Kamber and Tondeur [30] discusses general foliated bundles (there referred to as flat bundles).

In this section, we obtain conditions which are sufficient to imply that return equivalence implies homeomorphism, which yields a converse to Theorem 4.8 for foliated Cantor bundles. These results are used in the following sections to prove Theorem 1.4.

We recall some of the basic properties of the construction of foliated bundles, as needed in the following. Let $\mathfrak{F}$ be a compact topological space, and let $\mathrm{Homeo}(\mathfrak{F})$ denote its group of homeomorphisms. Given a closed manifold B, choose a basepoint $b_0 \in B$ and let $\Lambda = \pi_1(B, b_0)$ be the fundamental group, whose elements are represented by the endpoint-homotopic classes of closed curves in B with endpoints at b_0. Let $\widetilde{B}$ denote the universal covering of B, defined as the endpoint-homotopy classes of paths in B starting at b_0. The group Λ acts on the right on $\widetilde{B}$ by pre-composing paths representing elements of $\widetilde{B}$ with paths representing elements of Λ. This yields the action of Λ on $\widetilde{B}$ by deck transformations. Let $\varphi \colon \Lambda \to \mathrm{Homeo}(\mathfrak{F})$ be a representation, which defines a left-action of Λ on $\mathfrak{F}$ by homeomorphisms. Define the quotient space

$$\mathfrak{M}_\varphi = (\widetilde{B} \times \mathfrak{F})/\{(x \cdot \gamma, \omega) \sim (x, \varphi(\gamma) \cdot \omega\} \quad , \quad x \in \widetilde{B} \ , \ \omega \in \mathfrak{F} \ , \ \gamma \in \Lambda \tag{11}$$

The images of the slices $\widetilde{B} \times \{\omega\} \subset \widetilde{B} \times \mathfrak{F}$ in $\mathfrak{M}_\varphi$ form the leaves of the suspension foliation $\mathcal{F}_\varphi$ and gives $\mathfrak{M}_\varphi$ the structure of a foliated space. The projection $\widetilde{\pi} \colon \widetilde{B} \times \mathfrak{F} \to \widetilde{B}$ is equivariant with respect to the action of Λ, so descends to a fibration map $\pi \colon \mathfrak{M}_\varphi \to B$. Thus, $\mathfrak{M}_\varphi$ is a foliated bundle. The next result implies that all foliated bundles are of this form.

PROPOSITION 5.1. *Let $\pi \colon \mathfrak{M} \to B$ be a foliated bundle, $b_0 \in B$ a basepoint, and let $\mathfrak{F}_0 = \pi^{-1}(b_0)$ be the fiber. Then there is a well-defined* global holonomy *map $\varphi \colon \Lambda \to \mathrm{Homeo}(\mathfrak{F}_0)$ and a natural homeomorphism of foliated bundles, $\Phi_{\mathcal{F}} \colon \mathfrak{M}_\varphi \to \mathfrak{M}$.*

Proof. We sketch the construction of the maps φ and $\Phi_{\mathcal{F}}$, as the construction is standard. Given $\lambda \in \Lambda$, let $\gamma \colon [0,1] \to B$ denote a continuous path with $\gamma(0) = \gamma(1) = b_0$ representing λ. Given $\omega \in \mathfrak{F}_0$ let L_ω be the leaf of $\mathcal{F}$ containing ω, then $\pi \colon L_\omega \to B$ is a covering, so there exists a lift $\widetilde{\gamma}_\omega$ of γ with $\widetilde{\gamma}_\omega(0) = \omega$. Then set $\varphi(\lambda) \cdot \omega = \widetilde{\gamma}(1)$. By the properties of holonomy for foliations, this defines a homeomorphism of the fiber $\mathfrak{F}_0$. The properties of path lifting implies that $\varphi \colon \Lambda \to \mathrm{Homeo}(\mathfrak{F}_0)$ is a homomorphism.

Define $\Phi_{\mathcal{F}} \colon \widetilde{B} \times \mathfrak{F}_0 \to \mathfrak{M}$ as follows: for $\omega \in \mathfrak{F}_0$ and $\gamma_0 \in \widetilde{B}$ with $\gamma_0(0) = b_0$ and $\gamma_0(1) = b$, then let γ_ω be the lift of γ_0 to the leaf $L_\omega \subset \mathfrak{M}$ with $\gamma_\omega(0) = \omega$. Set $\widetilde{\Phi}_{\mathcal{F}}(b, \omega) = \gamma_\omega(1) \in \mathfrak{M}$. Note that by the definition of φ we have, for all $\gamma \in \pi_1(B, b_0)$, that $\widetilde{\Phi}_{\mathcal{F}}(x \cdot \gamma, \omega) = \widetilde{\Phi}_{\mathcal{F}}(x, \varphi(\gamma) \cdot \omega) \in \mathfrak{M}$ so $\widetilde{\Phi}_{\mathcal{F}}$ descends to a map $\Phi_{\mathcal{F}} \colon \mathfrak{M}_\varphi \to \mathfrak{M}$, which is checked to be a homeomorphism. □

Next we consider two types of maps between foliated bundles. The following results are proved using the path-lifting property of foliated bundles, in a manner similar to the proof of Proposition 5.1.

First, let $f\colon B' \to B$ be a diffeomorphism of closed manifolds B and B'. Let $b_0 \in B$ be a basepoint, and let $b_0' = f^{-1}(b_0) \in B_0'$ be the basepoint for B_0'. Set $\Lambda' = \pi_1(B_0', b_0')$, and let $f_\#\colon \Lambda' \to \Lambda$ be the induced isomorphism of fundamental groups.

Given a representation $\varphi\colon \Lambda \to \mathrm{Homeo}(\mathfrak{F})$, set $\varphi' = \varphi \circ f_\#\colon \Lambda' \to \mathrm{Homeo}(\mathfrak{F})$. Then we obtain an associated foliated bundle

$$\mathfrak{M}'_{\varphi'} = (\widetilde{B}' \times \mathfrak{F})/\{(x' \cdot \gamma', \omega) \sim (x', \varphi'(\gamma')(\omega)\} \quad , \quad x' \in \widetilde{B}' \ , \ \omega \in \mathfrak{C} \ , \ \gamma' \in \Lambda'.$$

PROPOSITION 5.2. *There is a foliated bundle isomorphism $F\colon \mathfrak{M}'_{\varphi'} \to \mathfrak{M}_\varphi$.* □

Next, let $h\colon \mathfrak{F}' \to \mathfrak{F}$ be a homeomorphism, and let $\varphi\colon \Lambda \to \mathrm{Homeo}(\mathfrak{F})$ be a representation. Define the representation $\varphi^h\colon \Lambda \to \mathrm{Homeo}(\mathfrak{F}')$ by setting $\varphi^h = h^{-1} \circ \varphi \circ h$. Then have

PROPOSITION 5.3. *There is a foliated bundle isomorphism $F\colon \mathfrak{M}_{\varphi^h} \to \mathfrak{M}_\varphi$.* □

In the case of foliated Cantor bundles, there is yet another method to induce homeomorphisms between their total spaces. This uses the following notion:

DEFINITION 5.4. *Let $\mathfrak{M}_\varphi$ be a foliated Cantor bundle, with projection map $\pi\colon \mathfrak{M}_\varphi \to B$, $b_0 \in B$ a basepoint, Cantor fiber $\mathfrak{F}_0 = \pi^{-1}(b_0)$, and global holonomy map $\varphi\colon \Lambda \to \mathrm{Homeo}(\mathfrak{F}_0)$. A clopen subset $W \subset \mathfrak{F}_0$ is* collapsible *if $\tau(W)$ is a fiber of a bundle projection $\pi'\colon \mathfrak{M}_\varphi \to B'$ such that there is a finite covering map $\pi_W\colon B' \to B$ that makes the following diagram commute:*

(12)
$$\begin{tikzcd} \mathfrak{M}_\varphi \arrow[d, "\pi"'] \arrow[dr, "\pi'"] & \\ B & B' \arrow[l, "\pi_W"] \end{tikzcd}$$

We say that $\mathfrak{M}_\varphi$ is infinitely collapsible *if every clopen subset of $W \subset \mathfrak{F}_0$ contains a collapsible clopen subset.*

The following gives effective criteria for when a clopen set is collapsible.

PROPOSITION 5.5. *Let $\mathfrak{M}_\varphi$ be a foliated Cantor bundle, with projection map $\pi\colon \mathfrak{M}_\varphi \to B$, $b_0 \in B$ a basepoint, Cantor fiber $\mathfrak{F}_0 = \pi^{-1}(b_0)$, and global holonomy map $\varphi\colon \Lambda \to \mathrm{Homeo}(\mathfrak{F}_0)$. Then the clopen subset $W \subset \mathfrak{F}_0$ is collapsible if and only if the collection $\{\varphi(\gamma) \cdot W \mid \gamma \in \Lambda\}$ is a finite partition of $\mathfrak{F}_0$ into clopen subsets.*

Proof. Suppose that the clopen subset $W \subset \mathfrak{F}_0$ is collapsible, and there is a diagram (12). Label the points in the preimage of b_0 by $X_W = \pi_W^{-1}(b_0) = \{b_1, \ldots, b_k\}$, and the corresponding fibers of π' by $W_i = (\pi')^{-1}(b_i) \subset \mathfrak{F}_0$ for $1 \leq i \leq b_k$. We can assume without loss that that $W = W_1$. It follows from the commutativity of the diagram (12) that these sets form a clopen partition of $\mathfrak{X}_0$, $\mathfrak{X}_0 = W_1 \cup \cdots \cup W_k$. Let $\Lambda_W \subset \Lambda$ be the covering group for π_W which is the image of the map

$$(\pi_W)_\#\colon \pi_1(B', b_1) \to \pi_1(B, b_0) = \Lambda.$$

Then the monodromy action of Λ_W on the fiber $\mathfrak{F}_0$ permutes the clopen sets W_i for $1 \leq i \leq k$. It follows that there is a homeomorphism

$$\mathfrak{M}_\varphi \cong (\widetilde{B} \times \mathfrak{F}_0)/\{(x \cdot \gamma, \omega) \sim (x, \varphi(\gamma) \cdot \omega\} \quad , \quad x \in \widetilde{B} \ , \ \omega \in W_1 \ , \ \gamma \in \Lambda_W. \tag{13}$$

