PERSPECTIVES ON KUPERBERG FLOWS

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Dedicated to Professor Krystyna Kuperberg on the occasion of her 70th birthday

Abstract. The "Seifert Conjecture" stated, "Every non-singular vector field on the 3-sphere $\mathbb{S}^3$ has a periodic orbit". In a celebrated work, Krystyna Kuperberg gave a construction of a smooth aperiodic vector field on a plug, which is then used to construct counter-examples to the Seifert Conjecture for smooth flows on the 3-sphere, and on compact 3-manifolds in general. The dynamics of the flows in these plugs have been extensively studied, with more precise results known in special "generic" cases of the construction. Moreover, the dynamical properties of smooth perturbations of Kuperberg's construction have been considered. In this work, we recall some of the results obtained to date for the Kuperberg flows and their perturbations. Then the main point of this work is to focus attention on how the known results for Kuperberg flows depend on the assumptions imposed on the flows, and to discuss some of the many interesting questions and problems that remain open about their dynamical and ergodic properties.

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1. Introduction

In his 1950 work [59], Seifert introduced an invariant for deformations of non-singular flows with a closed orbit on a 3-manifold, which he used to show that every sufficiently small deformation of the Hopf flow on the 3-sphere $\mathbb{S}^3$ must have a closed orbit. He also remarked in Section 5 of this work:

> *It is unknown if every continuous (non-singular) vector field on the three-dimensional sphere contains a closed integral curve.*

This remark became the basis for what is known as the "Seifert Conjecture":

> *Every non-singular vector field on the* 3*-sphere* $\mathbb{S}^3$ *has a periodic orbit.*

The Séminaire Bourbaki article [22] by Ghys discusses the background and obstacles to the construction of aperiodic flows in 3-dimensions. We recall some of the key developments.

In his 1966 work [67], and see also [53], Wilson showed that every closed 3-manifold M has a flow with only finitely many closed orbits. The Seifert Conjecture was thus reduced to showing that given a flow on a 3-manifold with only a finite number of periodic orbits, it can be modified to one with no periodic orbits.

Wilson introduced the notion of a "plug" for a flow on M, which is a neighborhood $P \subset M$ diffeomorphic to a disk cross an interval, $P \cong \mathbb{D}^2 \times [-1, 1]$, where orbits enter one end of the plug, $\partial^- P = \mathbb{D}^2 \times \{-1\}$, and exit the other end, $\partial^+ P = \mathbb{D}^2 \times \{+1\}$. Wilson showed how to modify the flow inside a plug so that it contains exactly two closed orbits. These *Wilson plugs* are used to modify a given non-singular flow on a closed 3-manifold, to obtain one with only isolated periodic orbits.

In his 1974 work [58], Schweitzer gave a modification of the construction of the Wilson plug, replacing the periodic orbits in the plug with invariant Denjoy-type minimal sets for the modified flow in the plug, to obtain a C^1-flow with no periodic orbits. He used this plug to show that there exists a non-singular C^1-vector field without periodic orbits on any closed orientable 3-manifold M. In her 1988 work [26], Harrison gave a modification of the construction of the Schweitzer plug, which yielded an aperiodic plug with a C^2-flow.

The Seifert Conjecture was settled for all degrees of smoothness by Krystyna Kuperberg in 1994:

THEOREM 1.1 (Kuperberg [39])**.** *On every closed oriented 3-manifold M, there exists a C^∞ non-vanishing vector field without periodic orbits.*

Kuperberg introduced a construction of aperiodic 3-dimensional plugs which was notable for its simplicity and beauty, and remains the only general method to date to construct C^∞-flows without periodic orbits.

Following Kuperberg's original work, there was a sequence of three works explaining in further detail the dynamical properties for the Kuperberg flows:

- the Séminaire Bourbaki lecture [22] by Étienne Ghys;
- the notes by Shigenori Matsumoto [45];
- the joint paper [40] by Greg and Krystyna Kuperberg.

Also, a brief overview of the construction was given in [42]. These works showed that a Kuperberg flow:

- has zero topological entropy, which follows by an observation of Ghys in [22] that an aperiodic flow on a 3-manifold has topological entropy equal to zero, as a consequence of a result of Katok in [36];
- has a unique minimal set Σ, whose topological structure is unknown in general;
- has an open set of non-recurrent points that limit on its minimal set Σ, a result of Matsumoto [45];
- preserves a 2-dimensional compact lamination $\mathfrak{M}$ with boundary, that is contained in the interior of the plug and contains the minimal set Σ.

The two minimal sets for the Wilson flow are each a closed circle, and the two minimal sets for the Schweitzer plug are each homeomorphic to a Denjoy minimal set in the 2-torus. In contrast, the topological type of the unique minimal set Σ for a Kuperberg flow is extraordinarily complicated, and requires further study.

Moreover, there are many choices made in the construction of the flows in the Kuperberg plugs, any of which may strongly influence their global dynamics. In his survey [22, page 302], Ghys wrote:

> *Par ailleurs, on peut construire beaucoup de pièges de Kuperberg et il n'est pas clair qu'ils aient le même dynamique.*

One of the goals of this paper is to bring into focus some of the dynamical properties of Kuperberg flows which depend on imposing additional hypotheses in their construction. For example, the monograph [29] introduced the notion of a *generic* Kuperberg plug, as recalled in Section 4 below. It was shown in [29] that for a generic Kuperberg flow, the minimal set Σ is equal to the lamination $\mathfrak{M}$. A detailed analysis of the dynamical and topological structure of $\mathfrak{M}$ was used to give a geometric proof that the topological entropy of the generic Kuperberg flow is zero, and also to show that $\mathfrak{M}$ has unstable shape.

The statement by Seifert quoted above can be given alternate interpretations as well. For example, one is to find conditions on a smooth flow on a closed 3-manifold which guarantees the existence of a periodic orbit. It has been shown by Hofer in [27], that the flow of a Reeb vector field on $\mathbb{S}^3$ must have a periodic orbit. More generally, Taubes showed in [63] that the flow of a Reeb vector field on a closed 3-manifold has periodic orbits. This theorem was extended to geodesible volume preserving flows (also known as Reeb vector field of stable hamiltonian structures) to manifolds that are not torus bundles over the circle by Hutchings and Taubes [33, 34], and Rechtman [49], and for real analytic geodesible flows by Rechtman [57]. The existence of a periodic orbit is also established for real analytic solutions of the Euler equation by Etnyre and Ghrist [17]. Frankel proved that quasi-geodesic flows on hyperbolic 3-manifolds always have periodic orbits [19].

Another interpretation of Seifert's work is to ask about the dynamical properties of flows which are close to an aperiodic flow. For example, the work [30] by the authors constructs smooth families of variations of the Kuperberg flow which are not aperiodic, and have invariant horseshoes in their dynamics. We also show that there are smooth families of variations of the Kuperberg plug with simple dynamics and exactly two periodic orbits, but the limit of the family is an aperiodic flow. In such a family, the period of the periodic orbits blows up at the limit. Palis and Shub in [50] asked whether this dynamical phenomenon can occur in families of smooth flows on closed manifolds, and called a closed orbit whose length "blows up" to infinity under deformation a "blue sky catastrophe". The first examples of a family of flows with this property was found by Medvedev in [47]. The constructions in [30] show that deformations of Kuperberg flows provide a new class of examples. The work of A. Shilnikov, L. Shilnikov and D. Turaev in [60] gives a discussion of stronger formulations of the blue sky catastrophe phenomenon.

Thus, Kuperberg flows are special in that they are aperiodic, have many further special dynamical properties and lie at the "boundary of chaos" in the C^∞-topology on flows. These properties suggest topics for further study, and this is the theme of this survey:

> *There are many interesting open problems concerning Kuperberg flows!*

We first describe the construction of the Kuperberg plugs. Section 2 gives the construction of the modified Wilson plugs, which provides the foundation of the construction. Section 3 then gives the construction of the Kuperberg plugs. These constructions are given in a succinct manner, and the interested reader can consult the literature cited above for further details and discussions.

Section 4 introduces the *generic hypotheses* for the Kuperberg construction, as given in the works [29, 30]. We also introduce variations of these generic hypotheses, whose implications for the dynamics of the flows will be discussed in later sections.

It was observed in the works [22, 45] that any orbit not escaping the plug in forward or backward time, limits to the invariant set defined as the closure of the "special orbits" for the flow. One consequence is that the Kuperberg flow has a unique minimal set in the plug, denoted by Σ, as recalled in Theorem 5.1.

A remarkable aspect of the construction of a Kuperberg flow is that it preserves an embedded infinite surface with boundary, denoted by $\mathfrak{M}_0$. The boundary of $\mathfrak{M}$ contains the special orbits, and so its closure $\mathfrak{M} = \overline{\mathfrak{M}_0}$ is a type of lamination with boundary that contains the minimal set Σ. The relation between the two sets $\Sigma \subset \mathfrak{M}$ is an important theme in the study of the dynamics of Kuperberg flows. Section 6 gives an outline of the structure theory for the embedded surface $\mathfrak{M}_0$ in the case of generic flows, as developed in [29]. This structure theory, and the corresponding properties of the level function defined on $\mathfrak{M}_0$, are crucial for the proofs in [29].

In Section 7, we develop conditions on a flow which are sufficient to show that the inclusion of the minimal set Σ in the laminated space $\mathfrak{M}$ is an equality. This result is a type of "Denjoy Theorem" for laminations, and its proof in [29] relies on the generic hypotheses for the flow in fundamental ways. It is an interesting problem whether this is a special case of a more general formulation of a Denjoy Theorem for laminations, which we present here as it is independent of the development of Kuperberg flows.

PROBLEM (Problem 7.9)**.** *Let $\mathcal{L}$ be a compact, connected 2-dimensional lamination which is embedded in a compact 3-manifold M, and let $\mathcal{X}$ be a smooth vector field tangent to the leaves of $\mathcal{L}$. Suppose that the lamination $\mathcal{L}$ is minimal, that is, every 2-dimensional leaf of $\mathcal{L}$ is dense in $\mathcal{L}$. Also assume that the flow of $\mathcal{X}$ has no periodic orbits. Show that every orbit of $\mathcal{X}$ is dense in $\mathcal{L}$.*

Ghys observed in [22] that an aperiodic flow must have entropy equal to zero, using a well-known result of Katok [36], and thus a Kuperberg flow restricted to the lamination $\mathfrak{M}$ must have zero entropy. The authors showed in [29], that while the usual entropy of the flow vanishes, the "slow entropy" as defined in [9, 37] of a generic Kuperberg flow is positive for exponent $\alpha = 1/2$. This calculation used two special properties of the Kuperberg construction. One is that the embedded surface $\mathfrak{M}_0 \subset \mathbb{K}$ has subexponential but not polynomial growth rate, which follows from the structure theory developed for $\mathfrak{M}_0$ in the generic case. The second is that the surface $\mathfrak{M}_0$ is the union of two infinite surfaces, corresponding to the two insertion maps used in the construction of the plug, and the flow along these surfaces separates points when they encounter the upper and lower insertions. It is an interesting problem to study the entropy-like invariants for all Kuperberg flows, not just the generic flows. Analogously, it is important to estimate the Hausdorff dimensions of the sets Σ and $\mathfrak{M}$, and how they depend on the choices used in the construction of the Kuperberg flow. These and related questions are addressed in Section 8.

A standard problem in topological dynamical systems theory is to describe the topological type of the closed attractors for the system, and for closed invariant transitive subsets more generally. Attractors often have very complicated topological description, and the theory of shape for spaces [43, 44] is used to describe them. Describing the shape of a dynamically defined invariant set of an arbitrary flow is typically quite difficult, but also can be highly revealing about the dynamical properties of the flow.

In Section 9, we discuss the shape properties of the unique minimal set for Kuperberg flows. For a generic Kuperberg flow, the shape of the minimal set Σ was shown in [29] to be *not stable*, as defined in Definition 9.3, but to satisfy a *Mittag-Leffler Property* on its homology groups, as defined in Proposition 9.10. The proofs of these assertions require the structure theory for the invariant set $\mathfrak{M}$, and a key idea is the construction of shape approximations to $\mathfrak{M}$ using its level hierarchy, as developed in the monograph [29].

The construction of shape approximations of the minimal set Σ suggests a relationship between the entropy of the flow and its shape properties. The following problem can be stated for general flows, and the motivation is given in Section 9 and Problem 9.8.

PROBLEM. *Assume that a flow of a closed 3-manifold M has an exceptional minimal set Σ whose shape is not stable. Is the slow entropy of the flow positive?*

Recall that a minimal set is said to be exceptional if it is not a submanifold of the ambient manifold M. When M has low dimension, this implies that the minimal set has "small" dimension, that is it has dimension 1 or 2. The assumption that the shape of Σ is not stable implies that the topological type of its shape approximations keep changing as they become increasingly fine, while the assumption that every orbit of the flow in Σ is dense implies a type of recurrence for the topology of the shape approximations. The problem is then asking if these assumptions are sufficient to guarantee that the topological type of the approximations exhibit a form of self-similarity in their topological type, which implies that there are exponential separation of the points in the orbit of the flow, at some possibly subexponential rate, as for Kuperberg flows.

Section 10 discusses a variety of questions about the flows which are C^∞-close to Kuperberg flows. The *Derived from Kuperberg* flows, or DK–flows, were introduced in [30], and are obtained by varying the construction of the usual Kuperberg flows, to obtain smooth families of flows containing Kuperberg flows, so are of central interest from the point of view of the properties of Kuperberg flows in the space of flows. The work [30] gave constructions of DK–flows which in fact have countably many independent horseshoe subsystems, and thus have positive topological entropy. Moreover, these examples can be constructed arbitrarily close to the generic Kuperberg flows. It is notable that the horseshoes generated by a variation of the Kuperberg construction are shown to exist using the shape approximations discussed in Section 9, providing more reasons to explore the relation between shape and entropy for flows.

It seems a good moment to comment on a question related to volume preserving flows on 3-manifolds. There are examples with a finite number of periodic orbits and C^1 examples without periodic orbits on any closed 3-manifold. These were constructed in [41] by Greg Kuperberg using plugs. Is it possible to build a C^∞ volume preserving aperiodic plug? Can we *a priori* say something about its minimal and invariant sets?

The authors dedicate this work to Krystyna Kuperberg for her discovery of the class of dynamical systems introduced in her celebrated works on aperiodic flows. We are grateful for her comments and suggestions to the authors that have inspired our continued fascination with "Kuperberg flows".

2. Modified Wilson plugs

In this section, we present the construction of the Wilson plugs which are the foundation for the construction of the Kuperberg plugs, with commentary on the choices made. First, we recall that a "plug" is a manifold with boundary endowed with a flow, that enables the modification of a given flow on a 3-manifold inside a flow-box. The idea is that after modification by insertion of a plug, a periodic orbit for the given flow is "broken open" – it enters the plug and never exits. Moreover, Kuperberg's construction does this modification without introducing additional periodic orbits. The first step is to construct Kuperberg's modified Wilson plug, which is analogous to the modified Wilson plug used by Schweitzer in [58].

The notion of a "plug" to be inserted in a flow on a 3-manifold was introduced by Wilson [67, 53]. A 3-dimensional plug is a manifold P endowed with a vector field $\mathcal{X}$ satisfying the following conditions. The 3-manifold P is of the form $D \times [-2, 2]$, where D is a compact 2-manifold with boundary ∂D. Set

$$\partial_v P = \partial D \times [-2, 2] \quad , \quad \partial_h^- P = D \times \{-2\} \quad , \quad \partial_h^+ P = D \times \{2\} \ .$$

Then the boundary of P has a decomposition

$$\partial P \ = \ \partial_v P \cup \partial_h P \ = \ \partial_v P \cup \partial_h^- P \cup \partial_h^+ P \ .$$

Let $\frac{\partial}{\partial z}$ be the *vertical* vector field on P, where z is the coordinate of the interval $[-2, 2]$.

The vector field $\mathcal{X}$ must satisfy the conditions:

- (P1) *vertical at the boundary*: $\mathcal{X} = \frac{\partial}{\partial z}$ in a neighborhood of ∂P; thus, $\partial_h^- P$ and $\partial_h^+ P$ are the entry and exit regions of P for the flow of $\mathcal{X}$, respectively;
- (P2) *entry-exit condition*: if a point $(x, -2)$ is in the same trajectory as $(y, 2)$, then $x = y$. That is, an orbit that traverses P, exits just in front of its entry point;
- (P3) *trapped orbit*: there is at least one entry point whose entire *forward* orbit is contained in P; we will say that its orbit is *trapped* by P;
- (P4) *tame*: there is an embedding $i \colon P \to \mathbb{R}^3$ that preserves the vertical direction.

Note that conditions (P2) and (P3) imply that if the forward orbit of a point $(x, -2)$ is trapped, then the backward orbit of $(x, 2)$ is also trapped.

A *semi-plug* is a manifold P endowed with a vector field $\mathcal{X}$ as above, satisfying conditions (P1), (P3) and (P4), but not necessarily (P2). The concatenation of a semi-plug with an inverted copy of it, that is a copy where the direction of the flow is inverted, is then a plug.