Conversely, suppose that $W \subset \mathfrak{F}_0$ is a clopen set, such that the collection $\{\varphi(\gamma) \cdot W \mid \gamma \in \Lambda\}$ is a finite partition of $\mathfrak{F}_0$ into clopen subsets. Set $W_1 = W$, and choose $\gamma_i \in \Lambda$ for $1 < i \leq k$ so that for $W_i = \varphi(\gamma_i) \cdot W$, the collection $\mathcal{W} = \{W_1, W_2, \ldots, W_k\}$ is a clopen partition of $\mathfrak{X}_0$. Then define

$$\Lambda_W = \{\gamma \in \Lambda \mid \varphi(\gamma) \cdot W = W\}. \tag{14}$$

Note that as the collection of clopen sets $\mathcal{W}$ is permuted by the action of Λ, the subgroup Λ_W has finite index. Let $\pi_W\colon B' \to B$ be the finite covering of B associated to Λ_W. Then projection along the fiber in the decomposition of $\mathfrak{M}$ in (13) yields a projection map $\pi'\colon \mathfrak{M}_\varphi \to B'$ so that the diagram (12) commutes, as was to be shown. □

Next consider the properties of return equivalence and collapsibility in the context of foliated Cantor bundles. For $i = 1,2$, let $\mathfrak{M}_{\varphi_i}$ be minimal foliated Cantor bundles over the common base B. Let $\pi_i \colon \mathfrak{M}_{\varphi_i} \to B$ be the corresponding projection maps, $b_0 \in B$ a basepoint, and define the Cantor fibers $\mathfrak{F}_i = \pi_i^{-1}(b_0)$, with global holonomy maps $\varphi_i \colon \Lambda \to \mathrm{Homeo}(\mathfrak{F}_i)$. Assume there clopen sets $W_i \subset \mathfrak{F}_i$ such that $\mathcal{G}_{W_1}$ and $\mathcal{G}_{W_2}$ are return equivalent. Let $U_i \subset W_i$ be clopen sets and $\phi \colon U_1 \to U_2$ be a homeomorphism which induces an isomorphism of the restricted pseudogroups, $\Phi \colon \mathcal{G}_{U_1} \to \mathcal{G}_{U_2}$.

LEMMA 5.6. *Assume that the clopen set U_1 is collapsible, then the clopen set U_2 is collapsible.*

Proof. By Proposition 5.5, it suffices to show that the collection $\{\varphi_2(\gamma) \cdot U_2 \mid \gamma \in \Lambda\}$ is a clopen partition of $\mathfrak{F}_0$. First, let $\gamma \in \Lambda$ satisfy $U_2 \cap \varphi_2(\gamma) \cdot U_2 \neq \emptyset$. By assumption, $\varphi_2(\gamma)$ is conjugate to some $g \in \mathcal{G}_{U_1}$ for which $U_1 \cap g \cdot U_1 \neq \emptyset$. As U_1 is collapsible, this implies $g \cdot U_1 = U_1$, and thus $\varphi_2(\gamma) \cdot U_2 = U_2$.

Next, suppose there exists $\gamma_1, \gamma_2 \in \Lambda$ such that there exists $\{\varphi_2(\gamma_1) \cdot U_2\} \cap \{\varphi_2(\gamma_1) \cdot U_2\} \neq \emptyset$. Then $U_2 \cap \varphi_2(\gamma_1^{-1}\gamma_2) \cdot U_2 \neq \emptyset$, so by the above we have $\varphi_2(\gamma_1^{-1}\gamma_2) \cdot U_2 = U_2$ and thus $\varphi_2(\gamma_1) \cdot U_2 = \varphi_2(\gamma_1) \cdot U_2$.

The action φ_2 is assumed to be minimal, so the collection $\{\varphi_2(\gamma) \cdot U_2 \mid \gamma \in \Lambda\}$ is an open covering of the compact space $\mathfrak{F}_2$, and thus admits a finite subcovering. The covering is by disjoint closed sets, hence is a clopen covering, as was to be shown. □

The proof of Lemma 5.6 shows that

$$\Lambda_{U_1} \equiv \{\gamma \in \Lambda \mid \varphi_1(\gamma) \cdot U_1 = U_1\} = \{\gamma \in \Lambda \mid \varphi_2(\gamma) \cdot U_2 = U_2\} \equiv \Lambda_{U_2}. \tag{15}$$

Proposition 5.3 for $\varphi_2 | U_2 = \phi \circ \varphi_1 | U_1 \circ \phi^{-1}$ and the decompositions (13) for $\mathfrak{M}_1$ and $\mathfrak{M}_2$ then yield:

PROPOSITION 5.7. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be minimal foliated Cantor bundles over B as above, which have conjugate restricted pseudogroups on the collapsible clopen set $U_1 \subset \mathfrak{F}_1$. Then $\phi \colon U_1 \to U_2$ induces a homeomorphism $\widehat{\Phi} \colon \mathfrak{M}_1 \to \mathfrak{M}_2$.* □

We next consider the applications of these ideas to proving that two minimal foliated Cantor bundles are homeomorphic. Assume we are given, for $i = 1,2$, minimal foliated Cantor bundles $\mathfrak{M}_{\varphi_i}$. Let B_i denote the associated base manifolds, with basepoint $b_i \in B_i$, $\Lambda_i = \pi_1(B_i, b_i)$, and representations $\varphi_i \colon \Lambda \to \mathrm{Homeo}(\mathfrak{F}_i)$. Assume also that $\mathfrak{M}_{\varphi_1}$ and $\mathfrak{M}_{\varphi_2}$ are return equivalent, so there exists clopen subsets $U_1 \subset \mathfrak{F}_1$ and $U_2 \subset \mathfrak{F}_2$ and a homeomorphism $\phi \colon U_1 \to U_2$ which conjugates $\mathcal{G}_{U_1}$ to $\mathcal{G}_{U_2}$. Assume that U_1 is collapsible. Then observe that the proof of Lemma 5.6 does not require the base manifolds be the same, so we conclude that U_2 is also collapsible. Thus, for $i = 1,2$, we can define the isotropy subgroups and their restricted actions

$$\Lambda_{U_i} = \{\gamma \in \Lambda_i \mid \varphi_i(\gamma) \cdot U_i = U_i\} \quad , \quad \varphi_i \colon \Lambda_{U_i} \to \mathrm{Homeo}(U_i) \tag{16}$$

and the homeomorphism ϕ induces a conjugation on the *images* of these maps. Note that each subgroup $\Lambda_{U_i} \subset \Lambda_i$ has finite index, though it need not be normal. Let B_i' be the finite covering associated to the subgroup $\Lambda_{U_i} \subset \Lambda_i$.

DEFINITION 5.8. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be return equivalent, minimal matchbox manifolds. We say that they have a* common base *if there is a homeomorphism $\phi \colon U_1 \to U_2$ between clopen subsets which induces an isomorphism $\Phi \colon \mathcal{G}_{U_1} \to \mathcal{G}_{U_2}$, and there is a homeomorphism $h \colon B_1' \to B_2'$ such that for the induced map on fundamental groups, $h_\# \colon \Lambda_{U_1} = \pi_1(B_1', b_1') \to \pi_1(B_2', b_2') = \Lambda_{U_2}$ we have*

$$\varphi_2(h_\#(\gamma)) \cdot \omega = \phi(\varphi_1(\gamma) \cdot \phi^{-1}(\omega)) \ , \ \text{for all } \gamma \in \Lambda_1 \ , \ \omega \in U_1. \tag{17}$$

The following technical result is used in Section 7 to establish the "common base" hypothesis.

PROPOSITION 5.9. *Let $\pi_1 \colon \mathfrak{M}_1 \to B_1$ and $\mathfrak{M}_2$ be minimal foliated Cantor bundles, and suppose that there exist a simply connected leaf $L_2 \subset \mathfrak{M}_2$. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent, clopen sets $U_i \subset \mathfrak{F}_i$ with U_1 collapsible, and a homeomorphism $\phi \colon U_1 \to U_2$ which induces an isomorphism $\Phi \colon \mathcal{G}_{U_1} \to \mathcal{G}_{U_2}$, then there exists an isomorphism on fundamental groups,*

$$\mathcal{H}_\phi \colon \Lambda_{U_1} = \pi_1(B_1', b_1') \to \pi_1(B_2', b_2') = \Lambda_{U_2} \tag{18}$$

such that

$$\varphi_2(\mathcal{H}_\phi(\gamma)) \cdot \omega = \phi(\varphi_1(\gamma) \cdot \phi^{-1}(\omega)) \ , \ \text{for all } \gamma \in \Lambda_{U_1} \ , \ \omega \in U_2. \tag{19}$$

Proof. We are given a representation $\varphi_2 \colon \Lambda_{U_2} \to \mathrm{Homeo}(U_2)$ whose image is the restricted pseudogroup $\mathcal{G}_{U_2}$. Suppose that $\gamma \in \Lambda_{U_2}$ is mapped by φ_2 to the identity, then γ defines a closed loop in B_i which lifts to a closed loop in each leaf that intersects U_2. In particular, as $\mathcal{F}_2$ has all leaves dense, it defines a closed loop $\widetilde{\gamma} \subset L_2$. As L_2 is simply connected, the lift $\widetilde{\gamma}$ must be homotopic to a constant map. The restricted projection $\pi_2 \colon L_2 \to B_2$ is a covering map, so γ is also homotopic to a constant, hence is the trivial element of Λ_{U_2}. Thus the map φ_2 is injective on Λ_{U_2}.

Now, use the conjugation defined by ϕ between the images $\varphi_1(\Lambda_{U_1})$ and $\varphi_2(\Lambda_{U_2})$ to define $\mathcal{H}_\phi$. Then the property (19) holds by definition. □

6. Equicontinuous matchbox manifolds

The dynamics and topology of equicontinuous matchbox manifolds are studied in the work [12] by the first two authors. We recall three main results from this paper, which will be used in the proof of Theorem 1.4. Theorem 6.6 below may be of interest on its own.

Recall that a matchbox manifold $\mathfrak{M}$ is equicontinuous, as stated in Definition 3.6, if the action of $\mathcal{G}_{\mathcal{F}}$ on the transversal space $\mathfrak{X}_*$ is equicontinuous for the metric $d_{\mathfrak{X}}$.

THEOREM 6.1 (Theorem 4.2, [12])**.** *An equicontinuous matchbox manifold $\mathfrak{M}$ is minimal.*

The next result is a direct consequence of Theorem 8.9 of [12]:

THEOREM 6.2. *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a minimal foliated Cantor bundle. That is, there exists a Cantor space $\mathfrak{F}_0$, a compact triangulated topological manifold B with basepoint $b_0 \in B$, fundamental group $\Lambda = \pi_1(B, b_0)$, and representation $\varphi \colon \Lambda \to \mathrm{Homeo}(\mathfrak{F}_0)$ such that $\mathfrak{M} \cong \mathfrak{M}_\varphi$. Moreover, there is a metric $d_{\mathfrak{F}_0}$ on $\mathfrak{F}_0$ such that the minimal action of φ is equicontinuous with respect to $d_{\mathfrak{F}_0}$.*

The third main result follows from Theorem 8.9 of [12] and its proof:

THEOREM 6.3. *Let $\mathfrak{M}_\varphi$ be a suspension matchbox manifold, whose global holonomy is a minimal action $\varphi \colon \Lambda \to \mathrm{Homeo}(\mathfrak{F}_0)$ which is equicontinuous with respect to the metric $d_{\mathfrak{F}_0}$ on $\mathfrak{F}_0$. Then for any open set $W \subset \mathfrak{F}_0$, there exists a sequence of clopen sets $U_i \subset W$ with $U_{i+1} \subset U_i$ for all $i \geq 1$, such that the translates $\{\varphi(\gamma) \cdot U_i \mid \gamma \in \Lambda\}$ form a finite covering of $\mathfrak{F}_0$ by disjoint clopen subsets. Moreover, for $\epsilon_i = \max \{\mathrm{diam}_{\mathfrak{F}_0} \{\varphi(\gamma) \cdot U_i\} \mid \gamma \in \Lambda\}$ we have $\lim\limits_{i \to \infty} \ \epsilon_i = 0$.*

Proposition 5.5 and Theorem 6.3 then imply:

COROLLARY 6.4. *Let $\mathfrak{M}_\varphi$ be a suspension matchbox manifold, whose global holonomy is a minimal action $\varphi \colon \Lambda \to \mathrm{Homeo}(\mathfrak{F}_0)$ which is equicontinuous with respect to the metric $d_{\mathfrak{F}_0}$ on $\mathfrak{F}_0$. Then $\mathfrak{M}_\varphi$ is infinitely collapsible.*

The main result of [12] follows from a combination of these results:

THEOREM 6.5 (Theorem 1.4, [12])**.** *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then there exists a sequence of closed triangulated topological manifolds and triangulated covering maps, $\{q_{\ell+1} \colon B_{\ell+1} \to B_\ell \mid \ell \geq 0\}$ such that $\mathfrak{M}$ is homeomorphic to the inverse limit of this system of maps*

$$\mathfrak{M} \overset{\mathrm{top}}{\approx} \varprojlim \{ \, q_{\ell+1} \colon B_{\ell+1} \to B_\ell \mid \ell \geq 0 \} \, . \tag{20}$$

That is, an equicontinuous matchbox manifold $\mathfrak{M}$ is foliated homeomorphic to a *weak solenoid* in the sense of McCord [37] and Schori [43].

THEOREM 6.6. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be equicontinuous matchbox manifolds, which are return equivalent with a common base. Then there is a homeomorphism $\widehat{\Phi} \colon \mathfrak{M}_1 \to \mathfrak{M}_2$.* □

Proof. By Theorem 6.1, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are minimal. Theorem 6.2 implies there are homeomorphisms $\mathfrak{M}_i \cong \mathfrak{M}_{\varphi_i}$ where for $i = 1, 2$, $\mathfrak{M}_{\varphi_i}$ is a foliated Cantor bundle with notation as in Theorem 6.2. Then by Corollary 6.4 each $\mathfrak{M}_{\varphi_i}$ is infinitely collapsible. Finally, Propositions 5.2 and 5.7 imply that the local conjugacy ϕ over the common base induces a homeomorphism $\widehat{\Phi} \colon \mathfrak{M}_1 \to \mathfrak{M}_2$. □

7. Shape and the common base

In this section, we obtain general conditions on equicontinuous matchbox manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that return equivalence implies that they have a common base. First, we introduce a generalization of the notion of Y-like, and introduce the notion of an *aspherical* matchbox manifold.

DEFINITION 7.1. *Let $\mathcal{C}$ denote a collection of compact metric spaces. A metric space X is said to $\mathcal{C}$–*like *if for every $\epsilon > 0$, there exists $Y \in \mathcal{C}$ and a continuous surjection $f_Y \colon X \to Y$ such that the fiber $f_Y^{-1}(y)$ of each point $y \in Y$ has diameter less than ϵ.*

If the collection $\mathcal{C} = \{Y\}$ is a single space, then Definition 7.1 reduces to Definition 1.2.

Mardešić and Segal show in Theorem 1^* of [35] the following key result:

THEOREM 7.2. *Let $\mathcal{C}$ be a given class of finite polyhedra, and let X be a continuum. Then X is $\mathcal{C}$–like if and only if X admits a presentation as an inverse limit $X \overset{\mathrm{top}}{\approx} \varprojlim \{ q_{\ell+1} \colon Y_{\ell+1} \to Y_\ell \mid \ell \geq 0\}$ in which the bonding maps q_ℓ are continuous surjections, and $Y_\ell \in \mathcal{C}$ for all ℓ.*

Observe that in this result, the only conclusion about the bonding maps q_ℓ is that they are continuous surjections, and in general they satisfy no other conditions. In particular, they need not be coverings.

We apply Theorem 7.2 to the collection $\mathcal{A}$ which consists of CW-complexes which are aspherical. Recall that a CW-complex Y is *aspherical* if it is connected, and $\pi_n(Y)$ is trivial for all $n \geq 2$. Equivalently, Y is aspherical if it is connected and its universal covering space is contractible. Note that if Y is aspherical, then every finite covering Y' of Y is also aspherical.

Our first result concerns the presentations of weak solenoids which are $\mathcal{A}$–like.

PROPOSITION 7.3. *Let $\mathfrak{M}$ is a matchbox manifold which is homeomorphic to a weak solenoid of dimension $n \geq 1$. Assume that $\mathfrak{M}$ is $\mathcal{A}$–like, then $\mathfrak{M}$ admits a presentation*

$$\mathfrak{M} \overset{\mathrm{top}}{\approx} \varprojlim \{ q_{\ell+1} \colon B_{\ell+1} \to B_\ell \mid \ell \geq 0\} \tag{21}$$

in which each bonding map q_ℓ is a covering map, and each B_ℓ is a closed aspherical n-manifold.

Proof. We are given that $\mathfrak{M}$ admits a presentation as in (21), in which each bonding map q_ℓ is a covering map, and each B_ℓ is a closed n-manifold. We show that each B_ℓ is aspherical. It suffices to show that for some $\ell \geq 0$, the universal covering $\widetilde{Y}_\ell$ is contractible.

By Theorem 7.2, there is a presentation

$$\mathfrak{M} \overset{\mathrm{top}}{\approx} \varprojlim \{ r_{\ell+1} \colon A_{\ell+1} \to A_\ell \mid \ell \geq 0\} \tag{22}$$

in which each map r_ℓ is a continuous surjection, and $A_\ell \in \mathcal{A}$ for all ℓ. Thus, using the notation $\mathfrak{M}_1 \overset{\mathrm{def}}{=} \varprojlim \{ q_{\ell+1} \colon B_{\ell+1} \to B_\ell \mid \ell \geq 0\}$ and $\mathfrak{M}_2 \overset{\mathrm{def}}{=} \varprojlim \{ r_{\ell+1} \colon A_{\ell+1} \to A_\ell \mid \ell \in \mathbb{N}_0\}$, we have two homeomorphisms $h_1 \colon \mathfrak{M} \to \mathfrak{M}_1$ and $h_2 \colon \mathfrak{M} \to \mathfrak{M}_2$.

Fix a base point $x \in \mathfrak{M}$, and set $x_i \overset{\mathrm{def}}{=} h_i(x)$ for $i = 1, 2$.

Consider the pro-groups homotopy groups, for $k \geq 1$, denoted by *pro-*$\pi_k(\mathfrak{M}, x)$ and *pro-*$\pi_k(\mathfrak{M}_i, x)$. For each $k \geq 1$, these groups are shape (and thus topological) invariants of the pointed spaces $(\mathfrak{M}_i, x_i)$, as shown in [36, Chapter II, Theorem 6]. In fact, a map with homotopically trivial fibers induces isomorphisms of the pro–homotopy groups [19]. Thus, the homeomorphism $h_1 \circ h_2^{-1}$ induces isomorphisms of the corresponding pro–homotopy groups.