Note that condition (P4) implies that given any open ball $B(\vec{x}, \epsilon) \subset \mathbb{R}^3$ with $\epsilon > 0$, there exists a modified embedding $i' \colon P \to B(\vec{x}, \epsilon)$ which preserves the vertical direction again. Thus, a plug can be used to change a vector field $\mathcal{Z}$ on any 3-manifold M inside a flowbox, as follows. Let $\varphi\colon U_x \to (-1,1)^3$ be a coordinate chart which maps the vector field $\mathcal{Z}$ on M to the vertical vector field $\frac{\partial}{\partial z}$. Choose a modified embedding $i'\colon P \to B(\vec{x},\epsilon) \subset (-1,1)^3$, and then replace the flow $\frac{\partial}{\partial z}$ in the interior of $i'(P)$ with the image of $\mathcal{X}$. This results in a flow $\mathcal{Z}'$ on M.

The entry-exit condition implies that a periodic orbit of $\mathcal{Z}$ which meets $\partial_h P$ in a non-trapped point, will remain periodic after this modification. An orbit of $\mathcal{Z}$ which meets $\partial_h P$ in a trapped point never exits the plug P, hence after modification, limits to a closed invariant set contained in P. A closed invariant set contains a minimal set for the flow, and thus, a plug serves as a device to insert a minimal set into a flow.

In the work of Wilson [67], the basic plug has the shape of a solid cylinder, whose base $\partial_h^- P = D \times \{-2\}$ is a planar disk $\mathbb{D}^2$. Schweitzer introduced in [58] plugs for which the base is obtained from a 2-torus minus an open disk, so $\partial_h^- P \cong \mathbb{T}^2 - \mathbb{D}^2$, which has the homotopy type of a wedge of two circles. As we shall see below, a key idea behind the Kuperberg construction is to consider a base which is obtained from an annulus by adding two connecting strips, so has the homotopy type of three circles. The "modified Wilson Plug" is a flow on a cylinder minus its core. The flow has two periodic orbits, and the dynamics of the flow is not stable under perturbations. The instability of its dynamics is a key property of these modified plugs.

The first step in the construction is to define a flow on a rectangle as follows. The rectangle is defined by

$$\mathbf{R} = [1,3] \times [-2,2] = \{(r,z) \mid 1 \leq r \leq 3 \ \& -2 \leq z \leq 2\} \ . \tag{1}$$

For a constant $0 < g_0 \leq 1$, choose a C^∞-function $g\colon \mathbf{R} \to [0, g_0]$ which satisfies the "vertical" symmetry condition $g(r,z) = g(r,-z)$. Also, require that $g(2,-1) = g(2,1) = 0$, that $g(r,z) = g_0$ for (r,z) near the boundary of $\mathbf{R}$, and that $g(r,z) > 0$ otherwise.

Define the vector field $\mathcal{W}_v = g \cdot \frac{\partial}{\partial z}$. Note that $\mathcal{W}_v$ has two singularities, at $(2, \pm 1)$, and is otherwise everywhere vertical. The flow lines of this vector field are illustrated in Figure 1. The value of g_0 chosen influences the quantitative nature of the flow, as small values of g_0 result in a slower vertical climb for the flow, but does not alter the qualitative nature of the flow.

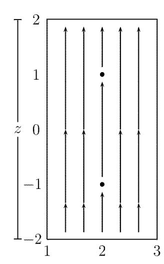


FIGURE 1. Vector field $\mathcal{W}_v$

The next step is to suspend the flow of the vector field $\mathcal{W}_v$ to obtain a flow on a a manifold with boundary:

$$\mathbb{W} = [1,3] \times \mathbb{S}^1 \times [-2,2] \cong \mathbf{R} \times \mathbb{S}^1 \tag{2}$$

with cylindrical coordinates $x = (r, \theta, z)$. That is, $\mathbb{W}$ is a solid cylinder with an open core removed, obtained by rotating the rectangle $\mathbf{R}$ defined in (1), considered as embedded in $\mathbb{R}^3$, around the z-axis.

This is done as follows, where we make more precise choices of the suspension flow than given in [39], though these choices do not matter so much. Choose a C^∞-function $f\colon \mathbf{R} \to [-1,1]$ satisfying the conditions:

(W1) $f(r,-z) = -f(r,z)$ [*anti-symmetry in* z]
(W2) $f(r,z) = 0$ for (r,z) near the boundary of $\mathbf{R}$
(W3) $f(r,z) \geq 0$ for $-2 \leq z \leq 0$.
(W4) $f(r,z) \leq 0$ for $0 \leq z \leq 2$.
(W5) $f(r,z) = 1$ for $5/4 \leq r \leq 11/4$ and $-7/4 \leq z \leq -1/4$.
(W6) $f(r,z) = -1$ for $5/4 \leq r \leq 11/4$ and $1/4 \leq z \leq 7/4$.

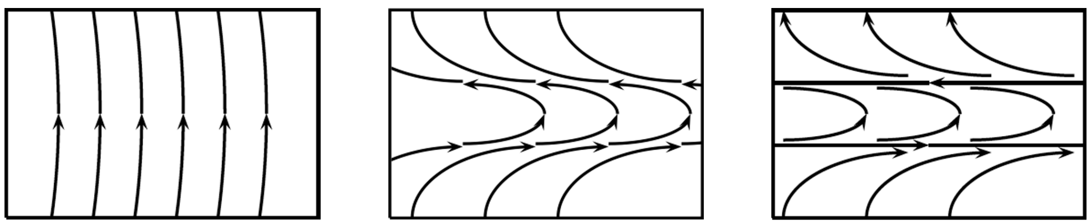
Condition (W1) implies that $f(r,0) = 0$ for all $1 \leq r \leq 3$. The other conditions (W2), (W3), and (W4) are assumed in the works [22, 39, 45] while (W5) and (W6) were imposed in [29] in order to facilitate the description of the dynamics of the Kuperberg flows, but do not qualitatively change the resulting dynamics.

Extend the functions f and g above to $\mathbb{W}$ by setting $f(r,\theta,z) = f(r,z)$ and $g(r,\theta,z) = g(r,z)$, so that they are invariant under rotations around the z-axis. The modified Wilson vector field on $\mathbb{W}$ is given by

$$\mathcal{W} = g(r,\theta,z)\frac{\partial}{\partial z} + f(r,\theta,z)\frac{\partial}{\partial \theta} \ . \tag{3}$$

Observe that the vector field $\mathcal{W}$ is vertical near the boundary of $\mathbb{W}$ and horizontal in the periodic orbits. Also, $\mathcal{W}$ is tangent to the cylinders $\{r = cst\}$.

Let Ψ_t denote the flow of $\mathcal{W}$ on $\mathbb{W}$. The flow of Ψ_t restricted to the cylinders $\{r = cst\}$ is illustrated (in cylindrical coordinate slices) by the lines in Figures 2 and 3. The flow of Ψ_t restricted to the cylinder $\{r = 2\}$ in Figure 2(C) is a called the *Reeb flow*, which Schweitzer remarks in [58] was the inspiration for his introduction of this variation on the Wilson plug.



(A) $r \approx 1, 3$ (B) $r \approx 2$ (C) $r = 2$

FIGURE 2. $\mathcal{W}$-orbits on the cylinders $\{r = cst\}$

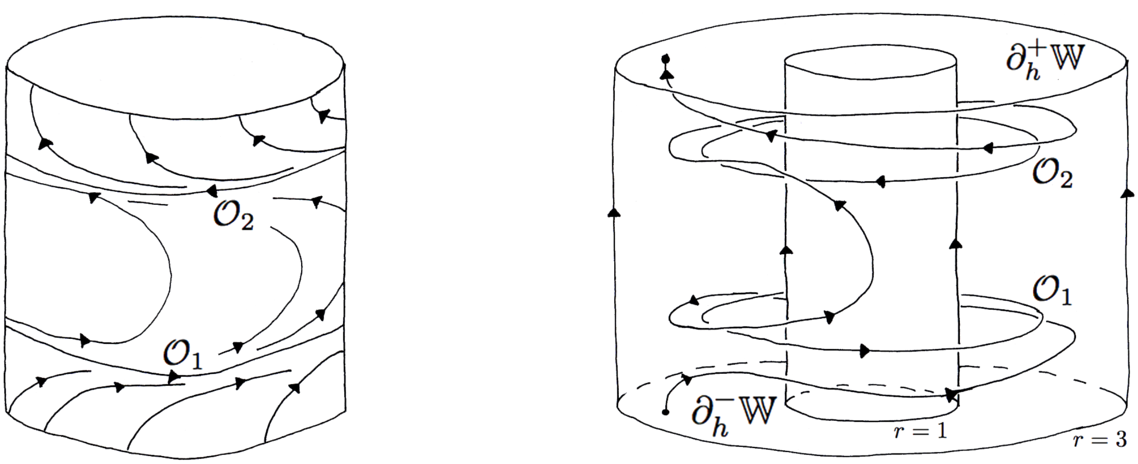


FIGURE 3. $\mathcal{W}$-orbits in the cylinder $\mathcal{C} = \{r = 2\}$ and in $\mathbb{W}$

We will make reference to the following sets in $\mathbb{W}$:

$$\begin{array}{rcl} \mathcal{C} & \equiv & \{r = 2\} \quad [\textit{The Full Cylinder}] \\ \mathcal{R} & \equiv & \{(2,\theta,z) \mid -1 \leq z \leq 1\} \quad [\textit{The Reeb Cylinder}] \\ \mathcal{A} & \equiv & \{z = 0\} \quad [\textit{The Center Annulus}] \\ \mathcal{O}_i & \equiv & \{(2,\theta,(-1)^i)\} \quad [\textit{Periodic Orbits, i=1,2}] \end{array}$$

Note that $\mathcal{O}_1$ is the lower boundary circle of the Reeb cylinder $\mathcal{R}$, and $\mathcal{O}_2$ is the upper boundary circle.

Let us also recall some of the basic properties of the modified Wilson flow, which follow from the construction of the vector field $\mathcal{W}$ and the conditions (W1) to (W4) on the suspension function f.

Let $R_\varphi \colon \mathbb{W} \to \mathbb{W}$ be rotation by the angle φ. That is, $R_\varphi(r,\theta,z) = (r,\theta+\varphi,z)$.

PROPOSITION 2.1. *Let Ψ_t be the flow on $\mathbb{W}$ defined above, then:*

1. *$R_\varphi \circ \Psi_t = \Psi_t \circ R_\varphi$ for all φ and t.*
2. *The flow Ψ_t preserves the cylinders $\{r = cst\}$ and in particular preserves the cylinders $\mathcal{R}$ and $\mathcal{C}$.*
3. *$\mathcal{O}_i$ for $i = 1, 2$ are the periodic orbits for Ψ_t.*
4. *For $x = (2, \theta, -2)$, the forward orbit $\Psi_t(x)$ for $t > 0$ is trapped.*
5. *For $x = (2, \theta, 2)$, the backward orbit $\Psi_t(x)$ for $t < 0$ is trapped.*
6. *For $x = (r, \theta, z)$ with $r \neq 2$, the orbit $\Psi_t(x)$ terminates in the top face $\partial_h^+\mathbb{W}$ for some $t \geq 0$, and terminates in $\partial_h^-\mathbb{W}$ for some $t \leq 0$.*
7. *The flow Ψ_t satisfies the entry-exit condition* (P2) *for plugs.*

The properties of the flow Ψ_t on $\mathbb{W}$ given in Proposition 2.1 are fundamental for showing that the Kuperberg flows constructed in the next section are aperiodic. On the other hand, the study of the further dynamical properties of the Kuperberg flows reveals the importance of the behavior of the flow Ψ_t in open neighborhoods of the periodic orbits $\mathcal{O}_1$ and $\mathcal{O}_2$. This behavior depends strongly on the properties of the function g in open neighborhoods of its vanishing points $(2, \pm 1)$, as will be discussed in later sections. In particular, note that if the function g is modified in arbitrarily small neighborhoods of the points $(2, -1)$ and $(2, 1)$, so that $g(r, z) > 0$ on $\mathbf{R}$, then the resulting flow on $\mathbb{W}$ will have no periodic orbits, and no trapped orbits.

3. The Kuperberg plugs

Kuperberg's construction in [39] of aperiodic smooth flows on plugs introduced a fundamental new idea, that of "geometric surgery" on the modified Wilson plug $\mathbb{W}$ constructed in the previous section, to obtain the *Kuperberg Plug* $\mathbb{K}$ as a quotient space, $\tau \colon \mathbb{W} \to \mathbb{K}$. The essence of the novel strategy behind the aperiodic property of Φ_t is perhaps best described by a quote from the paper by Matsumoto [45]:

> We therefore must demolish the two closed orbits in the Wilson Plug beforehand. But producing a new plug will take us back to the starting line. The idea of Kuperberg is to *let closed orbits demolish themselves.* We set up a trap within enemy lines and watch them settle their dispute while we take no active part.

There are many choices made in the implementation of this strategy, where all such choices result in aperiodic flows. On the other hand, some of the choices appear to impact the further dynamical properties of the Kuperberg flows, as will be discussed later. We indicate in this section these alternate choices, and later formulate some of the questions which appear to be important for further study. Finally, at the end of this section, we consider flows on plugs which violate the above strategy, where the traps for the periodic orbits are purposely not aligned. This results in what we call "Derived from Kuperberg" flows, or simply DK–flows for short, which were introduced in the work [30].

The construction of a Kuperberg Plug $\mathbb{K}$ begins with the modified Wilson Plug $\mathbb{W}$ with vector field $\mathcal{W}$ constructed in Section 2. The first step is to re-embed the manifold $\mathbb{W}$ in $\mathbb{R}^3$ as a *folded figure-eight*, as shown in Figure 4, preserving the vertical direction.

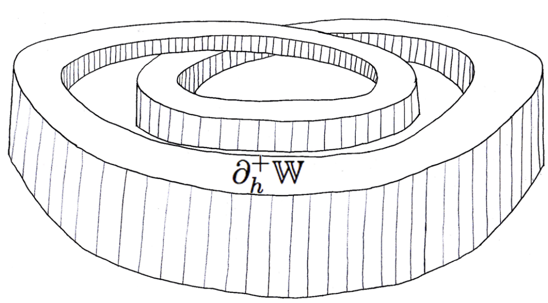


Figure 4. Embedding of Wilson Plug $\mathbb{W}$ as a *folded figure-eight*

The next step is to construct two (partial) insertions of $\mathbb{W}$ in itself, so that each periodic orbit of the Wilson flow is "broken open" by a trapped orbit on the self-insertion.

The construction begins with the choice in the annulus $[1,3] \times \mathbb{S}^1$ of two closed regions L_i, for $i = 1,2$, which are topological disks. Each region has boundary defined by two arcs: for $i = 1,2$, α_i' is the boundary contained in the interior of $[1,3] \times \mathbb{S}^1$ and α_i in the outer boundary contained in the circle $\{r = 3\}$, as depicted in Figure 5. In the work [29] the choices for these arcs are given more precisely, though for our discussion here this is not as important.

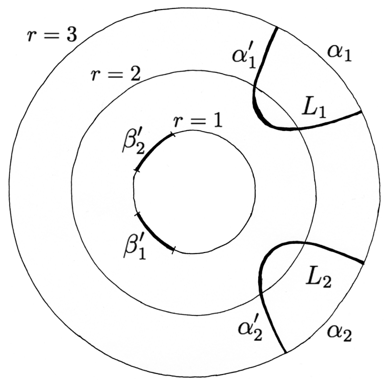


FIGURE 5. The disks L_1 and L_2

Consider the closed sets $D_i \equiv L_i \times [-2,2] \subset \mathbb{W}$, for $i = 1,2$. Note that each D_i is homeomorphic to a closed 3-ball, that $D_1 \cap D_2 = \emptyset$, and each D_i intersects the cylinder $\mathcal{C} = \{r = 2\}$ in a rectangle. Label the top and bottom faces of the closed sets D_i by

$$L_1^{\pm} = L_1 \times \{\pm 2\}\ ,\ L_2^{\pm} = L_2 \times \{\pm 2\}\ . \tag{4}$$

The next step is to define insertion maps $\sigma_i \colon D_i \to \mathbb{W}$, for $i = 1,2$, in such a way that the periodic orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ for the $\mathcal{W}$-flow intersect $\sigma_i(L_i^-)$ in points corresponding to $\mathcal{W}$-trapped points. Consider two disjoint arcs β_i' in the inner boundary circle $\{r = 1\}$ of $[1,3] \times \mathbb{S}^1$, also depicted in Figure 5. Now choose a smooth family of orientation preserving diffeomorphisms $\sigma_i \colon L_i^- \to \mathbb{W}$, $i = 1,2$. Extend these maps to smooth embeddings $\sigma_i \colon D_i \to \mathbb{W}$, for $i = 1,2$, as illustrated on the left-hand-side of Figure 6. We require the following conditions for $i = 1,2$:

- (K1) $\sigma_i(\alpha_i' \times z) = \beta_i' \times z$ for all $z \in [-2,2]$, the interior arc α_i' is mapped to a boundary arc β_i'.
- (K2) $\mathcal{D}_i = \sigma_i(D_i)$ then $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$;
- (K3) For every $x \in L_i$, the image $\mathcal{I}_{i,x} \equiv \sigma_i(x \times [-2,2])$ is an arc contained in a trajectory of $\mathcal{W}$;
- (K4) $\sigma_1(L_1^-) \subset \{z < 0\}$ and $\sigma_2(L_2^+) \subset \{z > 0\}$;
- (K5) Each slice $\sigma_i(L_i \times \{z\})$ is transverse to the vector field $\mathcal{W}$, for all $-2 \leq z \leq 2$.
- (K6) $\mathcal{D}_i$ intersects the periodic orbit $\mathcal{O}_i$ and not $\mathcal{O}_j$, for $i \neq j$.