One can find a general treatment of pro-homotopy groups in [36] or [8]. For our purposes, what is important is that these pro-groups can be obtained from any shape expansion of a space, such as is provided by the above inverse limit presentations in (21) and (22), and that isomorphisms of these towers have the form as described below.

By Theorem 3, Chapter 1 of [36], we can represent the isomorphism of pro-groups induced by a homeomorphism by a level morphism of isomorphic inverse sequences, in which the terms and bonding maps are derived from the original sequences. By Morita's lemma (see Chapter II, Theorem 5 in [36] or §2.1, Chapter III in [8]), this means that for each $k \geq 1$, there are subsequences $\{i_{(k,\ell)} \mid \ell \geq 1\}$ and $\{j_{(k,\ell)} \mid \ell \geq 1\}$, such that for each $\ell \geq 1$ we have a commutative diagram of homomorphisms:

$$\begin{tikzcd} D_{(k,\ell)}: \pi_k(A_{j_{(k,\ell)}}, x_{(2,j_{(k,\ell)})}) \arrow[d, "h_{(k,\ell)}"] & \pi_k(A_{j_{(k,s)}}, x_{(2,j_{(k,s)})}) \arrow[l] \arrow[d, "h_{(k,s)}"] \\ \pi_k(B_{i_{(k,\ell)}}, x_{(1,i_{(k,\ell)})}) & \pi_k(B_{i_{(k,s)}}, x_{(1,i_{(k,s)})}) \arrow[l] \arrow[ul, dashed, "g_{(k,s)}"] \end{tikzcd}$$

Here $x_{(i,j)}$ denotes the projection of x_i in the j–th factor space of the inverse sequence for $\mathfrak{M}_i$ and each s is some index greater than ℓ that depends on both k and ℓ. The horizontal maps are the homomorphisms induced from the composition of corresponding bonding maps, and the labeled maps are those resulting from the isomorphisms of pro–groups.

The bottom horizontal maps are injections since they result from covering maps, and hence each $g_{(k,s)}$ is also injective. Thus, for $k > 1$, the groups $\pi_k(B_{i_{(k,s)}}, x_{(1,i_{(k,s)})})$ as above are isomorphic to a subgroup of the group $\pi_k(A_{j_{(k,\ell)}}, x_{(2,j_{(k,\ell)})})$. By the definition of the class of spaces $\mathcal{A}$, each of these latter groups if trivial for $k > 1$, and thus the groups $\pi_k(B_{i_{(k,s)}}, x_{(1,i_{(k,s)})})$ are trivial as well.

We now show that all the spaces in the sequence B_ℓ are aspherical. Consider the universal covering $p : (\widetilde{B_0}, \widetilde{x_0}) \to (B_0, x_{(1,0)})$. We first show that $\widetilde{B_0}$ is contractible. By the above, for each $k > 1$ we know that for some s, we have that $\pi_k(B_{i_{(k,s)}}, x_{(1,i_{(k,s)})})$ is trivial. We then have for each k a commutative diagram of covering maps

$$\begin{tikzcd} (\widetilde{B_0}, \widetilde{x_0}) \arrow[d, "p"] \arrow[dr] & \\ (B_0, x_{(1,0)}) & (B_{i_{(k,s)}}, x_{(1,i_{(k,s)})}) \arrow[l] \end{tikzcd}$$

where the horizontal map is the composition of corresponding bonding maps and the diagonal covering map results from the universal property of p. Since covering maps induce monomorphisms of the corresponding homotopy groups, this shows that for each $k > 1$, the group $\pi_k(\widetilde{B_0}, \widetilde{x_0})$ factors through the trivial group and is therefore trivial. Since $(\widetilde{B_0}, \widetilde{x_0})$ is the universal covering space of a connected CW-complex, we then can conclude that it is contractible. Thus B_0 is aspherical, and hence so is each covering space of B_0, including each B_ℓ. □

DEFINITION 7.4. *A matchbox manifold* $\mathfrak{M}$ *is* aspherical *if pro-*$\pi_k(\mathfrak{M}, x) = 0$ *for all* $k > 1$.

The proof of Proposition 7.3 also shows the following.

PROPOSITION 7.5. *Let* $\mathfrak{M}$ *be a matchbox manifold which is* $\mathcal{A}$*–like, then* $\mathfrak{M}$ *is aspherical.* □

One of the important features of aspherical manifolds is given by the following standard result:

PROPOSITION 7.6. *If two closed aspherical manifolds* M_1 *and* M_2 *have isomorphic fundamental groups, then the isomorphism induces a homotopy equivalence between* M_1 *and* M_2*.*

Proof. The proof follows from standard obstruction theory for CW-complexes, as described in the proof of Theorem 2.1 in [34] for example. □

For the rest of this section, we consider the problem of showing that we have a homeomorphism between the bases of presentations for foliated Cantor bundles $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Proposition 7.6 is used to construct a homotopy equivalence between two bases, which must then be shown to yield a homeomorphism. While this conclusion is in general not true, it does hold for the special class of strongly Borel manifolds introduced in Definition 1.5. First, we show:

THEOREM 7.7. *Let $\mathcal{A}_B$ be a Borel collection of closed manifolds. If $\mathfrak{M}$ is an equicontinuous $\mathcal{A}_B$–like matchbox manifold, then $\mathfrak{M}$ admits a presentation $\mathfrak{M} \overset{\text{top}}{\approx} \varprojlim \{ q_{\ell+1} : B_{\ell+1} \to b_\ell \mid \ell \geq 0\}$ in which each bonding map q_ℓ is a covering map and $B_\ell \in \mathcal{A}_B$ for all ℓ.*

Proof. By Theorem 6.5, the equicontinuous matchbox manifold $\mathfrak{M}$ admits a presentation (20) in which each bonding map $q_{\ell+1}$ is a covering map, and each factor space B_ℓ is a closed manifold. We shall show that each closed manifold B_ℓ is an element of $\mathcal{A}_B$. Note that by Proposition 7.3 and condition 1) in Definition 1.5, each B_ℓ in this presentation is aspherical.

Now consider the diagrams $D_{(1,\ell)}$ as in the proof of Proposition 7.3. The diagram implies that for some s, $\pi_1(B_{i_s}, x_{(1,i_s)})$ is isomorphic to a finite indexed subgroup of $\pi_1(A_{j_{(k,\ell)}}, x_{(2,j_{(k,\ell)})})$ since the bottom horizontal map is an isomorphism onto a subgroup of $\pi_1(B_{i_{(k,\ell)}}, x_{(1,i_{(k,\ell)})})$ of finite index. Therefore, by the classification of covering spaces, $\pi_1(B_{i_s}, x_{(1,i_s)})$ is isomorphic to the fundamental group of a finite covering space of $A_{j_{(k,\ell)}}$. By conditions 2) and 3) in the definition of Borel collection, we can conclude that B_{i_s} is homeomorphic to some element in $\mathcal{A}_B$. By condition 2), we can conclude that for all $\ell \geq i_s$, M_ℓ is homeomorphic to an element of $\mathcal{A}_B$. Thus by truncating the terms before i_s and replacing each B_ℓ for $\ell \geq i_s$ with a homeomorphic element of $\mathcal{A}_B$ and adjusting the bonding maps accordingly, we obtain the desired presentation. □

Using the observation that $\mathcal{A}_B = \langle \mathbb{T}^n \rangle$ is a Borel collection, we immediately obtain:

COROLLARY 7.8. *If $\mathfrak{M}$ is an equicontinuous $\mathbb{T}^n$–like matchbox manifold, then $\mathfrak{M}$ admits a presentation $\mathfrak{M} \overset{\text{top}}{\approx} \varprojlim \{ q_{\ell+1} : \mathbb{T}^n \to \mathbb{T}^n \mid \ell \geq 0\}$ in which each bonding map q_ℓ is a covering map.*

Finally, we use the results shown previously to give the proofs of Theorems 1.4 and 1.6.

First, note that if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic, then they are return equivalent by Theorem 4.8, so it suffices to show the converse.

Assume that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are equicontinuous. Then Corollary 6.4 implies that both are infinitely collapsible, and Theorem 6.5 implies there is a presentation for each as in (20), which we label as:

$$\mathfrak{M}_1 \overset{\text{top}}{\approx} \varprojlim \left\{ q^1_{\ell+1} \colon B^1_{\ell+1} \to B^1_\ell \mid \ell \geq 0 \right\} \tag{23}$$

$$\mathfrak{M}_2 \overset{\text{top}}{\approx} \varprojlim \left\{ q^2_{\ell+1} \colon B^2_{\ell+1} \to B^2_\ell \mid \ell \geq 0 \right\} . \tag{24}$$

For $i = 1, 2$, let $b_i \in B^i_0$ be basepoints, let $\mathfrak{F}_i \subset \mathfrak{M}_i$ be the fiber over b_i and let $\Lambda_i = \pi_1(B_i, b_i)$ denote their fundamental groups. Let $\varphi_i \colon \Lambda_i \to \mathrm{Homeo}(\mathfrak{F}_i)$ be the global holonomy of each presentation.

The assumption that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent implies there exists clopen sets $U_i \subset \mathfrak{F}_i$ and a homeomorphism $\phi \colon U_1 \to U_2$ which induces an isomorphism $\Phi \colon \mathcal{G}_{U_1} \to \mathcal{G}_{U_2}$.

By Theorem 6.3, Lemma 5.6 and Proposition 5.5, we can assume that U_1 and U_2 are collapsible, and so are invariant under the action of the subgroups $\Lambda_{U_1} \subset \Lambda_1$ and $\Lambda_{U_2} \subset \Lambda_2$ as defined by (14).