The "horizontal faces" of the embedded regions $\mathcal{D}_i \subset \mathbb{W}$ are labeled by

$$\mathcal{L}_1^{\pm} = \sigma_1(L_1^{\pm})\ ,\ \mathcal{L}_2^{\pm} = \sigma_2(L_2^{\pm}). \tag{5}$$

Then the above assumptions imply that faces $\mathcal{L}_1^{\pm}$ of the lower insertion region intersect the first periodic orbit $\mathcal{O}_1$ and are disjoint from the second periodic orbit $\mathcal{O}_2$, while the faces $\mathcal{L}_2^{\pm}$ of the upper region intersect the second periodic orbit $\mathcal{O}_2$ and are disjoint from $\mathcal{O}_1$.

The embeddings σ_i are also required to satisfy two further conditions, which are the key to showing that the resulting Kuperberg flow Φ_t is *aperiodic*:

- (K7) For $i = 1,2$, the disk L_i contains a point $(2,\theta_i)$ such that the image under σ_i of the vertical segment $(2,\theta_i) \times [-2,2] \subset D_i \subset \mathbb{W}$ is an arc $\{r = 2\} \cap \{\theta_i^- \leq \theta \leq \theta_i^+\} \cap \{z = (-1)^i\}$ of the periodic orbit $\mathcal{O}_i$.
- (K8) *Radius Inequality*: For all $x' = (r',\theta',z') \in L_i \times [-2,2]$, let $x = (r,\theta,z) = \sigma_i(r',\theta',z') \in \mathcal{D}_i$, then $r < r'$ unless $x' = (2,\theta_i,z')$ and then $r = r' = 2$.

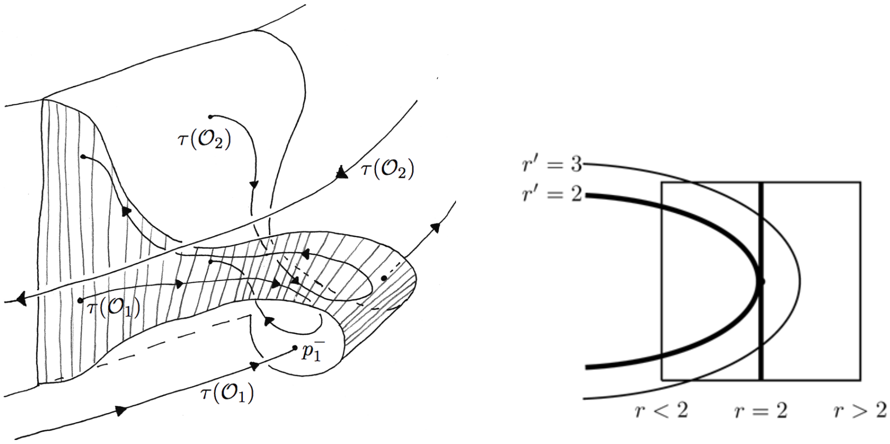


FIGURE 6. The image of $L_1 \times [-2,2]$ under σ_1 and the radius function

The Radius Inequality (K8) is one of the most fundamental concepts of Kuperberg's construction, and is illustrated by the graph on the right-hand-side of in Figure 6.

Condition (K4) and the fact that the flow of the vector field $\mathcal{W}$ on $\mathbb{W}$ preserves the radius coordinate on $\mathbb{W}$, allow restating (K8) for points in the faces $\mathcal{L}_i^-$ of the insertion regions $\mathcal{D}_i$. For $x = (r,\theta,z) = \sigma_i(r',\theta',z') \in \mathcal{D}_i$ we have

$$r(\sigma_i^{-1}(x)) \geq r \text{ for } x \in \mathcal{L}_i^- \ , \text{ with } r(\sigma_i^{-1}(x)) = r \text{ if and only if } x = \sigma_i(2,\theta_i,-2) \ . \tag{6}$$

The illustration of the radius inequality in Figure 6 is an "idealized" case, as it implicitly assumes that the relation between the values of r and r' is "quadratic" in a neighborhood of the special points $(2,\theta_i)$, which is not required in order that (K8) be satisfied.

Finally, define $\mathbb{K}$ to be the quotient manifold obtained from $\mathbb{W}$ by identifying the sets D_i with $\mathcal{D}_i$. That is, for each point $x \in D_i$ identify x with $\sigma_i(x) \in \mathbb{W}$, for $i = 1,2$. The restricted $\mathcal{W}$-flow on the inserted disk $\mathcal{D}_i = \sigma_i(D_i)$ is not compatible with the image of the restricted $\mathcal{W}$-flow on D_i. Thus, to obtain a smooth vector field $\mathcal{X}$ from this construction, it is necessary to modify $\mathcal{W}$ on each insertion region $\mathcal{D}_i$. The idea is to replace the vector field $\mathcal{W}$ in the interior of each region $\mathcal{D}_i$ with the image vector field, and smooth the resulting piecewise continuous flow [39, 22]. Then the vector field $\mathcal{W}'$ on $\mathbb{W}'$ descends to a smooth vector field on $\mathbb{K}$ denoted by $\mathcal{K}$, whose flow is denoted by Φ_t. The family of *Kuperberg Plugs* is the resulting space $\mathbb{K} \subset \mathbb{R}^3$, as illustrated in Figure 7.

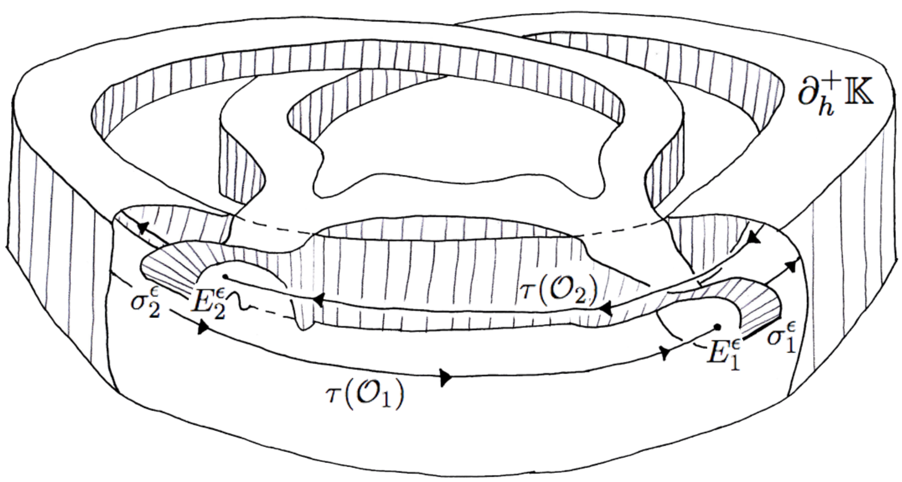


FIGURE 7. The Kuperberg Plug $\mathbb{K}_\epsilon$

The images in $\mathbb{K}$ of the cut-open periodic orbits from the Wilson flow Ψ_t on $\mathbb{W}$, generate two orbits for the Kuperberg flow Φ_t on $\mathbb{K}$, which are called the *special orbits* for Φ_t. These two special orbits play an absolutely central role in the study of the dynamics of the flow Φ_t. We now state Kuperberg's main result:

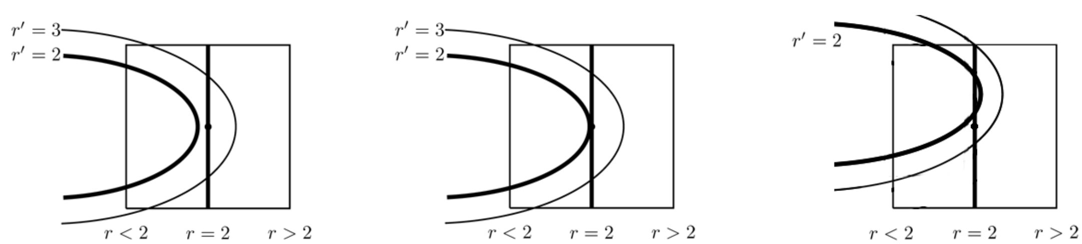
THEOREM 3.1. [39] *The flow Φ_t on $\mathbb{K}$ satisfies the conditions on a plug, and has no periodic orbits.*

The papers [22, 39] remark that a Kuperberg Plug can also be constructed for which the manifold $\mathbb{K}$ and its flow $\mathcal{K}$ are real analytic. An explicit construction of such a flow is given in [40, Section 6]. There is the added difficulty that the insertion of the plug in an analytic manifold must also be analytic, which requires some subtlety. This is discussed in [40, Section 6], and also in the second author's Ph.D. Thesis [56, Section 1.1.1].

Finally, we introduce a modification to the above construction, for which the periodic orbits of the Wilson flow are not necessarily broken open by the trapped orbits of the inserted regions. Let ϵ be a fixed small constant, positive or negative. Choose smooth embeddings $\sigma_i^\epsilon \colon D_i \to \mathbb{W}$, for $i = 1, 2$, again as illustrated on the left-hand-side of Figure 6, which satisfy the conditions (K1) to (K6). In place of the conditions (K7) and (K8), we impose the modified conditions:

- (K7ϵ) For $i = 1, 2$, the disk L_i contains a point $(2, \theta_i)$ such that the image under σ_i^ϵ of the vertical segment $(2, \theta_i) \times [-2, 2] \subset D_i \subset \mathbb{W}$ is contained in $\{r = 2 + \epsilon\} \cap \{\theta_i^- \leq \theta \leq \theta_i^+\}$, and for $\epsilon = 0$ it is contained in $\{r = 2\} \cap \{\theta_i^- \leq \theta \leq \theta_i^+\} \cap \{z = (-1)^i\}$.
- (K8ϵ) *Parametrized Radius Inequality*: For all $x' = (r', \theta', -2) \in L_i^-$, let $x = (r, \theta, z) = \sigma_i^\epsilon(r', \theta', -2) \in \mathcal{L}_i^{\epsilon-}$, then $r < r' + \epsilon$ unless $x' = (2, \theta_i, -2)$ and then $r = 2 + \epsilon$.

Observe that for $\epsilon = 0$, we recover the Radius Inequality (K8). Figure 8 represents the radius inequality for the three cases where $\epsilon < 0$, $\epsilon = 0$, and $\epsilon > 0$. Note that in the third illustration (c) for the case $\epsilon > 0$, the insertion as illustrated has a vertical shift upwards. This is not required by conditions (K7ϵ) and (K8ϵ), but it is used to prove Theorem 10.1 as explained in [30].



(A) $\epsilon < 0$ (B) $\epsilon = 0$ (C) $\epsilon > 0$

FIGURE 8. The modified radius inequality for the cases $\epsilon < 0$, $\epsilon = 0$ and $\epsilon > 0$

Again, define $\mathbb{K}_\epsilon$ to be the quotient manifold obtained from $\mathbb{W}$ by identifying the sets D_i with $\mathcal{D}_i^\epsilon = \sigma_i^\epsilon(D_i)$. Replace the vector field $\mathcal{W}$ on the interior of each region $\mathcal{D}_i^\epsilon$ with the image vector field and smooth the resulting piecewise continuous flow, so that we obtain a smooth vector field on $\mathbb{K}_\epsilon$ denoted by $\mathcal{K}_\epsilon$, whose flow is denoted by Φ_t^ϵ. We say that Φ_t^ϵ is a *Derived from Kuperberg flow*, or a DK–flow.

The dynamics of a DK–flow is actually quite simple in the case when $\epsilon < 0$, as shown by the following result.

THEOREM 3.2. [30] *Let $\epsilon < 0$ and Φ_t^ϵ be a DK–flow on $\mathbb{K}_e$. Then it satisfies the conditions on a plug, and moreover the flow in the plug $\mathbb{K}_\epsilon$ has two periodic orbits that bound an embedded invariant cylinder, and every other orbit belongs to the wandering set.*

The proof of Theorem 3.2 in [30] uses the same technical tools as developed in the previous works [39, 40, 22, 45, 29] for the study of the dynamics of Kuperberg flows. In contrast, the dynamics of a DK–flow when $\epsilon > 0$ can be quite chaotic: the flow has positive topological entropy and has an abundance of periodic orbits, as shown by the construction of examples in [30].

4. Generic hypotheses

The construction of aperiodic Kuperberg flows involve multiple choices, which do not change whether the resulting flows are aperiodic, but do impact other dynamical properties of these flows. In this section, we discuss these choices in more detail, and introduce the generic assumptions that were imposed in the works [29, 30]. The implications of these choices will be discussed in subsequent sections. We first discuss the choices made in constructing the modified Wilson plug, then consider the even wider range of choices involved with the construction of the insertion maps. We discuss first the case for the traditional Kuperberg flows, and afterwards discuss the variations of the construction.

Recall that the modified Wilson vector field on $\mathbb{W}$ is given in (3) by

$$\mathcal{W} = g(r,\theta,z)\frac{\partial}{\partial z} + f(r,\theta,z)\frac{\partial}{\partial \theta}$$

where the function $g(r,\theta,z) = g(r,z)$ is the suspension of the function $g\colon \mathbf{R} \to [0,g_0]$ which is non-negative, vanishing only at the points $(2,\pm 1)$, and symmetric about the line $\{z=0\}$. The function $f(r,\theta,z) = f(r,z)$ is assumed to satisfy the conditions (W1) to (W6), though conditions (W5) and (W6) are imposed to simplify calculations, and do not impact the aperiodic conclusion for the Kuperberg flows.

Figure 2 illustrates the dynamics of the flow of $\mathcal{W}$ restricted to the cylinders $\{r = cst\}$ in $\mathbb{W}$, for various values of the radius. It is clear from these pictures that the "interesting" part of the dynamics of this flow occurs on the cylinders with radius near to 2, and near the periodic orbits $\mathcal{O}_i$ for $i=1,2$.

The points $(2,\pm 1) \in \mathbf{R}$ are the local minima for the function g, and thus its matrix of first derivatives must also vanish at these points, and the Hessian matrix of second derivatives must be positive semi-definite. The generic property for such a function is that the Hessian matrix for g at these points is positive definite. In the works [29, 30], a more precise version of this was formulated:

HYPOTHESIS 4.1. *The function g satisfies the following conditions:*

$$g(r,z) = g_0 \quad \textit{for} \quad (r-2)^2 + (|z|-1)^2 \geq \epsilon_0^2 \tag{7}$$

where $0 < \epsilon_0 < 1/4$ is sufficiently small. Moreover, we require that the Hessian matrices of second partial derivatives for g at the vanishing points $(2,\pm 1)$ are positive definite. In addition, we require that $g(r,z)$ is monotone increasing as a function of the distance $\sqrt{(r-2)^2 + (|z|-1)^2}$ from the points $(2,\pm 1)$.

The conclusions of Proposition 2.1 do not require Hypothesis 4.1, and so Theorem 3.1 does not require it. On the other hand, many of the results in [29, 30] do require this generic hypothesis for their proofs, as it allows making estimates on the "speed of ascent" for the orbits of the Wilson flow near the periodic orbits.

Hypothesis 4.1 implies a local quadratic estimate on the function g near the points $(2,\pm 1)$ which is given as estimate (94) in [29]. We formulate a more general version of this local estimate for g.

HYPOTHESIS 4.2. *Let $n \geq 2$ be an even integer. We say that the vector field $\mathcal{W}$ on* $\mathbb{W}$ vanishes with order n *if there exists constants $\lambda_2 \geq \lambda_1 > 0$ and $\epsilon_0 > 0$ such that*

$$\lambda_1\left((r-2)^n + (|z|-1)^n\right) \leq g(r,z) \leq \lambda_2\left((r-2)^n + (|z|-1)^n\right) \quad \text{for} \ \left((r-2)^n + (|z|-1)^n\right) \leq \epsilon_0^{n+2} \ . \tag{8}$$

Hypothesis 4.1 implies that Hypothesis 4.2 holds for $n=2$. This yields an estimate on the speed which the orbits of $\mathcal{W}$ in $\mathbb{W}$ for points with $z \neq \pm 1$ approach the periodic orbits $\mathcal{O}_i$ in forward or backward time, as discussed in detail in [29, Chapter 17]. When $n > 2$, this speed of approach becomes slower and slower as n gets larger. We can also allow for the case where g has all partial derivatives vanishing at the points $(2,\pm 1)$, in which case we say that the function g vanishes to infinite order at the critical points, and we say that the resulting vector field $\mathcal{W}$ on $\mathbb{W}$ is *infinitely flat* at $\mathcal{O}_i$ for $i=1,2$. In that case, the speed of approach of orbits of $\mathcal{W}$ in $\mathbb{W}$ become arbitrarily slow towards the periodic orbits.

The choices for the embeddings $\sigma_i\colon D_i \to \mathbb{W}$, for $i=1,2$, as illustrated on the left-hand-side of Figure 6, are more wide-ranging, and have a fundamental influence on the dynamics of the resulting Kuperberg flows on the quotient space $\mathbb{K}$. We first impose a "normal form" condition on the insertions, which does not have significant impact on the dynamics, but allows a more straightforward formulation of the other properties of the insertion maps.

Let $(r,\theta,z) = \sigma_i(x') \in \mathcal{D}_i$ for $i = 1,2$, where $x' = (r',\theta',z') \in D_i$ is a point in the domain of σ_i. Let $\pi_z(r,\theta,z) = (r,\theta,-2)$ denote the projection of $\mathbb{W}$ along the z-coordinate. We assume that σ_i restricted to the bottom face, $\sigma_i \colon L_i^- \to \mathbb{W}$, has image transverse to the vertical fibers of π_z. This normal form can be achieved by an isotopy of a given embedding along the flow lines of the vector field $\mathcal{W}$, so does not change the orbit structure of the resulting vector field on the plug $\mathbb{K}$.

The above transversality assumption implies that $\pi_z \circ \sigma_i \colon L_i^- \to \mathbb{W}$ is a diffeomorphism into the face $\partial_h^- \mathbb{W}$, with image denoted by $\mathfrak{D}_i \subset \partial_h^- \mathbb{W}$. Then let $\vartheta_i = (\pi_z \circ \sigma_i)^{-1} \colon \mathfrak{D}_i \to L_i^-$ denote the inverse map, so we have:

$$\vartheta_i(r,\theta,-2) = (r(\vartheta_i(r,\theta,-2)), \theta(\vartheta_i(r,\theta,-2)), -2) = (R_{i,r}(\theta), \Theta_{i,r}(\theta), -2) \ . \tag{9}$$

We can then formalize in terms of the maps ϑ_i the assumptions on the insertion maps σ_i that are intuitively implicit in Figure 6, and will be assumed for all insertion maps considered.

HYPOTHESIS 4.3 (*Strong Radius Inequality*)**.** *For $i = 1,2$, assume that:*

(1) $\sigma_i \colon L_i^- \to \mathbb{W}$ is transverse to the fibers of π_z;
(2) $r = r(\sigma_i(r',\theta',z')) < r'$, except for $(2,\theta_i,z')$ and then $z(\sigma_i(2,\theta_i,z')) = (-1)^i$;
(3) $\Theta_{i,r}(\theta) = \theta(\vartheta_i(r,\theta,-2))$ is an increasing function of θ for each fixed r;
(4) $R_{i,r}(\theta) = r(\vartheta_i(r,\theta,-2))$ has non-vanishing derivative for $r = 2$, except for the case of $\overline{\theta_i}$ defined by $\vartheta_i(2,\overline{\theta_i},-2) = (2,\theta_i,-2)$;
(5) For r sufficiently close to 2, we require that the θ derivative of $R_{i,r}(\theta)$ vanish at a unique point denoted by $\overline{\theta}(i,r)$.

Consequently, each surface $\mathcal{L}_i^-$ is transverse to the coordinate vector fields $\partial/\partial\theta$ and $\partial/\partial z$ on $\mathbb{W}$.

The illustration of the image of the curves $r' = 2$ and $r' = 3$ on the right-hand-side of Figure 6 suggests that these curves have "parabolic shape". We formulate this notion more precisely using the function $\vartheta_i(r,\theta,-2)$ defined by (9), and introduce the more general hypotheses they may satisfy. Recall that $\epsilon_0 > 0$ was introduced in Hypothesis 4.1.

HYPOTHESIS 4.4. *Let $n \geq 2$ be an even integer. For $i = 1,2$, $2 \leq r_0 \leq 2+\epsilon_0$ and $\theta_i - \epsilon_0 \leq \theta \leq \theta_i + \epsilon_0$, assume that*

$$\frac{d}{d\theta}\Theta_{i,r_0}(\theta) > 0 \quad , \quad \frac{d^n}{d\theta^n}R_{i,r_0}(\theta) > 0 \quad , \quad \frac{d}{d\theta}R_{i,r_0}(\overline{\theta_i}) = 0 \quad , \quad \frac{d^\ell}{d\theta^\ell}R_{i,r_0}(\overline{\theta_i}) = 0 \text{ for } 1 \leq \ell < n \ . \tag{10}$$

where $\overline{\theta_i}$ satisfies $\vartheta_i(2,\overline{\theta_i},-2) = (2,\theta_i,-2)$. Thus for $2 \leq r_0 \leq 2+\epsilon_0$, the graph of $R_{i,r_0}(\theta)$ is convex upwards with vertex at $\theta = \overline{\theta_i}$.

In the case where $n = 2$, Hypothesis 4.4 implies that all of the level curves $r' = c$, for $2 \leq c \leq 2+\epsilon_0$, have parabolic shape, as the illustration in Figure 6 suggests. On the other hand, for $n > 2$ the level curves $r' = c$ have higher order contact with the vertical lines of constant radius in Figure 6, and in this case, many of the dynamical properties of the resulting flow Φ_t on $\mathbb{K}$ are not well-understood.

We can now define what is called a generic Kuperberg flow in the work [29].

DEFINITION 4.5. *A Kuperberg flow Φ_t is* generic *if the Wilson flow $\mathcal{W}$ used in the construction of the vector field $\mathcal{K}$ satisfies Hypothesis 4.1, and the insertion maps σ_i for $i = 1,2$ used in the construction of $\mathbb{K}$ satisfies Hypotheses 4.3, and Hypotheses 4.4 for $n = 2$. That is, the singularities for the vanishing of the vertical component $g \cdot \partial/\partial z$ of the vector field $\mathcal{W}$ are of quadratic type, and the insertion maps used to construct $\mathbb{K}$ yield quadratic radius functions near the special points.*

Recall that the insertion maps for a Derived from Kuperberg flow as introduced in Section 3 are denoted by $\sigma_i^\epsilon \colon D_i \to \mathbb{W}$, for $i = 1,2$. It is assumed that these maps satisfy the modified conditions (K7ϵ) and (K8ϵ). The illustrations of the radius inequality in Figure 8 again suggest that the images of the curves $r' = c$ are of "quadratic type", though the vertex of the image curves need no longer be at a special point. We again assume the insertion maps $\sigma_i^\epsilon \colon L_i^- \to \mathbb{W}$ are transverse to the fibers of the projection map $\pi_z \colon \mathbb{W} \to \partial_h^- \mathbb{W}$ along the z'-coordinate. Then we can define the inverse map $\vartheta_i^\epsilon = (\pi_z \circ \sigma_i^\epsilon)^{-1} \colon \mathfrak{D}_i \to L_i^-$ and express the inverse map $x' = \vartheta_i^\epsilon(x)$ in polar coordinates as:

$$x' = (r', \theta', -2) = \vartheta_i^\epsilon(r, \theta, -2) = (r(\vartheta_i^\epsilon(r, \theta, -2)), \theta(\vartheta_i^\epsilon(r, \theta, -2)), -2) = (R_{i,r}^\epsilon(\theta), \Theta_{i,r}^\epsilon(\theta), -2) \ . \tag{11}$$

Then the level curves $r' = c$ pictured in Figure 8 are given by the maps $\theta' \mapsto \pi_z(\sigma_i^\epsilon(c, \theta', -2)) \in \partial_h^- \mathbb{W}$.

We note a straightforward consequence of the Parametrized Radius Inequality (K8ϵ). Recall that θ_i is the radian coordinate specified in (K8ϵ) such that for $x' = (2, \theta_i, -2) \in L_i^-$ we have $r(\sigma_i^\epsilon(2, \theta_i, -2)) = 2 + \epsilon$.

LEMMA 4.6. [30, Lemma 6.1] *For $\epsilon > 0$ there exists $2 + \epsilon < r_\epsilon < 3$ such that $r(\sigma_i^\epsilon(r_\epsilon, \theta_i, -2)) = r_\epsilon$.*

We then add an additional assumption on the insertion maps σ_i^ϵ for $i = 1, 2$ which specifies the qualitative behavior of the radius function for $r \geq r_\epsilon$.

HYPOTHESIS 4.7. *If r_ϵ is the smallest $2 + \epsilon < r_\epsilon < 3$ such that $r(\sigma_i^\epsilon(r_\epsilon, \theta_i, -2)) = r_\epsilon$. Assume that $r(\sigma_i^\epsilon(r, \theta_i, -2)) < r$ for $r > r_\epsilon$.*

The conclusion of Hypothesis 4.7 is implied by the Radius Inequality for the case $\epsilon = 0$, but does not follow from the condition (K8ϵ) when $\epsilon > 0$. It is imposed to eliminate some of the possible pathologies in the behavior of the orbits of the DK–flows.

We can now formulate the analog for DK–flows of the Hypothesis 4.3, which imposes uniform conditions on the derivatives of the maps ϑ_i^ϵ. Recall that $0 < \epsilon_0 < 1/4$ was specified in Hypothesis 4.1, and we assume that $0 < \epsilon < \epsilon_0$.

HYPOTHESIS 4.8 (*Strong Radius Inequality*)**.** *For $i = 1, 2$, assume that:*

1. *$\sigma_i^\epsilon \colon L_i^- \to \mathbb{W}$ is transverse to the fibers of π_z;*
2. *$r = r(\sigma_i^\epsilon(r', \theta', z)) < r + \epsilon$, except for $x' = (2, \theta_i, z)$ and then $r = 2 + \epsilon$;*
3. *$\Theta_{i,r}^\epsilon(\theta)$ is an increasing function of θ for each fixed r;*
4. *For $2 - \epsilon_0 \leq r \leq 2 + \epsilon_0$ and $i = 1, 2$, assume that $R_{i,r}^\epsilon(\theta)$ has non-vanishing derivative, except when $\theta = \overline{\theta_i}$ as defined by $\vartheta_i^\epsilon(2 + \epsilon, \overline{\theta}_i, -2) = (2, \theta_i, -2)$;*
5. *For r sufficiently close to $2 + \epsilon$, we require that the θ derivative of $R_{i,r}^\epsilon(\theta)$ vanishes at a unique point denoted by $\overline{\theta}(i, r)$.*

Note that Hypotheses 4.7 and 4.8 combined imply that r_ϵ is the unique value of $2 + \epsilon < r_\epsilon < 3$ for which $r(\sigma_i^\epsilon(r_\epsilon, \theta_i, -2)) = r_\epsilon$. We can then formulate the analog of Hypothesis 4.4.

HYPOTHESIS 4.9. *Let $n \geq 2$ be an even integer. For $2 - \epsilon_0 \leq r_0 \leq 2 + \epsilon_0$ and $\theta_i - \epsilon_0 \leq \theta \leq \theta_i + \epsilon_0$, assume that*

$$\frac{d}{d\theta} \Theta_{i,r_0}^\epsilon(\theta) > 0 \quad , \quad \frac{d^n}{d\theta^n} R_{i,r_0}^\epsilon(\theta) > 0 \quad , \quad \frac{d}{d\theta} R_{i,r_0}^\epsilon(\overline{\theta_i}) = 0 \quad , \quad \frac{d^\ell}{d\theta^\ell} R_{i,r_0}^\epsilon(\overline{\theta_i}) = 0 \text{ for } 1 \leq \ell < n \ . \tag{12}$$

where $\overline{\theta_i}$ satisfies $\vartheta_i^\epsilon(2, \overline{\theta_i}, -2) = (2, \theta_i, -2)$. Thus for $2 - \epsilon_0 \leq r_0 \leq 2 + \epsilon_0$, the graph of $R_{i,r_0}^\epsilon(\theta)$ is convex upwards with vertex at $\theta = \overline{\theta_i}$.

Finally, we have the definition of the generic DK–flows studied in [30].

DEFINITION 4.10. *A DK–flow Φ_t^ϵ is* generic *if the Wilson flow $\mathcal{W}$ used in the construction of the vector field $\mathcal{K}_\epsilon$ satisfies Hypothesis 4.1, and the insertion maps σ_i^ϵ for $i = 1, 2$ used in the construction of $\mathbb{K}_\epsilon$ satisfies Hypotheses 4.8, and Hypotheses 4.9 for $n = 2$.*

5. Wandering and minimal sets

We next discuss some of the basic topological dynamical properties of the Kuperberg flows. Our main interest is in the asymptotic behavior of their orbits, especially the non-wandering and wandering sets for the flow. There is an additional subtlety in these considerations, in that many orbits for the flow in a plug may escape from the plug, while other orbits are trapped in either the forward or backward directions, or possibly both. We also recall the results about the uniqueness of the minimal set. First we recall some of the basic concepts for the flow in a plug.

Recall that $\mathcal{D}_i = \sigma_i(D_i)$ for $i = 1, 2$ are solid 3-disks embedded in $\mathbb{W}$. Introduce the sets:

$$\mathbb{W}' \equiv \mathbb{W} - \{\mathcal{D}_1 \cup \mathcal{D}_2\} \quad , \quad \widehat{\mathbb{W}} \equiv \overline{\mathbb{W} - \{\mathcal{D}_1 \cup \mathcal{D}_2\}} \ . \tag{13}$$

The compact space $\widehat{\mathbb{W}} \subset \mathbb{W}$ is the result of "drilling out" the interiors of $\mathcal{D}_1$ and $\mathcal{D}_2$.

Let $\tau \colon \mathbb{W} \to \mathbb{K}$ denote the quotient map. Note that the restriction $\tau' \colon \mathbb{W}' \to \mathbb{K}$ is injective and onto, while for $i = 1, 2$, the map τ identifies a point $x \in D_i$ with its image $\sigma_i(x) \in \mathcal{D}_i$. Let $(\tau')^{-1} \colon \mathbb{K} \to \mathbb{W}'$ denote the inverse map, which followed by the inclusion $\mathbb{W}' \subset \mathbb{W}$, yields the (discontinuous) map $\tau^{-1} \colon \mathbb{K} \to \mathbb{W}$, where $i = 1, 2$, we have:

$$\tau^{-1}(\tau(x)) = x \text{ for } x \in D_i \ , \text{ and } \sigma_i(\tau^{-1}(\tau(x))) = x \text{ for } x \in \mathcal{D}_i \ . \tag{14}$$

Consider the embedded disks $\mathcal{L}_i^{\pm} \subset \mathbb{W}$ defined by (5), which appear as the faces of the insertions in $\mathbb{W}$. Their images in the quotient manifold $\mathbb{K}$ are denoted by:

$$E_1 = \tau(\mathcal{L}_1^-) \ , \ S_1 = \tau(\mathcal{L}_1^+) \ , \ E_2 = \tau(\mathcal{L}_2^-) \ , \ S_2 = \tau(\mathcal{L}_2^+) \ . \tag{15}$$

Note that $\tau^{-1}(E_i) = L_i^-$, while $\tau^{-1}(S_i) = L_i^+$.

The *transition points* of an orbit of Φ_t are those points where the orbit intersects one of the sets E_i or S_i for $i = 1, 2$, or is contained in a boundary component $\partial_h^- \mathbb{K}$ or $\partial_h^+ \mathbb{K}$. The transition points are classified as either *primary* or *secondary*, where $x \in \mathbb{K}$ is:

- a *primary entry point* if $x \in \partial_h^- \mathbb{K}$;
- a *primary exit point* if $x \in \partial_h^+ \mathbb{K}$;
- a *secondary entry point* if $x \in E_1 \cup E_2$;
- a *secondary exit point* $x \in S_1 \cup S_2$.

If a Φ_t-orbit of a point $x \in \mathbb{K}$ contains no transition points, then the restriction $\tau^{-1}(\Phi_t(x))$ is a continuous function of t, and is contained in the Ψ_t-orbit of $x' = \tau^{-1}(x) \in \mathbb{W}$.

Recall that $r \colon \mathbb{W} \to [1, 3]$ is the radius coordinate on $\mathbb{W}$. Define the (discontinuous) radius coordinate $r \colon \mathbb{K} \to [1, 3]$, where for $x \in \mathbb{K}$ set $r(x) = r(\tau^{-1}(x))$. Then for $x \in \mathbb{K}$ set $\rho_x(t) \equiv r(\Phi_t(x))$, which is the radius coordinate function along the $\mathcal{K}$-orbit of x. Note that if $\Phi_{t_0}(x)$ is not an entry/exit point, then the function $\rho_x(t)$ is locally constant at t_0. On the other hand, if t_0 is a point of discontinuity for $\Phi_t(x)$, then $x_0 = \Phi_{t_0}(x)$ must be a secondary entry or exit point.

These properties of the radius function along orbits of the flow Φ_t gives a strategy for the study of the dynamics of the flow, and provides the key technique in [39] used to prove that the flow is aperiodic. A key idea is to index the points along the orbit of a point $x \in \mathbb{K}$ by the intersections with the sets $E_1 \cup E_2$, for which the index increases by $+1$, or their intersection with the sets $S_1 \cup S_2$, for which the index decreases by -1. This yields the integer-valued level function $n_x(t)$ which has $n_x(0) = 0$.