Then by Theorem 6.6, it suffices to show these restricted actions have a common base. For $i = 1, 2$, let B'_i denote the finite covering of B_i associated to the subgroup Λ_i. That is, by Definition 5.8, we must show there exists a homeomorphism $h \colon B'_1 \to B'_2$ such that for the induced map on fundamental groups, $h_\# \colon \Lambda_{U_1} = \pi_1(B'_1, b'_1) \to \pi_1(B'_2, b'_2) = \Lambda_{U_2}$ we have

$$\varphi_2(h_\#(\gamma)) \cdot \omega = \phi(\varphi_1(\gamma) \cdot \phi^{-1}(\omega)) \ , \text{ for all } \gamma \in \Lambda_1 \ , \ \omega \in U_1. \tag{25}$$

The idea is that we show the existence of a map

$$\mathcal{H}_\phi \colon \Lambda_{U_1} = \pi_1(B_1', b_1') \to \pi_1(B_2', b_2') = \Lambda_{U_2} \tag{26}$$

so that (25) holds for $h_\# = \mathcal{H}_\phi$, and then construct the homeomorphism h. To implement this, we require the assumption that $\mathfrak{M}_1$ is Y-like, for an appropriate choice of Y.

7.1. **Proof of Theorem 1.4.** We are given that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are $\mathbb{T}^n$-like, where $n \geq 1$ is the dimension of the leaves of $\mathcal{F}_i$. By Corollary 7.8, each $\mathfrak{M}_i$ then admits a presentation as in (23) and (24), where $B^i_\ell = \mathbb{T}^n$ for $\ell \geq 0$.

For $i = 1, 2$, introduce $\mathcal{K}_i = ker\{\varphi_i \colon \Lambda_{U_i} \to \mathrm{Homeo}(U_i)\} \subset \Lambda_{U_i} \cong \mathbb{Z}^n$. For simplicity of notation, identify $\Lambda_{U_i} = \mathbb{Z}^n$. As $\mathbb{Z}^n$ is free abelian, $\mathcal{K}_i$ is a free abelian subgroup with rank $0 \leq r_i < n$.

The quotient $\mathbb{Z}^n/\mathcal{K}_i$ is abelian. Let $\mathcal{A}_i \subset \mathbb{Z}^n/\mathcal{K}_i$ denote the subgroup of torsion elements. By the structure theory of abelian groups, $\mathcal{A}_i$ is an interior direct sum of cyclic subgroups, so there exists elements $\{a^i_1, \ldots, a^i_{d_i}\} \subset \mathbb{Z}^n$ whose images in $\mathbb{Z}^n/\mathcal{K}_i$ form a minimal basis for $\mathcal{A}_i$. Observe that $0 \leq d_i \leq r_i$. The conjugacy ϕ maps torsion elements in the image $\varphi_1(\Lambda_{U_1}) \subset \mathrm{Homeo}(U_1)$ to torsion elements of $\varphi_2(\Lambda_{U_2}) \subset \mathrm{Homeo}(U_2)$, so $\mathbb{Z}^n/\mathcal{K}_1 \cong \mathbb{Z}^n/\mathcal{K}_2$ and thus $d_1 = d_2$ hence $r_1 = r_2$.

Define the group isomorphism $\mathcal{H}_\phi \colon \mathbb{Z}^n \to \mathbb{Z}^n$ by defining its value on bases of the domain and range as follows. First, we can assume without loss of generality that $\varphi_1(a^1_\ell)$ and $\varphi_2(a^2_\ell)$ generate isomorphic cyclic subgroups under the conjugacy ϕ, for $1 \leq \ell \leq d_i$. Then set $\mathcal{H}_\phi(a^1_i) = a^2_i$.

Next, for each $1 = 1, 2$, choose elements $\{a^i_{d_i+1}, \ldots, a^i_{r_i}\} \subset \mathcal{K}_i \subset \mathbb{Z}^n$ which span the complement in $\mathcal{K}_i$ of the subgroup $\langle a^i_1, \ldots, a^i_{d_i}\rangle \cap \mathcal{K}_i$ generated by the torsion generators. Then the span $\langle a^i_1, \ldots, a^i_{r_i}\rangle$ is a subgroup of $\mathbb{Z}^n$. Note that the action $\varphi_i(a^i_\ell)$ is trivial for $d_i < \ell \leq r_i$, and we set $\mathcal{H}_\phi(a^1_i) = a^2_i$.

Finally, note that the quotient of $\mathbb{Z}^n$ by $\langle a^i_1, \ldots, a^i_{r_i}\rangle$ is free abelian with rank $n - r_i$. As the quotient is free, each set $\{a^i_1, \ldots, a^i_{r_i}\}$ admits an extension to a basis of $\mathbb{Z}^n$. First, choose an extension for $i = 1$, say $\{a^1_1, \ldots, a^1_n\}$. Then for $r_i < \ell \leq n$, choose $a^2_\ell \in \mathbb{Z}^n$ so that $\varphi_2(a^2_\ell) = \phi \circ \varphi_1(a^1_\ell)$. Then set $\mathcal{H}_\phi(a^1_\ell) = a^2_\ell$. It follows from our choices that the map $\mathcal{H}_\phi \colon \mathbb{Z}^n \to \mathbb{Z}^n$ so defined satisfies the condition (25) holds for $h_\# = \mathcal{H}_\phi$.

Finally, the map $\mathcal{H}_\phi$ defines a linear map $\widehat{\mathcal{H}_\phi} \colon \mathbb{R}^n \to \mathbb{R}^n$, and so induces a diffeomorphism of the quotient spaces, $h \colon \mathbb{T}^n \to \mathbb{T}^n$ so that condition (25) holds. Thus, we have shown that the presentations (23) and (24) have a common base. Theorem 1.4 then follows from Theorem 6.6.

7.2. **Proof of Theorem 1.6.** We are given that Y is strongly Borel, and $\mathfrak{M}$ is equicontinuous and Y-like. In addition, it is assumed that each of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have a leaf which is simply connected. By Theorem 6.1 each leaf is dense, so in particular, every transversal clopen set intersects a leaf with trivial fundamental group. By the proof of Theorem 7.7, each of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ admits a presentation in which each bonding map is a finite covering map.

The assumption that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ which are return equivalent, implies by Proposition 5.9 that there is an induced map $\mathcal{H}_\phi \colon \Lambda_{U_1} = \pi_1(B_1', b_1') \to \pi_1(B_2', b_2') = \Lambda_{U_2}$ such that (26)holds. It follows from Proposition 7.6 that for the covering $B_1' \to B_1$ associated to $\Lambda_{U_1} \subset \Lambda_1$ and the covering $B_2' \to B_2$ associated to $\Lambda_{U_2} \subset \Lambda_2$, the isomorphism $\mathcal{H}_\phi$ on fundamental groups induces a homotopy equivalence $\widehat{h} \colon B_1' \to B_2'$ such that $\widehat{h}_\# = \mathcal{H}_\phi$. Each of the manifolds B_1' and B_2' can be assumed to be coverings of Y. As Y is assumed to be strongly Borel, the homotopy equivalence induces a homeomorphism h such that (25) is satisfied. That is, they have a common base, so it follows from Theorem 6.5 that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic. This proves the claim of Theorem 1.6.

REMARK 7.9. Given the choice of the clopen sets U_1 and U_2 in the above proofs, these sets are infinitely collapsible, so by refinement, we can assume that the conjugacy ϕ is induced on an arbitrary covering of $\mathbb{T}^n$ for Theorems 1.4, or Y for Theorems 1.6. As remarked in [16], the homeomorphism h that is obtained from the solutions of the Borel Conjecture can be assume to be smooth for a sufficiently large finite covering. Thus, we conclude that the homeomorphism $\Phi \colon \mathfrak{M}_1 \to \mathfrak{M}_2$ obtained above can be chosen to be smooth along leaves.

8. Examples and counter-examples

In this section, we give applications of the results in the previous sections. First, we describe a general construction of examples of equicontinuous matchbox manifolds for which the hypotheses of Theorem 1.6 are satisfied. These constructions are based on the notion of *non-co-Hopfian* manifolds, which is closely related to the Y-like property of Definition 1.2. Using these ideas, it is then clear how to construct classes of examples of equicontinuous matchbox manifolds for which return equivalence does not imply homeomorphism, as the Y-like hypothesis Theorem 1.6 is not satisfied.

Recall that a group G is *co-Hopfian* if there does not exist an embedding of G to a proper subgroup of itself, and *non-co-Hopfian* otherwise. A closed manifold Y is co-Hopfian if every covering map $\pi\colon Y \to Y$ is a diffeomorphism, and non-co-Hopfian if Y admits proper self-coverings. Clearly, Y is co-Hopfian if and only if its fundamental group is co-Hopfian.

The co-Hopfian concept for groups was first studied by Reinhold Baer in [4], where they are referred to as “S-groups”. More recently, the paper of Delgado and Timm [17] considers the co-Hopfian condition for the fundamental groups of connected finite complexes, and the paper by Endimioni and Robinson [20] gives some sufficient conditions for a group to be co-Hopfian or non-co-Hopfian. The paper by Belegradek [6] considers which finitely-generated nilpotent groups are non-co-Hopfian.

A finitely generated infinite group G is called *scale-invariant* if there is a nested sequence of finite index subgroups G_n that are all isomorphic to G and whose intersection is a finite group. The paper by Nekrashevych [39] gives natural conditions for which the semi-direct product G of a countable scale-invariant group H with a countable automorphism group A of G is scale-invariant, providing classes of examples of non-co-Hopfian groups which do not have polynomial word growth.

The product $G = G_1 \times G_2$ of any group G_1 with a non-co-Hopfian group G_2 is again non-co-Hopfian, though it may happen that the product of two co-Hopfian groups is non-co-Hopfian [33].