Recall that $\mathcal{O}_i$ for $i = 1, 2$ denotes the periodic orbits for the Wilson flow on $\mathbb{W}$, so that each intersection $\mathcal{O}_i \cap \mathbb{W}'$ consists of an open connected arc with endpoints $\mathcal{L}_i^{\pm} \cap \mathcal{O}_i$. The *special entry/exit points* for the flow Φ_t are the images, for $i = 1, 2$,

$$p_i^- = \tau(\mathcal{O}_i \cap \mathcal{L}_i^-) \in E_i \ , \ p_i^+ = \tau(\mathcal{O}_i \cap \mathcal{L}_i^+) \in S_i \ . \tag{16}$$

Note that by definitions and the Radius Inequality, we have $r(p_i^{\pm}) = 2$ for $i = 1, 2$.

We now recall the results for the minimal set of aperiodic Kuperberg flows based on the combined results from the works [22, 39, 40, 45]. It was observed by Kuperberg in [39] that for $x \in \mathbb{K}$ with $r(x) = 2$, then either its forward orbit $\{\Phi_t(x) \mid t \geq 0\}$ contains a special point in its closure, or this is true for the backward orbit $\{\Phi_t(x) \mid t \leq 0\}$, or both conditions hold. Also, for $x \in \mathbb{K}$ if the radius function $\rho_x(t) \geq c$ for some $c > 2$, then the orbit of x escapes in finite time in both forward and backward directions. It follows from this that for $x \in \mathbb{K}$ with $r(x) > 2$ and whose orbit is infinite in either forward or backward directions, then its orbit closure must contain at least one of the special orbits.

It was observed in Matsumoto [45] that there is an open set of primary entry points with radius less than 2 whose forward orbits are non-recurrent and yet accumulate on the special orbits. Ghys showed in [22, Théorème, page 301] that if $x \in \mathbb{K}$ does not escape from $\mathbb{K}$ in a finite time, either forward or backward, then the orbit of the point accumulates on the special orbits. These results combined imply that a Kuperberg flow has a unique minimal set contained in the interior of $\mathbb{K}$.

We state these results more succinctly as follows. Define the following orbit closures in $\mathbb{K}$:

$$\Sigma_1 \equiv \overline{\{\Phi_t(p_1^-) \mid -\infty < t < \infty\}} \quad , \quad \Sigma_2 \equiv \overline{\{\Phi_t(p_2^-) \mid -\infty < t < \infty\}} \ . \tag{17}$$

THEOREM 5.1. [29, Theorem 8.2] *For the closed sets Σ_i for $i = 1, 2$ we have:*

(1) Σ_i is Φ_t-invariant;
(2) $r(x) \geq 2$ for all $x \in \Sigma_i$;
(3) $\Sigma_1 = \Sigma_2$ and we set $\Sigma = \Sigma_1 = \Sigma_2$;
(4) Let $\mathcal{Z} \subset \mathbb{K}$ be a closed invariant set for Φ_t contained in the interior of $\mathbb{K}$, then $\Sigma \subset \mathcal{Z}$;
(5) Σ is the unique minimal set for Φ_t.

The orbits of the Kuperberg flow are divided into those which are finite, forward or backward trapped, or trapped in both directions and so infinite. A point $x \in \mathbb{K}$ is *forward wandering* if there exists an open set $x \in U \subset \mathbb{K}$ and $T_U > 0$ so that for all $t \geq T_U$ we have $\Phi_t(U) \cap U = \emptyset$. Similarly, x is *backward wandering* if there exists an open set $x \in U \subset \mathbb{K}$ and $T_U < 0$ so that for all $t \leq T_U$ we have $\Phi_t(U) \cap U = \emptyset$. A point x with infinite orbit is *wandering* if it is forward and backward wandering. Define the following subsets of $\mathbb{K}$:

$$\begin{aligned}
\mathfrak{W}^0 &\equiv \{x \in \mathbb{K} \mid x \text{ orbit is finite}\} \\
\mathfrak{W}^+ &\equiv \{x \in \mathbb{K} \mid x \text{ orbit is forward wandering}\} \\
\mathfrak{W}^- &\equiv \{x \in \mathbb{K} \mid x \text{ orbit is backward wandering}\} \\
\mathfrak{W}^\infty &\equiv \{x \in \mathbb{K} \mid x \ \text{ is forward and backward wandering}\}
\end{aligned}$$

Note that $x \in \mathfrak{W}^0$ if and only if the orbit of x escapes through $\partial_h^+\mathbb{K}$ in forward time, and escapes though $\partial_h^-\mathbb{K}$ in backward time. Define

$$\mathfrak{W} \ = \ \mathfrak{W}^0 \cup \mathfrak{W}^+ \cup \mathfrak{W}^- \cup \mathfrak{W}^\infty \quad ; \quad \Omega = \mathbb{K} - \mathfrak{W}. \tag{18}$$

The set Ω is called the *non-wandering* set for Φ_t, is closed and Φ_t-invariant. A point x with forward trapped orbit is characterized by the property: $x \in \Omega$ if for all $\epsilon > 0$ and $T > 0$, there exists y and $t > T$ such that $d_{\mathbb{K}}(x, y) < \epsilon$ and $d_{\mathbb{K}}(x, \Phi_t(y)) < \epsilon$, where $d_{\mathbb{K}}$ is a distance function on $\mathbb{K}$. There are obvious corresponding statements for points which are backward trapped or infinite. Here are some of the properties of the wandering and non-wandering sets for Kuperberg flows. The proofs can be found in [29, Chapter 8].

LEMMA 5.2. *If $x \in \mathbb{K}$ is a primary entry or exit point, then $x \in \mathfrak{W}^+$ or $\mathfrak{W}^-$.*

LEMMA 5.3. *For each $x \in \Omega$, the Φ_t-orbit of x is infinite.*

PROPOSITION 5.4. *$\Sigma \subset \Omega \subset \{x \in \mathbb{K} \mid r(x) \geq 2\}$.*

Finally, let us recall a result of Matsumoto:

THEOREM 5.5. [45, Theorem 7.1(b)] *The sets $\mathfrak{W}^\pm$ contain interior points.*

This implies the following important consequence:

COROLLARY 5.6. *The flow Φ_t cannot preserve any smooth invariant measure on $\mathbb{K}$ which assigns positive mass to any open neighborhood of a special point.*

6. Zippered laminations

We next introduce the Φ_t-invariant embedded surface $\mathfrak{M}_0$ and its closure $\mathfrak{M}$, and discuss the relation between the minimal set Σ and the space $\mathfrak{M}$. The existence of this compact connected subset $\mathfrak{M}$, which is invariant for the Kuperberg flow Φ_t, is a remarkable consequence of the construction of Φ_t, and is the key to a deeper understanding of the properties of the minimal set Σ of Φ_t. We then give an overview of the structure theory for $\mathfrak{M}_0$ which plays a fundamental role in analyzing the dynamical properties of Kuperberg flows.

Recall that the Reeb cylinder $\mathcal{R} \subset \mathbb{W}$ is bounded by the two periodic orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ for the Wilson flow Ψ_t on $\mathbb{W}$. The cylinder $\mathcal{R}$ is itself invariant under this flow, and for a point $x \in \mathbb{W}$ with $r(x)$ close to 2, the Ψ_t-orbit of x has increasingly long orbit segments which shadow the periodic orbits.

Introduce the *notched Reeb cylinder*, $\mathcal{R}' = \mathcal{R} \cap \mathbb{W}'$, which has two closed "notches" removed from $\mathcal{R}$ where it intersects the closed insertions $\mathcal{D}_i \subset \mathbb{W}$ for $i = 1, 2$. Figure 9 illustrates the cylinder $\mathcal{R}'$ inside $\mathbb{W}$. The boundary segments γ' and λ' labeled in Figure 9 satisfy $\gamma' \subset \mathcal{L}_1^-$ and $\lambda' \subset \mathcal{L}_2^-$, while the boundary segments $\overline{\gamma}'$ and $\overline{\lambda}'$ labeled in Figure 9 satisfy $\overline{\gamma}' \subset \mathcal{L}_1^+$ and $\overline{\lambda}' \subset \mathcal{L}_2^+$. A basic observation is that these curves are each transverse to the restriction of the Wilson flow to the cylinder $\mathcal{R}$.

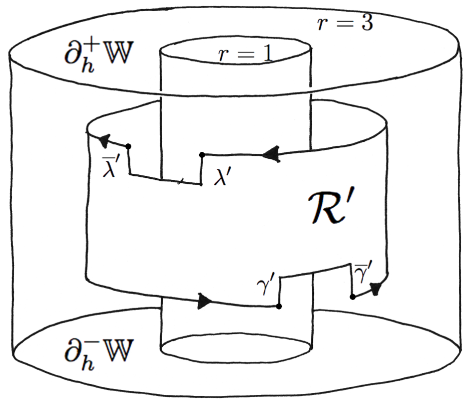


Figure 9. The notched cylinder $\mathcal{R}'$ embedded in $\mathbb{W}$

The map $\tau \colon \mathcal{R}' \to \mathbb{K}$ is an embedding, so the Φ_t-flow of $\tau(\mathcal{R}') \subset \mathbb{K}$ is an embedded surface,

$$\mathfrak{M}_0 \ \equiv \ \{\Phi_t(\tau(\mathcal{R}')) \mid -\infty < t < \infty\} \ . \tag{19}$$

The special orbits in $\mathbb{K}$ contain the intersection $\tau(\mathcal{O}_i \cap \mathbb{W}')$ for $i = 1, 2$, hence the "boundary" of $\mathfrak{M}_0$ consists of the two special orbits in $\mathbb{K}$ obtained by the Φ_t-flows of the arcs $\tau(\mathcal{O}_i \cap \mathbb{W}')$, so that $\mathfrak{M}_0$ is an "infinite bordism" between the two special orbits of the flow Φ_t. Thus, the closure $\mathfrak{M} = \overline{\mathfrak{M}_0}$ is a flow invariant, compact connected subset of $\mathbb{K}$, which contains the closure of the special orbits, hence by Theorem 5.1, the minimal set $\Sigma \subset \mathfrak{M}$.

A fundamental problem is to give a description of the topology and geometry of the space $\mathfrak{M}$. The question of when $\Sigma = \mathfrak{M}$ is treated in Section 7, while in this section we concentrate on the properties of $\mathfrak{M}$.

The key to understanding the structure of the space $\mathfrak{M}$ is to analyze the structure of $\mathfrak{M}_0$ and its embedding in $\mathbb{K}$. This analysis is based on a simple observation, that the images $\tau(\gamma'), \tau(\lambda') \subset \mathfrak{M}$ are curves transverse to the flow Φ_t and contained in the region $\{x \in \mathfrak{M} \mid r(x) \geq 2\}$. Moreover, for a point $x \in \tau(\gamma')$ with $r(x) > 2$, there is a finite $t_x > 0$ such that $\Phi_{t_x}(x) \in \tau(\overline{\gamma}')$. That is, the flow across the notch in $\tau(\mathcal{R}')$ with boundary curve $\tau(\gamma')$ closes up by returning to the facing boundary curve $\tau(\overline{\gamma}')$, *unless* $r(x) = 2$ *and then* x *is the special point* p_1^-. A similar remark holds for the notch in $\mathcal{R}'$ with boundary curves $\lambda', \overline{\lambda}'$. It follows from the proof of the above remarks that we can analyze the submanifold $\mathfrak{M}_0$ using a recursive approach, decomposing the space into the flows in $\mathbb{K}$ of the curves of successive intersections with the entry/exit surfaces E_i and S_i.

PROPOSITION 6.1. [29, Proposition 10.1] *There is a well-defined level function*

$$n_0 \colon \mathfrak{M}_0 \to \mathbb{N} = \{0, 1, 2, \ldots\} \ , \tag{20}$$

where the preimage $n_0^{-1}(0) = \tau(\mathcal{R}')$, *the preimage* $n_0^{-1}(1)$ *in the union of two infinite notched propellers which are asymptotic to* $\tau(\mathcal{R}')$, *and for* $\ell > 1$ *the preimage* $n_0^{-1}(\ell)$ *is an infinite union of finite notched propellers.*

The precise description of propellers, both finite and infinite, is given in [29, Chapters 11, 12], and the decomposition is made precise there. We give a general sketch of the idea.

A *propeller* is an embedded surface in $\mathbb{W}$ that results from the Wilson flow Ψ_t of a curve $\gamma \subset \partial_h^- \mathbb{W}$ in the bottom face of $\mathbb{W}$. Such a surface has the form of a "tongue" wrapping around the core cylinder $\mathcal{C}$. Figure 10 illustrates a "typical" finite propeller P_γ as a compact "flattened" propeller on the right, and its embedding in $\mathbb{W}$ on the left. Observe that in this case, for any $x \in \gamma$, the radius of x is strictly bigger than 2.

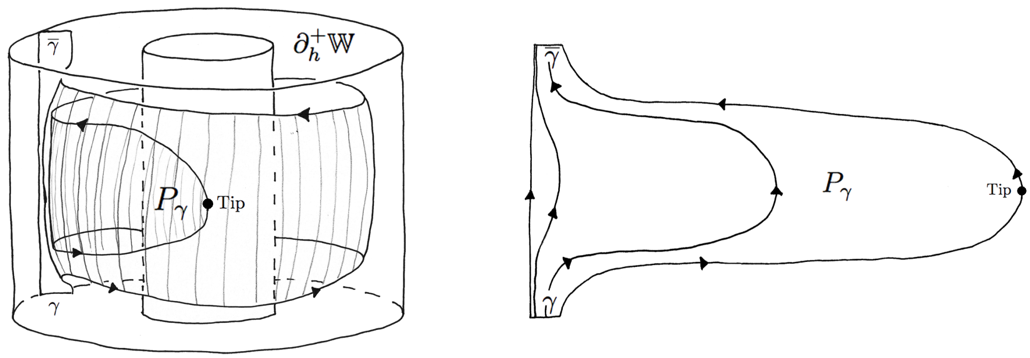


FIGURE 10. Embedded and flattened finite propeller

An infinite propeller is the result of flowing a curve γ which has endpoint on the cylinder $r = 2$, hence the flow of the compact curve is not closed, as its boundary curve is the orbit of an entry point with radius 2, hence limits on the Reeb cylinder $\mathcal{R}$. The embedding of an infinite propeller is highly dependent on the shape of the curve γ near the cylinder $\mathcal{C}$, and on the dynamics of the Wilson flow near its periodic orbits.

Figure 11 gives a model for $\mathfrak{M}_0$, though the distances along propellers are not to scale, and there is a hidden simplification in that there may be "bubbles" in the surfaces which are suppressed in the illustration. A bubble is a compact surface with boundary attached to the interior regions of a propeller along its boundary, and are analyzed in Chapters 15 and 18 of [29]. Also, all the propellers represented in Figure 11 have roughly the same width when embedded in $\mathbb{K}$, which is the width of the Reeb cylinder.

We make some further comments on the properties of $\mathfrak{M}_0$ as illustrated in Figure 11. The upper horizontal band the figure represents the notched Reeb cylinder. The flow of the special point p_1^- is the curve along the bottom edge of the image $\tau(\mathcal{R}')$. When the flow crosses the curve $\tau(\gamma) = \tau(\mathcal{R}') \cap E_1$, it turns to the right and enters the infinite propeller at level 1, and follows the left edge of this vertical strip downward, along the Wilson flow of a point with $r = 2$ until it intersects the secondary entry surface E_1 again. It then turns to the right in the flow direction, and enters a finite propeller at level 2. In the case pictured, it then flows upward until it crosses the annulus $\mathcal{A} = \{z = 0\} \subset \mathbb{W}$, which corresponds to the tip of the propeller. It then reverses direction and flows until it crosses the secondary exit surface S_1, and resumes flowing downward along the infinite level 1 propeller. However, as this is following a Wilson orbit, the z-values of this part of the orbit are increasing towards -1.

This procedure continues repeatedly, though as the curve moves further down the level 1 propeller, the z-values get closer to -1, and hence the flow in the side level 2 propellers intersects the secondary entry region E_1 increasingly often, before flowing through the tip of the corresponding level 2 propeller, and reversing its march through either a secondary exit surface S_i or another secondary entry face E_i. This process can be viewed as a geometric model for the recursive description of the flow dynamics as described using programming language in [40, Section 5]. A key point is that the lengths of the side branches, while finite, increase in length and branching complexity as the orbit moves downwards along the vertical level 1 propeller. A similar scenario plays out when following the upper infinite propeller, whose initial segment is all that is illustrated in Figure 11.

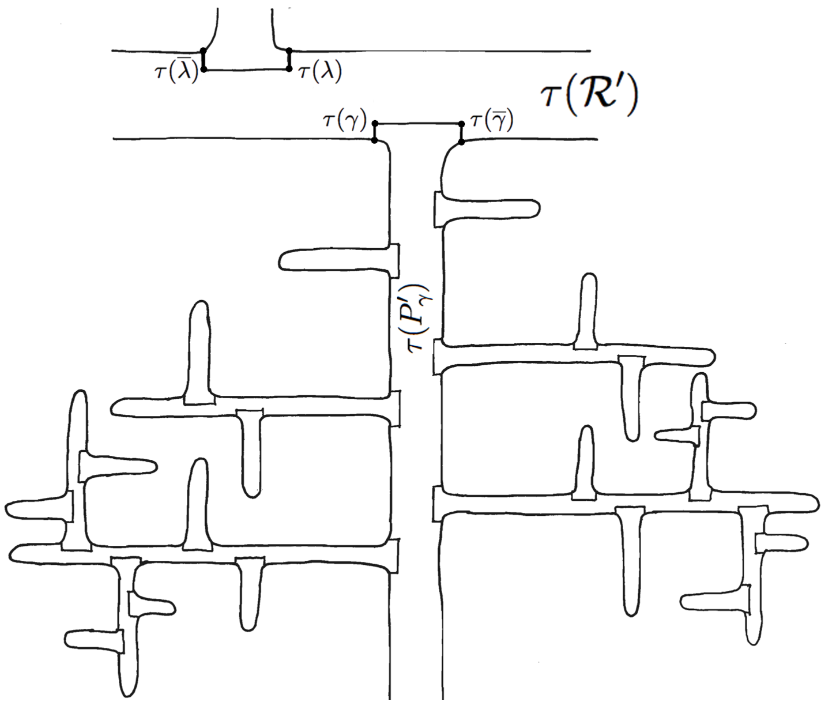


FIGURE 11. Flattened part of $\mathfrak{M}_0$

The two infinite propellers which constitute $n_0^{-1}(1)$ are well-understood, but the finite propellers which constitute the sets $n_0^{-1}(\ell)$ for $\ell > 1$ (pictured as the side branching surfaces in Figure 11) may defy a systematic description without imposing some form of generic hypotheses on the construction of the flow. On the other hand, for a generic Kuperberg flow as in Definition 4.5, the work [29] gives a reasonably complete description of the components of the level decomposition of $\mathfrak{M}_0$. These results are used to show:

THEOREM 6.2. [29, Theorem 19.1] *If Φ_t is a generic Kuperberg flow on $\mathbb{K}$, then the closure $\mathfrak{M}$ in $\mathbb{W}$ of $\mathfrak{M}_0$ is a* zippered lamination.

The definition of a zippered lamination is technical, and given in [29, Definition 19.3]. The notion can be summarized by the conditions that $\mathfrak{M}$ is a union of 2-dimensional submanifolds of $\mathfrak{M}$, and admits a finite cover by special foliation charts which are maps of subsets of $\mathfrak{M}$ to a measurable product of a disk with boundary in $\mathbb{R}^2$ with a Cantor set. In particular, this covering property enables the construction of the transverse holonomy maps along the leaves of the lamination $\mathfrak{M}$. The structure of the submanifold $\mathfrak{M}_0$ is key to understanding the entropy invariants of the flow, and also conjecturally the Hausdorff dimensions of its closed invariant sets, as will be discussed further in Section 8.

7. Denjoy Theory for laminations

The fundamental problem in the study of the dynamics of a Kuperberg flow is to understand the ergodic and topological structure of its unique minimal set Σ. Surprisingly, the ergodic properties of a flow Φ_t is the least well-understood aspect of its dynamics. Since there is a unique minimal set, here is the basic question:

PROBLEM 7.1. *Show that the restriction of a Kuperberg flow to its minimal set is uniquely ergodic. If not, characterize the invariant probability measures for the flow.*

We have that Φ_t is a zero entropy flow on a compact space of dimension at most two, so the problem bears some resemblance to the problem of showing that the horocycle flow for a 2-dimensional Riemann surface of negative curvature is uniquely ergodic, which was proven by Furstenberg [20]. However, the dynamics of the flow Φ_t seems to be much more irregular than for a horocycle flow, as the orbits of Φ_t cluster around the deleted Reeb cylinder $\mathcal{R}'$ for long orbit segments, before wandering out far away from $\mathcal{R}'$ in the plug and then returning. In this sense, the problem may bear more resemblance to the result of Dani and Smillie [13] that the horocycle flow is uniquely ergodic for a Fuchsian group. In any case, the question of whether the Kuperberg flows are uniquely ergodic is very basic.

The remainder of this section will consider questions about the topological dynamics of the flow Φ_t restricted to the invariant space $\mathfrak{M}$. In Section 9 of the paper [40], the authors assert:

> *Although most aperiodic self-inserted Wilson-type plugs have 2-dimensional minimal sets, a carefully chosen self-intersection may result in a 1-dimensional minimal set.*

Since we always have $\Sigma \subset \mathfrak{M}$, the above remark highlights the importance of the following problem:

PROBLEM 7.2. *Give conditions on a Kuperberg flow which imply that $\Sigma = \mathfrak{M}$.*

The equality $\Sigma = \mathfrak{M}$ is a remarkable conclusion, as the flow of the special orbits $p_i^{\pm} \in \mathbb{K}$ are dense in Σ and constitute the boundary of the submanifold $\mathfrak{M}_0$, so the equality $\Sigma = \mathfrak{M}$ implies that the boundaries of a path connected component of $\mathfrak{M}$ are dense in the space itself! This property seems highly improbable.

The result [22, Théorème, page 302] states that there exist Kuperberg flows for which $\Sigma = \mathfrak{M}$, and hence the minimal set Σ is 2-dimensional. The result [40, Theorem 17] gives an explicit analytic flow for which $\Sigma = \mathfrak{M}$. The idea behind these examples is based on the observation that the orbit $\{\Phi_t(p_1^-) \mid -\infty < t < \infty\}$ of a special point p_1^- contains the boundary of all the level 2 propellers represented in Figure 11, thus it contains the tips of these propellers. As the level 2 propellers get longer, the tips have smaller radius that tends to 2. The points corresponding to the tips are contained in the annulus $\tau(\mathcal{A}) = \tau(\{z = 0\})$, and thus accumulate on the Reeb cylinder $\tau(\mathcal{R}')$, and hence on all of $\mathfrak{M}$.

The proof of the following result was inspired by the proof of [40, Theorem 17], and uses these ideas to show:

THEOREM 7.3. [29, Theorem 17.1] *Let Φ_t be a generic Kuperberg flow on $\mathbb{K}$, then $\Sigma = \mathfrak{M}$.*

The proof of Theorem 7.3 uses the generic hypotheses on both the Wilson flow and the insertion maps, to obtain estimates on the density of the orbit $\{\Phi_t(p_1^-) \mid -\infty < t < \infty\}$ near to $\tau(\mathcal{R}')$. While the calculations in [29] use the quadratic assumptions on the maps, it seems reasonable to expect that the required estimates can be achieved for more general cases.

PROBLEM 7.4. *Let Φ_t be a Kuperberg flow on $\mathbb{K}$ which satisfies Hypothesis 4.2 for some even $n \geq 2$, Hypothesis 4.4 for some possibly different value of $n \geq 2$, and otherwise satisfies the generic hypotheses. Show that $\Sigma = \mathfrak{M}$.*

The hypotheses of Problem 7.4 are essentially satisfied for a real analytic Kuperberg flow, so as a variation on Problem 7.4, we ask whether all real-analytic flows have 2-dimensional minimal sets:

PROBLEM 7.5. *Show that $\Sigma = \mathfrak{M}$ if Φ_t is a real analytic Kuperberg flow on $\mathbb{K}$.*

We mention another problem concerning real analytic Kuperberg flows.

PROBLEM 7.6. *Find dynamical properties of Kuperberg flows which distinguish the real analytic flows from the smooth (possibly non-generic) flows.*

Next, consider the case where the inclusion $\Sigma \subset \mathfrak{M}$ of invariant sets is proper. Then Σ is a closed 1-dimensional invariant set in a lamination, which must have some remarkable properties as a subspace of $\mathfrak{M}$. It is not known if such examples can exist for C^1-flows, for example, so we propose:

PROBLEM 7.7. *Construct a C^1 Kuperberg flow for which the minimal set Σ is 1-dimensional.*

One approach might be to perturb a smooth Kuperberg flow in the C^1-topology to obtain a Denjoy-type minimal set contained in the lamination $\mathfrak{M}$, which would be a "wild" version of a Schweitzer plug.

The notion of piecewise-linear (PL)-flows in a plug was developed in Section 8 of [40], and these flows yield a class of dynamical systems on a plug which are distinct from the class of smooth flows. The proof of

Theorem 19 of [40] shows that there are PL-flows on plugs with 1-dimensional minimal sets, though it should be noted that the examples these authors give introduces a modification of the standard construction of the Kuperberg plug as given in Section 3. They also describe an example of a PL-flow for which the minimal set is 2-dimensional. These results suggest that the following project should be very interesting:

PROBLEM 7.8. *Study the dynamical properties of PL-flows in a Kuperberg plug. Find conditions on the constructions of such flows which ensure that the minimal set* Σ *is* 1*-dimensional.*

A good place to start would be to elaborate on the methods introduced in Sections 8 and 9 of [40], and investigate the dynamics of the resulting PL-flows.

Finally, the study of the relationship between Σ and $\mathfrak{M}$ suggests considering a more general problem, whether there exists a type of "Denjoy Theorem" for 2-dimensional laminations, or matchbox manifolds in the terminology of [7].

PROBLEM 7.9. *Let* $\mathcal{L}$ *be a compact connected* 2*-dimensional lamination, possibly with boundary, and let* $\mathcal{X}$ *be a smooth vector field tangent to the leaves of* $\mathcal{L}$*. If the boundaries of the leaves of* $\mathcal{L}$ *are non-empty, we also assume that* $\mathcal{X}$ *is tangent to the boundary. Suppose that* $\mathcal{L}$ *is a minimal lamination, and the flow of* $\mathcal{X}$ *has no periodic orbits, then show that every orbit is dense.*

The question is whether the equality $\Sigma = \mathfrak{M}$ for Kuperberg flows might follow from a more general "Denjoy Principle" which is independent of the embedding of the space $\mathfrak{M} \subset \mathbb{K}$. For example, can the proof of the traditional Denjoy Theorem for C^2-flows on the 2-torus $\mathbb{T}^2$ be adapted to work for laminations? If so, what are the minimal hypotheses required to obtain such a result?

8. Growth, slow entropy, and Hausdorff dimension

We next consider invariants of Kuperberg flows derived from the choice of a Riemannian metric on $\mathbb{K}$. These include the *growth rate of area* for the embedded surface $\mathfrak{M}_0 \subset \mathbb{K}$, the *slow entropy* of the flow Φ_t on $\mathbb{K}$, and the *Hausdorff dimension*s of the closed invariant sets Σ and $\mathfrak{M}$. The authors work [29] gives partial results on the entropy properties for generic flows, and the work of Ingebretson [35] studies the Hausdorff dimension of $\mathfrak{M}$, but almost nothing is known about these dynamical invariants for the case of non-generic flows.

We first consider the geometric properties of the invariant set $\mathfrak{M}$. Recall that $\mathfrak{M}$ is defined as the closure of the connected infinite surface $\mathfrak{M}_0$ with boundary, where $\mathfrak{M}_0$ is defined in (19) as the infinite flow of the notched Reeb cylinder $\mathcal{R}'$. In general, one approach to studying the dynamics of a smooth flow on a compact manifold M is to consider the action of the flow on embedded surfaces in M. For example, Yomdim's proof of the Shub Entropy Conjecture [24, 61, 68] studies the growth rates of the action of the flow on smooth simplices $\Delta^k \subset M$, where Δ^k is a simplex of dimension $0 < k < dim(M)$. For a Kuperberg flow, the essential case is for simplices contained in the laminated space $\mathfrak{M}$. The study of the geometric properties of $\mathfrak{M}_0$ captures all of this information.

Choose a Riemannian metric on $\mathbb{K}$, then the smooth embedded submanifold $\mathfrak{M}_0 \subset \mathbb{K}$ with boundary inherits a Riemannian metric. Let $d_{\mathfrak{M}}$ denote the associated path-distance function on $\mathfrak{M}_0$. Fix the basepoint $\omega_0 = (2, \pi, 0) \in \tau(\mathcal{R}')$ and let $B_{\omega_0}(s) = \{x \in \mathfrak{M}_0 \mid d_{\mathfrak{M}}(\omega_0, x) \leq s\}$ be the closed ball of radius s about the basepoint ω_0. Let $\mathrm{Area}(X)$ denote the Riemannian area of a Borel subset $X \subset \mathfrak{M}_0$. Then $\mathrm{Gr}(\mathfrak{M}_0, s) = \mathrm{Area}(B_{\omega_0}(s))$ is called the *growth function* of $\mathfrak{M}_0$.

Given functions $f_1, f_2 \colon [0, \infty) \to [0, \infty)$, we say that $f_1 \lesssim f_2$ if there exists constants $A, B, C > 0$ such that for all $s \geq 0$, we have that $f_2(s) \leq A \cdot f_1(B \cdot s) + C$. Say that $f_1 \sim f_2$ if both $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ hold. This defines an equivalence relation on functions, which is used to define their *growth type*.

The growth function $\mathrm{Gr}(\mathfrak{M}_0, s)$ for $\mathfrak{M}_0$ depends upon the choice of Riemannian metric on $\mathbb{K}$ and basepoint $\omega_0 \in \mathfrak{M}_0$, however the growth type of $\mathrm{Gr}(\mathfrak{M}_0, s)$ is independent of these choices.

We say that $\mathfrak{M}_0$ has *exponential growth type* if $\mathrm{Gr}(\mathfrak{M}_0, s) \sim \exp(s)$. Note that $\exp(\lambda\, s) \sim \exp(s)$ for any $\lambda > 0$, so there is only one growth class of "exponential type". We say that $\mathfrak{M}_0$ has *nonexponential growth type* if $\mathrm{Gr}(\mathfrak{M}_0, s) \lesssim \exp(s)$ but $\exp(s) \not\lesssim \mathrm{Gr}(\mathfrak{M}_0, s)$. We also have the subclass of nonexponential growth type, where $\mathfrak{M}_0$ has *quasi-polynomial growth type* if there exists $d \geq 0$ such that $\mathrm{Gr}(\mathfrak{M}_0, s) \lesssim s^d$. The growth type of a leaf of a foliation or lamination is an entropy-type invariant of its dynamics, as discussed in [28].

For an embedded propeller $P_\gamma \subset \mathbb{K}$ the area of the propeller increases as it makes successive revolutions around the core cylinder, as illustrated in Figure 10, and this increase is proportional, with uniform bounds above and below, to the number of revolutions times the area of the Reeb cylinder $\mathcal{R}$. Thus, the growth type of $\mathrm{Gr}(\mathfrak{M}_0, s)$ is a measure of the number of branches and their length in $\mathfrak{M}_0$ within a given distance s from ω_0 along the surface. It is thus a measure of the complexity of the recursive procedure which is used to define the level decomposition of $\mathfrak{M}_0$. We first state the most general problem concerning $\mathfrak{M}_0$.

PROBLEM 8.1. *How does the growth type of* $\mathrm{Gr}(\mathfrak{M}_0, s)$ *for a Kuperberg flow depend on the geometry of the insertion maps, and the germ of the Wilson vector field in a neighborhood of the periodic orbits?*

Here is a more precise question about the growth function:

PROBLEM 8.2. *Show that the growth type of* $\mathrm{Gr}(\mathfrak{M}_0, s)$ *for a Kuperberg flow is always nonexponential.*

This problem was answered in [29] in the case where the flow is generic. Under the additional hypothesis on the insertion maps σ_i for $i = 1, 2$, which is that they have "slow growth", the following result is proved.

THEOREM 8.3. [29, Theorem 22.1] *Let* Φ_t *be a generic Kuperberg flow. If the insertion maps* σ_i *for* $i = 1, 2$ *have "slow growth", then the growth type of* $\mathfrak{M}_0$ *is nonexponential, and satisfies* $\exp(\sqrt{s}) \lesssim \mathrm{Gr}(\mathfrak{M}_0, s)$. *In particular,* $\mathfrak{M}_0$ *does not have quasi-polynomial growth type.*

The definition of "slow growth" for the insertion maps is given in [29, Definition 21.11], and will not be recalled here, as it requires some background preparations.

The previous theorem suggests two questions:

PROBLEM 8.4. *Show that the growth type of* $\mathrm{Gr}(\mathfrak{M}_0, s)$ *for a* generic *Kuperberg flow whose insertion maps have slow growth is precisely the growth type of the function* $\exp(\sqrt{s})$.

It seems reasonable to expect this problem has a positive answer. It would also be very interesting to know if the same growth estimate also applies in the case where the flow is real analytic.

PROBLEM 8.5. *Let* Φ_t *be a Kuperberg flow on* $\mathbb{K}$ *which satisfies Hypothesis 4.2 for some even* $n \geq 2$, *Hypothesis 4.4 for some possibly different value of* $n \geq 2$, *and otherwise satisfies the generic hypotheses. Calculate the growth type for* $\mathrm{Gr}(\mathfrak{M}_0, s)$.

One motivation for the study of the growth function $\mathrm{Gr}(\mathfrak{M}_0, s)$ is its relation to the topological entropy invariants for the flow Φ_t. We first discuss the entropy invariants of the flow Φ_t, then explain how these properties are related.