The paper by Ohshika and Potyagailo [40] gives examples of a freely indecomposable geometrically finite torsion-free non-elementary Kleinian group which are not co-Hopfian. The work of Delzant and Potyagailo [18] also studies which non-elementary geometrically finite Kleinian groups are co-Hopfian. The question of which compact 3-manifolds admit proper self-coverings has been studied in detail by González-Acuña, Litherland and Whitten in the works [24] and [25].

PROPOSITION 8.1. *Let G be a finitely generated, torsion-free group which admits a descending chain of groups $G_{\ell+1} \subset G_\ell$ each of finite index in G, whose intersection is the identity, and for some ℓ_0 we have G_ℓ is isomorphic to G_{ℓ_0} for all $\ell > \ell_0$. Let B_0 be a closed manifold whose fundamental group $G_0 = \pi(B_0, b_0)$ satisfies this condition. Let $p_\ell\colon B_\ell \to B_0$ be the finite covering associated to the subgroup G_ℓ, and set $Y = B_{\ell_0}$. Let $q_{\ell+1}\colon B_{\ell+1} \to B_\ell$ denote that covering induced by the inclusion $G_{\ell+1} \to G_\ell$. Let $\mathfrak{M}$ denote the weak solenoid defined as the inverse limit of the sequence of maps $q_{\ell+1}\colon B_{\ell+1} \to B_\ell$ for $\ell \geq 0$, so*

$$\mathfrak{M} \ \equiv\ \varprojlim \{ q_{\ell+1}\colon B_{\ell+1} \to B_\ell \mid \ell \geq 0\} \quad \subset \quad \prod_{\ell \geq 0} B_\ell . \tag{27}$$

Then $\mathfrak{M}$ is an equicontinuous matchbox manifold which is Y-like, and each leaf of the foliation $\mathcal{F}$ on $\mathfrak{M}$ is simply-connected.

Proof. Proposition 10.1 of [12] shows that $\mathfrak{M}$ is an equicontinuous matchbox manifold. For each $\ell \geq 0$, the definition of the inverse limit as a closed subset of the infinite product in (27) yields projection maps onto the factors, $\pi_\ell\colon \mathfrak{M} \to B_\ell$. By the definition of the product metric topology, for all $b \in B_0$, the diameters of the fibers $\pi_\ell^{-1}(b)$ tend to zero as $\ell \to \infty$. Given that $Y \cong B_\ell$ for all $\ell \geq \ell_0$ it follows that $\mathfrak{M}$ is Y-like. For a leaf $L \subset \mathfrak{M}$, its fundamental group is isomorphic to the intersection of the subgroups $G_\ell = \pi_1(B_\ell, b_\ell)$ for $\ell \geq 0$, which is the trivial group by assumption. □

The proof of Proposition 7.3 shows the close connection between the Y-like hypothesis and the non-co-Hopfian property for the fundamental groups in the presentation (27). In fact, the Y-like hypothesis on a solenoid is a type of homotopy version of the non-co-Hopfian property for manifolds.

8.1. **Examples for dimension** $n = 1$**.** The circle is the prototypical example of a non-co-Hopfian space, and Theorem 1.4 applies to the classical Vietoris solenoids with base $\mathbb{S}^1$. We examine this case in detail, recalling the classical classification of these spaces.

Let $\vec{m} = (m_1, m_2, \dots)$ denote a sequence of positive integers with each $m_i \geq 2$. Set $m_0 = 1$, then there is then the corresponding profinite group

$$
\begin{aligned}
\mathfrak{G}_{\vec{m}} &\stackrel{\text{def}}{=} \varprojlim \, \{ \, q_{\ell+1} \colon \mathbb{Z}/m_1 \cdots m_{\ell+1}\mathbb{Z} \to \mathbb{Z}/m_0 m_1 \cdots m_\ell \mathbb{Z} \mid \ell \geq 1 \, \} \\
&= \varprojlim \, \left\{ \mathbb{Z}/\mathbb{Z} \xleftarrow{m_1} \mathbb{Z}/m_1\mathbb{Z} \xleftarrow{m_2} \mathbb{Z}/m_1 m_2 \mathbb{Z} \xleftarrow{m_3} \mathbb{Z}/m_1 m_2 m_3 \mathbb{Z} \xleftarrow{m_4} \cdots \right\}
\end{aligned}
\tag{28}
$$

where $q_{\ell+1}$ is the quotient map of degree $m_{\ell+1}$. Each of the profinite groups $\mathfrak{G}_{\vec{m}}$ contains a copy of $\mathbb{Z}$ embedded as a dense subgroup by $z \to ([z]_0, [z]_1, ..., [z]_k, ...)$, where $[z]_k$ corresponds to the class of z in the quotient group $\mathbb{Z}/m_0 \cdots m_k \mathbb{Z}$. There is a homeomorphism $a_{\vec{m}} \colon \mathfrak{G}_{\vec{m}} \to \mathfrak{G}_{\vec{m}}$ given by "addition of 1" in each finite factor group. The dynamics of $a_{\vec{m}}$ acting on $\mathfrak{G}_{\vec{m}}$ is referred to as an *adding machine*, or equivalently as an *odometer*.

For a given sequence $\vec{m}$ as above, there is a corresponding Vietoris solenoid

$$
\mathcal{S}(\vec{m}) \stackrel{\text{def}}{=} \varprojlim \, \{ \, p_{\ell+1} : \mathbb{S}^1 \to \mathbb{S}^1 \mid \ell \geq 0 \} \tag{29}
$$

where $p_{\ell+1}$ is the covering map of $\mathbb{S}^1$ defined by multiplication of the covering space $\mathbb{R}$ by $m_{\ell+1}$.

It is well known that $\mathcal{S}(\vec{m})$ is homeomorphic to the suspension over $\mathbb{S}^1$ of the action by the map $a_{\vec{m}}$. Let $\pi_{\vec{m}} \colon \mathcal{S}(\vec{m}) \to \mathbb{S}^1$ denote projection onto the first factor, then $\mathcal{S}(\vec{m})$ is isomorphic as a topological group to the subgroup $\ker(\pi_{\vec{m}})$ (for example, see [1, 37].) Accordingly, $\mathcal{S}(\vec{m})$ is the total space of a principal $\mathfrak{G}_{\vec{m}}$–bundle $\xi_{\vec{m}} = (\mathcal{S}(\vec{m}), \pi_{\vec{m}}, \mathbb{S}^1)$ over $\mathbb{S}^1$.

The solenoids $\mathcal{S}(\vec{m})$ are classified using the following function, as shown in [7].

DEFINITION 8.2. *Given a sequence of integers $\vec{m}$ as above, let $C_{\vec{m}}$ denote the function from the set of prime numbers to the set of extended natural numbers $\{0, 1, 2, ..., \infty\}$ given by*

$$
C_{\vec{m}}(p) = \sum_1^\infty p_i,
$$

where p_i is the power of the prime p in the prime factorization of m_i.

DEFINITION 8.3. *Two sequences of integers $\vec{m}$ and $\vec{n}$ as above are* return equivalent*, denoted $\vec{m} \stackrel{\text{Ret}}{\sim} \vec{n}$ if and only if the following two conditions hold:*

(1) For all but finitely many primes p, $C_{\vec{m}}(p) = C_{\vec{n}}(p)$ and
(2) for all primes p, $C_{\vec{m}}(p) = \infty$ if and only if $C_{\vec{n}}(p) = \infty$.

The classical classification of the Vietoris solenoids up to homeomorphism then becomes:

THEOREM 8.4. [37, 1, 7] *The solenoids $\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{n})$ are homeomorphic if and only if $\vec{m} \stackrel{\text{Ret}}{\sim} \vec{n}$. Thus, they are return equivalent if and only if $\vec{m} \stackrel{\text{Ret}}{\sim} \vec{n}$.*

8.2. **Examples for dimension** $n = 2$**.** The simplest examples of *co-Hopfian* closed manifolds are the closed surfaces Σ_g of genus $g \geq 2$. The surface Σ_g has Euler characteristic $\chi(\Sigma_g) = 2 - 2g$, and the Euler characteristic is multiplicative for coverings. That is, if Σ_g' is a p-fold covering of Σ_g then $\chi(\Sigma_g') = p \cdot \chi(\Sigma_g)$. Thus, for $g > 1$, a proper covering Σ_g' of Σ_g is never homeomorphic to Σ_g. We use this remark to construct examples of weak solenoids with common base Σ_2 which are return equivalent but not homeomorphic.

Recall that the fundamental group of Σ_g has the standard finite presentation, for basepoint $\mathbf{x_0} \in \Sigma_g$:

$$
\pi_1(\Sigma_g, \mathbf{x_0}) \simeq \langle \, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \mid [\alpha_1 \beta_1] \cdots [\alpha_g \beta_g] \, \rangle \, .
$$

Define a homomorphism $h_0 \colon \pi_1(\Sigma_g, \mathbf{x_0}) \to \mathbb{Z}$ by setting $h_0(\alpha_1) = 1 \in \mathbb{Z}$, $h_0(\alpha_i) = 0$ for $1 < i \leq g$, and $h_0(\beta_i) = 0$ for $1 \leq i \leq g$. Then h_0 is induced by a continuous map, again denoted $h_0 \colon \Sigma_g \to \mathbb{S}^1$, which maps $\mathbf{x_0}$ to the basepoint $\theta_0 \in \mathbb{S}^1$. We use the map h_0 to form induced minimal Cantor bundles over Σ_g to obtain what we call $\vec{m}$*–adic surfaces*, as defined in the following.

For a given sequence $\vec{m}$ as above, and orientable surface Σ_g of genus $g \geq 1$, define an action $A_{\vec{m}}$ of $\pi_1(\Sigma_g, \mathbf{x_0})$ on the Cantor set $\mathfrak{G}_{\vec{m}}$ by composing the homomorphism $h_0 \colon \pi_1(\Sigma_g, \mathbf{x_0}) \to \mathbb{Z}$ with the action $a_{\vec{m}}$ of $\mathbb{Z}$ on $\mathfrak{G}_{\vec{m}}$. Note that the induced representation $A_{\vec{m}} \colon \pi_1(\Sigma_g, \mathbf{x_0}) \to \mathrm{Homeo}(\mathfrak{G}_{\vec{m}})$ thus constructed is never injective.

DEFINITION 8.5. *Given a closed, orientable surface Σ_g of genus $g \geq 1$ and a sequence of integers $\vec{m}$ as above, the $\vec{m}$–adic* surface *$\mathfrak{M}(\Sigma_g, \vec{m})$ is the Cantor bundle defined by the suspension of the action $A_{\vec{m}}$ as in (11), with $B = \Sigma_g$ and $\mathfrak{F} = \mathfrak{G}_{\vec{m}}$. As the action $a_{\vec{m}}$ is minimal, the matchbox manifold $\mathfrak{M}(\Sigma_g, \vec{m})$ is minimal.*

We next make some basic observations about the $\vec{m}$*–adic* surfaces $\mathfrak{M}(\Sigma_g, \vec{m})$.

Recall that the homomorphism $h_0 \colon \pi_1(\Sigma_g, \mathbf{x_0}) \to \mathbb{Z}$ is induced by a topological map $h_0 \colon \Sigma_g \to \mathbb{S}^1$. Then by general bundle theory [29, 30], the foliated Cantor bundle $\pi_* \colon \mathfrak{M}(\Sigma_g, \vec{m}) \to \Sigma_g$ is the pull-back of the Cantor bundle $\mathcal{S}(\vec{m}) \to \mathbb{S}^1$. The methods of Section 5 then yield:

LEMMA 8.6. *Let $\mathfrak{M}$ be a minimal matchbox manifold $\mathfrak{M}$ that is the total space of foliated bundle $\eta = \{\pi_* \colon \mathfrak{M} \to B\}$. Suppose that $f \colon B' \to B$ is a continuous map which induces a surjection of fundamental groups, where the dimensions of B and B' need not be the same. Then the total space $f^*(\mathfrak{M})$ of the induced bundle $f^*(\eta)$ over B' is return equivalent to $\mathfrak{M}$.* □

COROLLARY 8.7. *Given a closed, orientable surface Σ_g of genus $g \geq 1$ and a sequence of integers $\vec{m}$ as above, then the minimal matchbox manifolds $\mathcal{S}(\vec{m})$ and $\mathfrak{M}(\Sigma_g, \vec{m})$ are return equivalent.*

The geometric meaning of Corollary 8.7 is that the restricted pseudogroup of the $\vec{m}$*–adic* surface $\mathfrak{M}(\Sigma_g, \vec{m})$ does not "see" the trivial holonomy maps corresponding to loops in the base Σ_g that represent the classes $\alpha_{i>1}, \beta_j$.

Note that the dimensions of the leaves for $\mathcal{S}(\vec{m})$ and $\mathfrak{M}(\Sigma_g, \vec{m})$ differ, so they cannot possibly be homeomorphic. We obtain examples with the same leaf dimensions by applying Lemma 8.6, Theorem 8.4 and Proposition 4.6 to obtain the following result.

COROLLARY 8.8. *Given closed orientable surfaces Σ_{g_1} and Σ_{g_2} of genus $g_i \geq 1$ for $i = 1, 2$, and sequences $\vec{m}$ and $\vec{n}$, then $\mathfrak{M}(\Sigma_{g_1}, \vec{m})$ is return equivalent to $\mathfrak{M}(\Sigma_{g_2}, \vec{m})$ if and only if $\vec{m} \overset{\mathrm{Ret}}{\sim} \vec{n}$.*

Corollary 8.8 poses the problem, given *adic*-surfaces $\mathfrak{M}(\Sigma_{g_1}, \vec{m})$ and $\mathfrak{M}(\Sigma_{g_2}, \vec{n})$ with $\vec{m} \overset{\mathrm{Ret}}{\sim} \vec{n}$, when are they homeomorphic as matchbox manifolds? First, consider the case of genus $g_1 = g_2 = 1$ so that $\Sigma_{g_1} = \Sigma_{g_2} = \mathbb{T}^2$. Then Theorem 1.4 and Corollary 8.8 yield:

THEOREM 8.9. *$\mathfrak{M}(\mathbb{T}^2, \vec{m})$ and $\mathfrak{M}(\mathbb{T}^2, \vec{n})$ are homeomorphic if and only if $\vec{m} \overset{\mathrm{Ret}}{\sim} \vec{n}$.* □

For the general case, where at least one base manifold has higher genus, we have:

THEOREM 8.10. *Let $\mathfrak{M}_1 = \mathfrak{M}(\Sigma_{g_1}, \vec{m})$ and $\mathfrak{M}_2 = \mathfrak{M}(\Sigma_{g_2}, \vec{n})$ be* adic*-surfaces.*

(1) If $g_1 > 1$ and $g_2 = 1$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are never homeomorphic.
(2) If $g_1 = g_2 > 1$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic if and only if $C_{\vec{m}} = C_{\vec{n}}$.
(3) If $g_1 = g_2 > 1$, then there exists $\vec{m}$, $\vec{n}$ such that $\vec{m} \overset{\mathrm{Ret}}{\sim} \vec{n}$, but $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$.

Proof. First, consider the case where $g = g_1 = g_2 > 1$ and $C_{\vec{m}} = C_{\vec{n}}$. Then the Cantor bundles $\pi_{\vec{m}} \colon \mathcal{S}(\vec{m}) \to \mathbb{S}^1$ and $\pi_{\vec{n}} \colon \mathcal{S}(\vec{n}) \to \mathbb{S}^1$ are homeomorphic as bundles over $\mathbb{S}^1$ (see [7, Corollary 2.8]) and therefore their pull-back bundles under the map $h_0 \colon \Sigma_g \to \mathbb{S}^1$ are homeomorphic as bundles over Σ_g which is a stronger conclusion than the statement that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic.

For the proofs of parts 1) and 3) and also to show the converse conclusion in 2), assume there is a homeomorphism $\mathcal{H} \colon \mathfrak{M}_1 \to \mathfrak{M}_2$. By the results of Rogers and Tollefson in [41, 42], the map $\mathcal{H}$ is homotopic to a homeomorphism $\widetilde{\mathcal{H}}$ which is induced by a map of the inverse limit representations of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ as in (27). Let $X_j \equiv \mathfrak{M}(\Sigma_{g_1}, \vec{m}; j)$ denote the $j-th$ stage in (27) of the inverse limit representation for $\mathfrak{M}(\Sigma_{g_1}, \vec{m})$, and similarly set $Y_j \equiv \mathfrak{M}(\Sigma_{g_2}, \vec{n}, j)$. Then there exists an increasing integer-valued function $k \to \ell_k$ for $k \geq 0$, and covering maps $\widetilde{\mathcal{H}}_k \colon X_{\ell_k} \to Y_k$ where the collection of maps $\{\widetilde{\mathcal{H}}_k \mid k \geq k_0\}$ form a commutative diagram:

$$\begin{tikzcd} X_{\ell_0} \arrow[d, "\widetilde{\mathcal{H}}_0"'] & X_{\ell_1} \arrow[l, "f_{n_1}^{n_2}"'] \arrow[d, "\widetilde{h_1}"] & \cdots \arrow[l] & X_{\ell_k} \arrow[l] \arrow[d, "\widetilde{\mathcal{H}}_k"] & X_{\ell_{k+1}} \arrow[l, "f_{n_k+1}^{n_{k+1}-1}"'] \arrow[d, "\widetilde{\mathcal{H}}k+1"] & \cdots \arrow[l] \\ Y_0 & Y_1 \arrow[l, "g_1"] & \cdots \arrow[l] & Y_k \arrow[l] & Y_{k+1} \arrow[l, "g_k"] & \cdots \arrow[l] \end{tikzcd} \tag{30}$$

where the f_k and g_k are the bonding maps in the inverse limit representation of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and $f_{n_k+1}^{n_{k+1}-1}$ denotes the corresponding composition of bonding maps f_k.

Note that all of the maps in the diagram (30) are covering maps by construction. Thus, the Euler classes of all surfaces there are related by the covering degrees. For example, $\chi(X_{\ell_k}) = d_k \cdot \chi(Y_k)$ where d_k is the covering degree of $\widetilde{\mathcal{H}}_k$.

To show 1) we assume that a homeomorphism $\mathcal{H}$ exists, and so we have diagram (30) as above. Observe that $g_2 = 1$ implies that $\chi(\Sigma_2) = \chi(\mathbb{T}^2) = 0$, hence the covering $\chi(Y_k) = 0$ for all $k \geq 0$. Then as $d_k \geq 1$ for all k, we obtain $\chi(X_{\ell_k}) = 0$. But this contradicts the assumption that $g_1 > 1$ hence $\chi(X_{\ell_k}) < 0$ as X_{ℓ_k} is a covering of Σ_1 which has $\chi(\Sigma_1) < 0$. Thus $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$.

To show the converse in 2) assume that a homeomorphism $\mathcal{H}$ exists, and suppose that for some prime p we have $C_{\vec{m}}(p) \neq C_{\vec{n}}(p)$. We assume without loss of generality that $C_{\vec{m}}(p) < C_{\vec{n}}(p)$. Then as $\chi(\Sigma_1) = \chi(\Sigma_2)$, for sufficiently large k the prime factorization of the Euler characteristic $\chi(X_{\ell_k})$ contains a lower power of p than the prime factorization of $\chi(Y_k)$. But this contradicts that $\chi(X_{\ell_k}) = d_k \cdot \chi(Y_k)$ where d_k is the covering degree of $\widetilde{\mathcal{H}}_k$.