Entropy is defined using a variation of the Bowen formulation of topological entropy [5, 64] for a flow φ_t on a compact metric space (X, d_X), with the definition symmetric in the role of the time variable t. For $\epsilon > 0$, two points $p, q \in X$ are said to be (φ_t, T, ϵ)*-separated* if

$$d_X(\varphi_t(p), \varphi_t(q)) > \epsilon \quad \text{for some} \quad -T \leq t \leq T \ . \tag{21}$$

A set $E \subset X$ is (φ_t, T, ϵ)*-separated* if all pairs of distinct points in E are (φ_t, T, ϵ)-separated. Let $s(\varphi_t, T, \epsilon)$ be the maximal cardinality of a (φ_t, T, ϵ)-separated set in X. The growth type of the function $s(\varphi_t, T, \epsilon)$ is called the ϵ-growth type of φ_t, and we can then study the behavior of the growth type as $\epsilon \to 0$.

The topological entropy of the flow φ_t is then defined by

$$h_{top}(\varphi_t) = \frac{1}{2} \cdot \lim_{\epsilon \to 0} \left\{ \limsup_{T \to \infty} \frac{1}{T} \log(s(\varphi_t, T, \epsilon)) \right\} \ . \tag{22}$$

Moreover, for a compact space X, the entropy $h_{top}(\varphi_t)$ is independent of the choice of metric d_X.

A relative form of the topological entropy for a flow φ_t can be defined for any subset $Y \subset X$, by requiring that the collection of distinct (φ_t, T, ϵ)-separated points used in the definition (21) be contained in Y. The restricted topological entropy $h_{top}(\varphi_t | Y)$ is bounded above by $h_{top}(\varphi_t)$.

For a flow with zero entropy, de Carvalho [9], and Katok and Thouvenot [37], introduced the notion of slow entropy as a measure of the complexity of the flow. For $0 < \alpha < 1$, the slow entropy measures the subexponential growth of the ϵ-separated points, and is given by:

DEFINITION 8.6. *For a flow φ_t on X, and $\alpha > 0$, the α-slow entropy of φ_t is given by*

$$h^{\alpha}_{top}(\varphi_t) = \frac{1}{2} \cdot \lim_{\epsilon \to 0} \left\{ \limsup_{T \to \infty} \frac{1}{T^{\alpha}} \log(s(\varphi_t, T, \epsilon)) \right\} \ . \tag{23}$$

In a later work, Dou, Huang and Park introduced in [14] the derived notion of the entropy dimension of a dynamical system, using the notion of slow entropy:

DEFINITION 8.7. *For a flow φ_t on X, the entropy dimension of φ_t is given by*

$$\mathrm{Dim}_h(\varphi_t) = \inf_{\alpha > 0} \left\{ h^{\alpha}_{top}(\varphi_t) \right\} = 0 \ . \tag{24}$$

For a smooth flow on a compact manifold M, we have $0 \leq \mathrm{Dim}_h(\varphi_t) \leq 1$. Observe that $\mathrm{Dim}_h(\varphi_t)$ is not related to the dimension of the ambient manifold M.

We now return to the consideration of entropy-type invariants for Kuperberg flows. Katok proved in [36, Corollary 4.4] that for a C^2-flow φ_t on a compact 3-manifold, its topological entropy $h_{top}(\varphi_t)$ is bounded above by the exponent of the rate of growth of its periodic orbits. In particular, Katok's result can be applied to an aperiodic flow obtained by inserting a Kuperberg plug, as mentioned in the introduction in Section 1, from which it follows that:

THEOREM 8.8. *Let Φ_t be a Kuperberg flow, then the restricted entropy $h_{top}(\Phi_t|\mathfrak{M}) = 0$.*

The proof of [36, Corollary 4.4] used the Pesin Theory for C^2-flows [1, 54] to obtain a relationship between topological entropy of a flow, its Lyapunov exponents, and the existence of invariant horseshoes in the dynamics of the system. The dynamics of a horseshoe system always includes a dense collection of periodic orbits, so if there are no periodic orbits, then the topological entropy must be zero.

The dynamics of generic Kuperberg flows are explicitly analyzed in [29], and a part of this analysis is to give a direct proof that the topological entropy of the flow vanishes. One key aspect of this analysis is the estimates on the growth function $\mathrm{Gr}(\mathfrak{M}_0, s)$ discussed above. A second key aspect is less obvious but fundamental. Let $\mathbf{R}_0$ be the 2-dimensional rectangular section to the flow, and let $\widehat{\mathfrak{M}}_0$ be the bi-infinite flow of the full cylinder $\mathcal{C}$ by the flow Φ_t introduced in [29, Chapter 13]. Then the intersection $\mathbf{R}_0 \cap \widehat{\mathfrak{M}}_0$ is an infinite family of nested closed curves. Moreover, the induced dynamics of the flow on $\mathbf{R}_0$ can be viewed as a type of iterated function system with an infinite number of generating maps, where each generating map moves one family of nested curves into another family. This is explained in systematic way in the work of Ingebretson [35]. The entropy of the flow restricted to $\mathfrak{M}$ can then be calculated in terms of this iterated function system, as shown in [32]. Using these ideas in [29], the authors showed:

THEOREM 8.9. [29, Theorem 1.7] *Let Φ_t be a generic Kuperberg flow, then $h_{top}(\Phi_t|\mathfrak{M}) = 0$.*

We list several problems about the entropy of Kuperberg flows, which can be considered as "work in progress":

PROBLEM 8.10. *Let Φ_t be a generic Kuperberg flow. Show that $\mathrm{Dim}_h(\Phi_t) = 1/2$.*

For the case of non-generic Kuperberg flows, we expect the following results to be true:

PROBLEM 8.11. *Let Φ_t be a Kuperberg flow, and suppose that the growth type of $\mathfrak{M}_0$ is at least that of the function n^{α}, for $0 < \alpha < 1$. Show that $\mathrm{Dim}_h(\Phi_t) \geq \alpha$.*

PROBLEM 8.12. *Let Φ_t be a Kuperberg flow, and suppose that the Wilson flow used in its construction is infinitely flat near its periodic orbits. Show that $\mathrm{Dim}_h(\Phi_t) = 0$.*

The entropy invariants defined above can also be considered for the for PL-versions of the Kuperberg construction, as in [40, Section 8]. If we allow the Wilson flow to have a discontinuity in its defining function g along the periodic orbits, then it is possible to construct Wilson flows for which the special points ω_i are hyperbolic attracting orbits. In this case, we propose:

PROBLEM 8.13. *Let Φ_t be a PL Kuperberg flow, constructed from a Wilson flow for which the periodic orbits are hyperbolic attracting when restricted to the cylinder $\mathcal{C}$. Show that $h_{top}(\Phi_t|\mathfrak{M}) > 0$.*

In general, it seems likely that the dynamical and ergodic theory properties of PL-versions of the Kuperberg construction will have a much wider range of possibilities, as suggested by the examples in the work [40].

Finally, we mention some natural questions about the Hausdorff dimension properties of Kuperberg flows.

PROBLEM 8.14. *Let Φ_t be a Kuperberg flow. Estimate the Hausdorff dimension $dim_H(\Sigma)$ of its minimal set Σ and the Hausdorff dimension $dim_H(\mathfrak{M})$ of the invariant set $\mathfrak{M}$.*

The best (and only) results to date are in the thesis work of Ingebretson [35]. The difficulty with estimating the Hausdorff dimensions of the invariant sets associated with a Kuperberg flow, is that the standard methods of estimating Hausdorff dimensions of Σ and $\mathfrak{M}$ do not seem to apply. On the other hand, the sets Σ and $\mathfrak{M}$ are both dynamically defined by the action of a pseudogroup $\mathcal{G}_K$ generated by its return map to a section $\mathbf{R}_0$ to the flow in the plug. Ingebretson has shown that for a generic flow, the induced dynamics of the return map on $\mathbf{R}_0$ can be viewed as a variant of an iterated function system with an infinite number of generating maps. One can then define extensions of the Bowen-Ruelle-Sinai pressure function for this type of iterated function system, as in Mauldin and Urbański [46] for example, to obtain estimates on the Hausdorff dimension for an invariant Cantor set in $\mathbf{R}_0$, which yields the above estimate.

For a general Kuperberg flow, it is natural to first consider the problem of estimating the Hausdorff dimension of the closed invariant lamination $\mathfrak{M}$ associated to the flow, as it shares many properties with the generic case, such as the existence of the nested structure for the intersections with a section $\mathbf{R}_0$. However, the structure of the set $\mathfrak{M} \cap \mathbf{R}_0$ as a generalized iterated function system is no longer well-understood, and other estimates that are required to define the Perron-Frobenius operator for the restricted for to $\mathfrak{M}$ are not known. In any case, the following problem seems like a reasonable test case:

PROBLEM 8.15. *Let Φ_t be a real analytic Kuperberg flow. Estimate the Hausdorff dimension $dim_H(\mathfrak{M})$ of the invariant lamination $\mathfrak{M}$.*

We conclude with a general question about the case of PL-flows:

PROBLEM 8.16. *Given $1 < \alpha < 3$, is it possible to construct a PL Kuperberg flow such that $\mathrm{HD}(\Sigma) = \alpha$?*

For a PL-flow, some aspects of the estimations of the pressure function for the set Σ may be much simpler than for the generic case. On the other hand, the structure of the set Σ is not well understood at all.

More generally, the PL-flows constructed using the standard Kuperberg plug construction, or possibly some class of its deformations of the DK type, represent a fascinating are of research. On one hand, the PL-nature of the flow implies a type of discreteness, which should simplify calculations. On the other hand, the deformations of these flows will admit horseshoes, which limit to the aperiodic flow. So one expects a rich class of dynamical systems to result from this construction.

9. Shape theory for the minimal set

Shape theory studies the topological properties of a topological space $\mathfrak{Z}$ using a form of Čech homotopy theory, and this is the natural framework for the study of topological properties of the minimal set Σ of a Kuperberg flow. For example, Krystyna Kuperberg posed the question whether Σ has stable shape? Stable shape is discussed below, and is about the nicest property one can expect for a minimal set that is not a compact submanifold. There are other more delicate shape properties of these spaces to consider. The known results about the shape properties of Σ are all for the generic case.

We first give a brief introduction to the notions of shape theory, and introduce stable shape and the movable property, then discuss the known results and some problems.

The definition of shape for a topological space $\mathfrak{Z}$ was introduced by Borsuk [2, 4]. Later developments and results of shape theory are discussed in the texts [15, 43] and the historical essay [44]. See also the works of Fox [18] and Morita [48].

Recall that a *continuum* is a compact, connected metrizable space. For example, the subspaces Σ and $\mathfrak{M}$ of $\mathbb{K}$ are compact and connected, so are continua. We discuss below shape theory for continua.

DEFINITION 9.1. *Let $\mathfrak{Z} \subset X$ be a continuum embedded in a metric space X. A* shape approximation *of $\mathfrak{Z}$ is a sequence $\mathfrak{U} = \{U_\ell \mid \ell = 1, 2, \ldots\}$ satisfying the conditions:*

(1) each U_ℓ is an open neighborhood of $\mathfrak{Z}$ in X which is homotopy equivalent to a compact polyhedron;
(2) $U_{\ell+1} \subset U_\ell$ for $\ell \geq 1$, and their closures satisfy $\bigcap_{\ell \geq 1} \overline{U}_\ell = \mathfrak{Z}$.

There is a notion of equivalence of shape approximations for continua $\mathfrak{Z}$ and $\mathfrak{Z}'$. In the case where these spaces are embedded in a manifold, the notion of equivalence is discussed in [29, Chapter 23]. Otherwise, any of the sources cited above give the more general definitions of equivalence of shape approximations.

DEFINITION 9.2. *Let $\mathfrak{Z} \subset X$ be a compact subset of a connected manifold X. Then the* shape of $\mathfrak{Z}$ *is defined to be the equivalence class of a shape approximation of $\mathfrak{Z}$ as above.*

It is a basic fact of shape theory that two homotopy equivalent continua have the same shape. Complete details and alternate approaches to defining the shape of a space are given in [15, 43]. An overview of shape theory for continua embedded in Riemannian manifolds is given in [6, Section 2].

For the purposes of defining the shape of the spaces Σ and $\mathfrak{M}$ for a Kuperberg flow, which are both embedded in $\mathbb{K}$, their shape can be defined using a shape approximation $\mathfrak{U}$ defined by a descending chain of open ϵ-neighborhoods in $\mathbb{K}$ of each set. For example, the open sets $U_\ell = \{x \in \mathbb{K} \mid d_{\mathbb{K}}(x, \Sigma) < \epsilon_\ell\}$ where we have $0 < \epsilon_{\ell+1} < \epsilon_\ell$ for all $\ell \geq 1$, and $\lim_{\ell \to \infty} \epsilon_\ell = 0$, give a shape approximation to Σ.

Now we define two basic properties of the shape of a space.

DEFINITION 9.3. *A continuum $\mathfrak{Z}$ has* stable shape *if it is* shape equivalent *to a finite polyhedron. That is, there exists a shape approximation $\mathfrak{U}$ such that each inclusion $\iota\colon U_{\ell+1} \hookrightarrow U_\ell$ induces a homotopy equivalence, and U_1 has the homotopy type of a finite polyhedron.*

Some examples of spaces with stable shape are compact connected manifolds, and more generally connected finite CW-complexes. A less obvious example is the minimal set for a Denjoy flow on $\mathbb{T}^2$, whose shape is equivalent to the wedge of two circles. In particular, the minimal set of an aperiodic C^1-flow on plugs as constructed by Schweitzer in [58] has stable shape. In contrast, the minimal set for a generic Kuperberg flow has very complicated shape, and in particular Theorem 1.5 in [29] shows:

THEOREM 9.4. *The minimal set Σ of a generic Kuperberg flow does not have stable shape.*

The proof of this result uses the detailed structure theory for the space $\mathfrak{M}_0$ developed in [29]. The shape approximations to $\mathfrak{M}$ are obtained by defining a sequence of open neighborhoods U_ℓ using the level decomposition of the double propeller space $\widehat{\mathfrak{M}}_0$ introduced above, to define a sequence of "filled propellers". As the level ℓ used to define the neighborhood U_ℓ increases on infinity, the number of connected components of the intersection with the rectangular section $\mathbf{R}_0 \cap U_\ell$ grows exponentially. The proof shows that these connected components contribute classes to the first homology group of the approximating open set U_ℓ, and moreover the behavior of these homology classes is unstable. One can thus hope for a solution to the following:

PROBLEM 9.5. *Find an alternate, simpler proof of Theorem 9.4 to that given in* [29, Chapter 23]*, to show that the minimal set Σ of a generic Kuperberg flow does not have stable shape.*

One reward of obtaining an alternate approach would be to gain insights towards a solution of the following:

PROBLEM 9.6. *Show that the minimal set Σ of a Kuperberg flow does not have stable shape.*

The idea of the proof would be to show that for other classes of Kuperberg flows, the surface $\widehat{\mathfrak{M}}_0$ is still well-defined, though it an explicit labeling system is not known. None-the-less, one expects that the number of connected components of the intersection with the rectangular section $\mathbf{R}_0 \cap U_\ell$ again tends to infinity, and each connected component contributes an independent class to the first homology group of the approximating open set U_ℓ, and moreover the behavior of these homology classes is unstable. An inspection of the proof in [29, Chapter 23] of Theorem 9.4 shows that the arguments required are quite involved, so an extension of these methods to non-generic flows may be quite difficult.

The construction of the shape approximations U_ℓ using the filled double propellers obtained from $\widehat{\mathfrak{M}}_0$ is very sensitive to the dynamical properties of the Kuperberg flow Φ_t near the Reeb cylinder $\mathcal{R}'$, and these properties depend in very sensitive way on the regularity of the flow near to $\mathcal{R}'$. Thus, one expects there are topological invariants of the shape system for the minimal set Σ which are dependent on the dynamical behavior of the flow. For example, one set of such invariants are the growth rates of the ranks of the homology groups $H_1(U_\ell, \mathbb{Z})$ of the approximating neighborhoods as ℓ tends to infinity. In general, we propose:

PROBLEM 9.7. *Find shape properties which distinguish the minimal set for a generic Kuperberg flow, from the minimal set for a non-generic Kuperberg flow.*

One of the surprise observation from the proof of Theorem 9.4 in [29, Chapter 23] is that the construction of the system of shape approximations $\{U_\ell \mid \ell \geq 1\}$ uses the same properties of the flow that were used to show that it has non-zero slow entropy. In fact, it is natural to speculate that this is not a coincidence:

PROBLEM 9.8. *Let Σ be the minimal set for a Kuperberg flow. Show that the existence of unstable shape approximations to Σ implies that the slow entropy $h^{\alpha}_{top}(\Phi_t) > 0$ for some $0 < \alpha < 1$.*

The idea for proving such a result is to observe that an unstable system of shape approximations to Σ yields families of classes in their homology groups $H_1(U_\ell, \mathbb{Z})$ which are necessarily represented by closed loops whose lengths tend to infinity. These closed loops are shadowed in an appropriate sense by orbits of the flow Φ_t, and hence give rise to families of ϵ-separated points. However, to obtain positive slow entropy for the flow, it must also be shown that the numbers of these loops grow at some rate greater than polynomial, but less than exponential. The hope is that the assumption that Σ is a minimal set for the flow can be used to obtain such growth estimates, due to the "quasi-self-similarity" that minimality implies for $\Sigma \cap \mathbf{R}_0$.