Finally, to show 3) let $\Sigma = \Sigma_{g_1} = \Sigma_{g_2}$ where $g = g_1 = g_2 > 1$. It suffices to define $\vec{m}$, $\vec{n}$ such that $\vec{m} \overset{\mathrm{Ret}}{\sim} \vec{n}$, but $C_{\vec{m}} \neq C_{\vec{n}}$. It then follows from 2) that $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$. Pick a prime $p_1 \geq 3$ and let $\vec{m}$ be any sequence such that $C_{\vec{m}}(p_1) = 0$. Then define $\vec{n}$ by setting $n_1 = p_1$ and $n_{k+1} = m_k$ for all $k \geq 1$.

Note that $C_{\vec{m}}(p_1) = 0 \neq 1 = C_{\vec{n}}(p_1)$ so $C_{\vec{m}}(p) \neq C_{\vec{n}}(p)$ is satisfied. But clearly $\vec{m} \overset{\mathrm{Ret}}{\sim} \vec{n}$, so the *adic*-surfaces $\mathfrak{M}(\Sigma_g, \vec{m})$ and $\mathfrak{M}(\Sigma_g, \vec{n})$ are return equivalent by Corollary 8.8, but are not homeomorphic by part 2) above. □

REMARK 8.11. The results of Theorem 8.10 are restricted to the case of the *adic*-surfaces introduced in Definition 8.5, which are inverse limits defined by a system of subgroups of finite index of $\Lambda = \pi_1(\Sigma_g, \mathbf{x_0})$ associated to the choice of the homomorphism $h_0 \colon \pi_1(\Sigma_g, \mathbf{x_0}) \to \mathbb{Z}$ and the sequence of integers $\vec{m}$. The proof of Theorem 8.10 uses the classification results of the 1-dimensional case in an essential manner.

There is a more general construction of 2-dimensional equicontinuous matchbox manifolds $\mathfrak{M}(\Sigma_g, \mathcal{L})$ obtained from a given infinite, partially-ordered collection of subgroups of finite index. Set $\mathcal{L} \equiv \{\Lambda_i \subset \Lambda \mid i \in \mathcal{I}\}$ where each Λ_i is a subgroup of finite index in Λ. The partial order on $\mathcal{L}$ is defined by setting where $\Lambda_i \lesssim \Lambda_j$ if $\Lambda_i \subset \Lambda_j$. Then $\mathfrak{M}(\Sigma_g, \mathcal{L})$ is the inverse limit of the finite coverings $\Sigma_{g,i} \to \Sigma_g$ associated to the subgroups in $\mathcal{L}$. In particular, let $\mathcal{L}^*$ denote the *universal* partially ordered lattice of subgroups, which includes all subgroups of Λ of finite index. The space $\mathfrak{M}(\Sigma_g, \mathcal{L}^*)$ was introduced by Sullivan in [44], where it was called the *universal Riemann surface lamination*, and used in the study of conformal geometries for Riemann surfaces. The techniques of this paper give no insights to the classification up to homeomorphism of these spaces, and suggest that a deeper understanding of their homeomorphism types will require fundamentally new techniques.

8.3. **Examples for dimension $n \geq 3$.** We briefly discuss the homeomorphism problem for the case of n-dimensional equicontinuous matchbox manifolds, for $n \geq 3$. The discussion above of the examples of non-co-Hopfian groups shows there are many classes of closed n-manifolds which are non-co-Hopfian and not covered by the torus $\mathbb{T}^n$, and are also strongly Borel. The papers [24, 25] apply especially to the case of closed 3-manifolds, where it seems that some of the above results for $n = 2$ can be extended to this case. The Euler characteristic of a closed 3-manifold is always zero, so the method above will not apply directly, as it used the Euler characteristic of the closed manifolds appearing in the inverse representation to show the matchbox manifolds defined by the inverse systems are not homeomorphic. On the other hand, the paper by Wang and Wu [48] gives invariants of coverings of 3-manifolds which give obstructions to a proper covering being diffeomorphic to its base, so it is likely this can be used to show the inverse limits are not homeomorphic in an analogous manner.

REMARK 8.12. We also note that it follows from the results of [11] that given $\epsilon > 0$, each $\mathfrak{M}(\Sigma_g, \vec{m})$ for $g \geq 1$ occurs as the minimal set of a C^∞ ϵ–perturbation of the product foliation of $\Sigma_g \times \mathbb{D}^2$, where $\mathbb{D}^2$ is the unit 2-dimensional disk. Thus, the examples we construct above are topologically wild, but not necessarily pathological, as they can occur naturally in the study of the dynamics of smooth foliations. See [28] for a further discussion of this topic.

9. Concluding remarks and a solenoidal Borel Conjecture

One of the key results required for the proofs of Theorems 1.4 and 1.6, is a form of the Borel Conjecture for solenoids that are approximated by strongly Borel manifolds. Here we show how our considerations lead to a generalized Borel Conjecture for equicontinuous matchbox manifolds.

It is known that two equicontinuous $\mathbb{T}^n$–like matchbox manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with equivalent shape (or even just isomorphic first Čech cohomology groups) are homeomorphic. Indeed, since these spaces admit an abelian topological group structure, the first Čech cohomology group of such a space is isomorphic to its character group, and Pontrjagin duality then shows that two such spaces are homeomorphic if and only if their first Čech cohomology groups are isomorphic. Considering this in a broader context leads to the following two related conjectures for the class $\mathcal{B}$ of closed aspherical manifolds to which the Borel conjecture applies. That is, any closed manifold M homotopy equivalent to some $B \in \mathcal{B}$ is in fact homeomorphic to B. We can then formulate two conjectures that would naturally generalize the Borel conjecture for aspherical manifolds to the setting of equicontinuous matchbox manifolds.

Consider two equicontinuous matchbox manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of the same leaf dimension $n \geq 2$ that are shape equivalent, which is the appropriate generalization to this setting of two closed manifolds being homotopy equivalent. The first problem we pose is a generalization of the classification of the compact abelian groups in terms of their shape, as mentioned in the introduction to this paper.

CONJECTURE 9.1. *Let $\mathcal{A}_B$ be a Borel collection of compact manifolds of dimension $n \geq 1$. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are equicontinuous, $\mathcal{A}_B$–like matchbox manifolds that are shape equivalent, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic.*

As indicated in Proposition 7.6, two aspherical manifolds with isomorphic fundamental groups are in fact homotopy equivalent. If one can show analogously that two equicontinuous $\mathcal{A}_B$–like matchbox manifolds $\mathfrak{M}$ and $\mathfrak{M}'$ that have isomorphic $\mathrm{pro} - \pi_1$ pro-groups are in fact shape equivalent, then a proof of the first conjecture would lead to a proof of the following stronger conjecture.

CONJECTURE 9.2. *If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are equicontinuous, $\mathcal{A}_B$–like matchbox manifolds that have isomorphic* $\mathrm{pro} - \pi_1$ *pro-groups, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic.*

The positive results we have obtained have been in the context of a class of matchbox manifolds that are the total space of a foliated bundle over the same base manifold. One of the shortcomings of using restricted pseudogroups for the classification problem, is that they do not distinguish paths that induce trivial maps in holonomy. This is seen in the hypothesis on Theorem 1.6 that there exists simply connected leaves, which eliminates this possibility. On the other hand, Theorem 1.4 does not impose this assumption, and uses the structure of free abelian groups to resolve the difficulties in the proof of homeomorphism which arise.

PROBLEM 9.3. *Let $\mathcal{A}_B$ be a Borel collection of infra-nil-manifolds of dimension $n \geq 3$. Show that if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are equicontinuous, $\mathcal{A}_B$–like matchbox manifolds which are return equivalent, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic.*

The techniques of this paper are based on the reduction of the classification problem to that for minimal Cantor fibrations over a closed base manifold. This is a strong restriction, and does not generally hold for the minimal sets of *Axiom A attractors* as discussed by Williams [49, 50, 51]. Even if one restricts oneself to the class of two–dimensional, orientable matchbox manifolds that occur as an expanding attractor of a diffeomorphism, Farrell and Jones in [21] show there are examples that do not fiber over any closed manifold.

In consideration of this more general situation, the authors established in [13, 14] the existence of decompositions of minimal matchbox manifolds $\mathfrak{M}$ that arises from the fibers of a projection onto a branched manifold $\pi \colon \mathfrak{M} \to B$.

PROBLEM 9.4. *Use the projection of a matchbox manifold onto a branched manifold $\pi \colon \mathfrak{M} \to B$ to develop a classification of minimal matchbox manifolds without holonomy.*

Starting with Williams [50, Section 7], there has been an attempt to use the fundamental group of the base B in this setting to classify special classes of one–dimensional attractors that have the structure of a matchbox manifold. However, this technique, even in dimension one, is fraught with difficulties. As discovered in the erratum to [5] and explored more fully in the paper [45], there are fundamental problems with these techniques. In general, the lack of a true bundle structure in this setting creates serious obstructions to applying the techniques we have developed in this work.

Finally, the return equivalence of 1-dimensional equicontinuous solenoids is described by a sequence of integers as discussed in Section 8. In the higher dimensional case, equicontinuous torus solenoids are described by sequences of integer matrices. By the results of [31] there is no reasonable way of describing the classification of the structures resulting from these sequences of matrices. However, it might nonetheless be possible to describe a condition in the spirit of Definition 8.3.

QUESTION 9.5. *Can one find a combinatorial description of return equivalence for sequences of integer matrices associated to equicontinuous toral solenoids of dimension $d > 1$?*

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