A minimal set is said to be exceptional if it is not a submanifold of the ambient manifold. The previous problem can be stated for any exceptional minimal set: if the minimal set has unstable shape, must the slow entropy of the flow positive for some α?

Finally, we discuss another, more delicate shape property that can be investigated for a continuum.

DEFINITION 9.9. *A continuum $\mathfrak{Z} \subset X$ is said to be* movable *in X if for every neighborhood U of $\mathfrak{Z}$, there exists a neighborhood $U_0 \subset U$ of $\mathfrak{Z}$ such that, for every neighborhood $W \subset U_0$ of $\mathfrak{Z}$, there is a continuous map $\varphi \colon U_0 \times [0,1] \to U$ satisfying the condition $\varphi(x,0) = x$ and $\varphi(x,1) \in W$ for every point $x \in U_0$.*

The notion of a movable continuum was introduced by Borsuk [3] as a generalization of spaces having the shape of an *absolute neighborhood retract* (ANR's). See [6, 15, 38, 43] for further discussions concerning movability. It is a subtle problem to construct continuum which are invariant sets for dynamical systems and which are movable, but do not have stable shape, such as given in [62]. Showing the movable property for a space requires the construction of a homotopy retract φ with the properties stated in the definition, whose existence can be difficult to achieve in practice. There is an alternate condition on homology groups, weaker than the movable condition.

PROPOSITION 9.10. *Let $\mathfrak{Z}$ be a movable continuum with shape approximation $\mathfrak{U}$. Then the homology groups satisfy the* Mittag-Leffler Condition*: For all $\ell \geq 1$, there exists $p \geq \ell$ such that for any $q \geq p$, the maps on homology groups for $m \geq 1$ induced by the inclusion maps satisfy*

$$\mathit{Image}\,\{H_m(U_p; \mathbb{Z}) \to H_m(U_\ell; \mathbb{Z})\} = \mathit{Image}\,\{H_m(U_q; \mathbb{Z}) \to H_m(U_\ell; \mathbb{Z})\} \quad . \tag{25}$$

This result is a special case of a more general Mittag-Leffler condition, as discussed in detail in [6]. For example, the above form of the Mittag-Leffler condition can be used to show that the Vietoris solenoid formed from the inverse limit of coverings of the circle is not movable.

We can now state an additional shape property for the minimal set of a generic Kuperberg flow.

THEOREM 9.11. [29, Theorem 1.6] *Let Σ be the minimal set for a generic Kuperberg flow. Then the Mittag-Leffler condition for homology groups is satisfied. That is, given a shape approximation $\mathfrak{U} = \{U_\ell\}$ for Σ, then for any $\ell \geq 1$ there exists $p > \ell$ such that for any $q \geq p$*

$$Image\{H_1(U_p; \mathbb{Z}) \to H_1(U_\ell; \mathbb{Z})\} = Image\{H_1(U_q; \mathbb{Z}) \to H_1(U_\ell; \mathbb{Z})\}. \tag{26}$$

The proof of Theorem 9.11 in [29, Chapter 23] is even more subtle than the proof of Theorem 9.4, but it suggests the following should be true:

PROBLEM 9.12. *Show that the minimal set Σ for a generic Kuperberg flow is movable.*

The authors believe that this result is true, based on explicit constructions of homotopies between the shape approximations constructed in [29, Chapter 23]. However, these calculations were not included in this work, as they demanded even more complicated and explicit constructions of an infinite sequences of maps. Furthermore, it is a conjecture that these constructions suffice to show that Σ is movable. It would be very pleasant if there was a more direct solution to Problem 9.12 using methods analogous to those in [16].

On the other hand, it would be very remarkable if the minimal set for all Kuperberg flows should be movable, so perhaps the following can be shown:

PROBLEM 9.13. *Construct an example of a Kuperberg flow such that the minimal set Σ is not movable.*

There are two approaches to this problem that might be reasonable. The first is to construct a generic Kuperberg flow which is "extremely regular", possibly this means real analytic, but for which the constructions of the shape approximations in [29, Chapter 23] can be worked with an analyzed more easily. The second suggestion is that as part of the study of the dynamics of PL-flows in a Kuperberg construction, it may again be possible to precisely control the construction of the approximating open neighborhoods U_ℓ for the minimal set, and so deduce that it has unstable shape.

10. Derived from Kuperberg flows

In this final section, we discuss the variant on the construction of the Kuperberg flows which was introduced in the authors' work [30], and discussed at the end of Section 3. This new class of flows are called "Derived from Kuperberg" flows, or DK–flows, as they are constructed using the same method as for the standard Kuperberg flows, and are in fact smooth variations of these flows. However, the resulting flows need not be aperiodic, and it is their other dynamical properties that motivate their study.

The DK–flows were constructed at the end of Section 3, and the generic hypotheses on these flows was formulated in Definition 4.10. Then in [30, Section 9.2] the *admissibility condition* was formulated for these flows. Briefly, this condition is that there exist a constant $C > 0$, which depends only on the generic Wilson flow used in the construction, so that if the vertical offset in the z-coordinate of the vertex of an insertion map σ_i^ϵ is $\delta > 0$, then we assume that the horizontal offset ϵ satisfies $0 < \epsilon < C \cdot \sqrt{\delta}$. Then we have:

THEOREM 10.1. [30, Theorem 9.5] *For $\epsilon > 0$, let Φ_t^ϵ be a generic DK–flow on $\mathbb{K}_\epsilon$ which satisfies the admissibility condition. Then Φ_t^ϵ has an invariant horseshoe dynamical system, and thus $h_{top}(\Phi_t^\epsilon) > 0$. In fact, each such DK–flow Φ_t^ϵ has countably many independent horseshoe subsystems.*

The proof of this result in [30] requires the introduction of the pseudogroup defined by the first return map of the flow Φ_t^ϵ to the rectangular cross-section $\mathbf{R}_0$, as constructed in [29, Chapter 9] or [30, Section 7]. The second reference gives just the bare minimum of details of the construction of this pseudogroup as required to prove Theorem 10.1, while the first reference has a comprehensive discussion of the pseudogroups associated to Kuperberg flows. Then the existence of a horseshoe dynamical subsystem embedded in these flows is shown to exist using the shape approximations $\{U_\ell\}$ introduced in [29, Chapter 23] and discussed in Section 9 above. The pseudogroup action induced on the connected components of $U_\ell \cap \mathbf{R}_0$ is shown to produce horseshoe subsystems if the perturbed flow Φ_t^ϵ satisfies the admissibility condition.

The conclusion of Theorem 10.1 is that a generic aperiodic Kuperberg flow is a zero entropy flow which is the limit of a smooth deformation of flows with positive entropy. The study of the generic properties of flows, as discussed for example in the works [8, 21, 51, 52, 55], shows that the aperiodic flows admit arbitrarily close flows with positive entropy, in the C^1-topology on flows, but the existence of smooth deformations is more special.

It is natural to ask whether the admissibility hypothesis in Theorem 10.1 is necessary.

PROBLEM 10.2. *Show that the topological entropy $h_{top}(\Phi_t^\epsilon) > 0$ for a DK–flow Φ_t^ϵ on $\mathbb{K}_\epsilon$ with $\epsilon > 0$.*

A solution of this may just require a technical extension of the ideas used in the proof of Theorem 10.1, though it is also possible that there are novel dynamical problems which arise in the study of it.

There is another variation on the DK construction, using various alternative surgery schemes. In Section 3 of [40], the authors introduce the notion of "bridge immersions" of a plug. Then in Section 5 of that work, they show that plugs with bridge immersions also yield 3-dimensional aperiodic plugs. The use of bridge immersions of a plug allows for many more variations of the constructions of DK-flows.

PROBLEM 10.3. *Study the dynamical properties of an aperiodic Kuperberg flow obtained via a bridge immersion, and for flows obtained by smooth variations of aperiodic flows from bridge immersions.*

Essentially nothing is know about the structure of the minimal sets for bridge immersed Kuperberg flows and their dynamical properties. Even less is known for the DK flows obtained by smooth variations of the insertion maps in bridge immersed flows.

Theorem 10.1 states that there exist smooth families of DK–flows with positive entropy, which limit on a given generic Kuperberg flow. The dynamical properties of the horseshoes for the perturbed flow Φ_t^ϵ are still unexplored. One fascinating question is the relation between the deformation of these dynamical systems to an aperiodic flow, and whether the limit flow is uniquely ergodic, as suggested in Problem 7.1.

PROBLEM 10.4. *Let $\{\Phi_t^\epsilon \mid 0 < \epsilon \leq \epsilon_0\}$ be a family of generic DK–flows on $\mathbb{K}_\epsilon$ which converge to a generic Kuperberg flow Φ_t in the C^∞-topology of flows. Study the limiting behavior of the periodic orbits for the invariant horseshoes for the flows Φ_t^ϵ as $\epsilon \to 0$? Describe the invariant measures on the minimal set Σ which arise as the limits of periodic orbits for Φ_t^ϵ.*

The construction of the horseshoe dynamics for the generic DK–flows in Theorem 10.1 are based on choosing appropriate compact branches of the embedded surface $\mathfrak{M}_0$ as discussed in [30]. These compact surfaces approximately generate the orbits which define the horseshoe, and so behave much like a template for the horseshoes created [65, 66].

PROBLEM 10.5. *Show that the horseshoes for a generic DK–flow with positive entropy are carried by templates derived from the compact pieces of $\mathfrak{M}_0$.*

The dynamics of the DK–flows appears to be reminiscent of the analysis of the dynamics of Lorenz attractors, as discussed for example in the survey by Ghys [23]. Moreover, the variation of the horseshoes for a smooth family for generic DK–flows with positive entropy suggests a comparison with the degeneration in the dynamics of the Lorenz attractors as studied by de Carvalho and Hall [10, 11, 12, 25]. The analogy between the dynamics of a family of generic DK–flows and a family of Lorenz attractors suggests that the topic is worth further investigation.

We conclude this discussion of the dynamics of the variations on the construction of Kuperberg flows with a curious problem. The periodic orbits for the modified Wilson plugs constructed in Section 2 can be destroyed if the function g used in the definition of the Wilson flow in (3) is perturbed so that it is non-vanishing. Then every orbit for the Wilson flow escapes from the plug. One can then construct a Kuperberg plug, as in Section 3, using such a Wilson plug that stops no orbits. However, inserting the plug no longer breaks open the periodic orbits. The problem is to describe the dynamical properties of a perturbed flow Φ_t^ϵ constructed using a non-stopping Wilson flow.

PROBLEM 10.6. *Let Φ_t^ϵ be a generic DK–flow on $\mathbb{K}_\epsilon$ with $\epsilon > 0$, and let ϕ_t be the flow obtained from the construction of Φ_t^ϵ by taking a sufficiently small smooth perturbation of the function g used in the construction of the Wilson flow which removes its vanishing points. Show that the flow ϕ_t has an invariant horseshoe dynamical system, and thus $h_{top}(\phi_t) > 0$.*

If these perturbed flows do contain horseshoe dynamical systems, then they have positive topological entropy. Thus, by taking a smooth deformation of the functions g above, one obtains another family of positive entropy flows which limit smoothly on the aperiodic Kuperberg flows. This problem is discussed further in [31].

References

[1] L. Barreira and Ya.B. Pesin, **Lyapunov exponents and smooth ergodic theory**, University Lecture Series. Vol. 23, American Mathematical Society, Providence, RI, 2002.
[2] K. Borsuk, *Concerning homotopy properties of compacta*, **Fund. Math.**, 62:223–254, 1968.
[3] K. Borsuk, *On movable compacta*, **Fund. Math.**, 66:137–146, 1969.
[4] K. Borsuk, **Theory of shape**, Monografie Mat., vol. 59, Polish Science Publ., Warszawa, 1975..
[5] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, **Trans. Amer. Math. Soc.**, 153:401–414, 1971.
[6] A. Clark and J. Hunton, *Tiling spaces, codimension one attractors and shape*, **New York J. Math.**, 18:765–796, 2012.
[7] A. Clark and S. Hurder, *Homogeneous matchbox manifolds*, **Trans. A.M.S**, 365:3151–3191, 2013.
[8] S. Crovisier and D. Yang, *On the density of singular hyperbolic three-dimensional vector fields: a conjecture of Palis*, **C. R. Math. Acad. Sci. Paris**, 353:85–88, 2015.
[9] A. de Carvalho, *Entropy dimension of dynamical systems*, **Portugal. Math.**, 54:19–40, 1997.
[10] A. de Carvalho, *Pruning fronts and the formation of horseshoes*, **Ergodic Theory Dynam. Systems**, 19:851–894, 1999.
[11] A. de Carvalho and T. Hall, *How to prune a horseshoe*, **Nonlinearity**, 15:R19–R68, 2002.
[12] A. de Carvalho and T. Hall, *The forcing relation for horseshoe braid types*, **Experiment. Math.**, 11:271–288, 2002.
[13] S.G. Dani and J. Smillie, *Uniform distribution of horocycle orbits for Fuchsian groups*, **Duke Math. J.**, 51:185–194, 1984.
[14] D. Dou, W. Huang and K.K. Park, *Entropy dimension of topological dynamical systems*, **Trans. Amer. Math. Soc.**, 363:659–680, 2011.
[15] J. Dydak and J. Segal, **Shape theory**, Lecture Notes in Mathematics Vol. 688, Springer, Berlin, 1978.
[16] J. Dydak and J. Segal, *Approximate polyhedra and Shape Theory*, **Topology Proceedings**, 6:279–286, 1982.
[17] J. Etnyre and R. Ghrist, *Contact topology and hydrodynamics. I. Beltrami fields and the Seifert conjecture*, **Nonlinearity**, 13:441–458, 2000.
[18] R.H. Fox, *On shape*, **Fund. Math.**, 74:47–71, 1972.
[19] S. Frankel, *Quasigeodesic flows and Möbius-like groups*, **J. Differential Geom.**, 93:401–429, 2013.
[20] H. Furtsenberg, *The unique ergodicity of the horocycle flow*, in **Recent advances in topological dynamics (Proc. Conf., Yale Univ., New Haven, Conn., 1972; in honor of Gustav Arnold Hedlund)**, Lecture Notes in Math., Vol. 318, 1973, pages 95–115.
[21] S. Gan and D. Yang, *Morse-Smale systems and horseshoes for three dimensional singular flows*, preprint, 2016.
[22] É Ghys, *Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)*, Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, **Astérisque**, Vol. 227: 283–307, 1995.
[23] É Ghys, *The Lorenz attractor, a paradigm for chaos*, in **Chaos**, Prog. Math. Phys. Vol. 66, Birkhäuser/Springer, Basel, 2013, pages 1–54.
[24] M. Gromov, *Entropy, homology and semialgebraic geometry*, **Séminaire Bourbaki, Exp. No. 663**, 1985–1986.
[25] T. Hall, *The creation of horseshoes*, **Nonlinearity**, 3:861–924, 1994.
[26] J.C. Harrison, *C^2 counterexamples to the Seifert conjecture*, **Topology**, 27:249–278, 1988.
[27] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, **Invent. Math.**, 114:515–563, 1993.
[28] S. Hurder, *Lectures on Foliation Dynamics: Barcelona 2010*, **Proceedings of Conference on Geometry and Topology of Foliations (C.R.M. 2010)**, 2014.
[29] S. Hurder and A. Rechtman, *The dynamics of generic Kuperberg flows*, **Astérisque**, Vol. 377:1–250, 2016.
[30] S. Hurder and A. Rechtman, *Aperiodicity at the boundary of chaos*, **Ergodic Theory Dyn. Systems**, to appear; arXiv:1603.07877.
[31] S. Hurder and A. Rechtman, *The dynamics of Derived from Kuperberg flows*, **in preparation**, 2017.
[32] S. Hurder, and D. Ingebretson, *Smooth flows with fractional entropy dimension*, **in preparation**, 2017.
[33] M. Hutchings and C. Taubes, *The Weinstein conjecture for stable Hamiltonian structures*, **Geom. Topol.**, 13:901–941, 2009.
[34] M. Hutchings, *Taubes's proof of the Weinstein conjecture in dimension three*, **Bull. Amer. Math. Soc. (N.S.)**, 47:73–125, 2010.
[35] D. Ingebretson, *Hausdorff dimension of generic Kuperberg minimal sets*, **Thesis**, University of Illinois at Chicago, 2017.
[36] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, **Inst. Hautes Études Sci. Publ. Math.**, 51:137–173, 1980.
[37] A. Katok and J.-P. Thouvenot, *Slow entropy type invariants and smooth realization of commuting measure-preserving transformations*, **Ann. Inst. H. Poincaré Probab. Statist.**, 33:323–338, 1997.
[38] J. Krasinkiewicz, *On pointed* 1*-movability and related notions*, **Fund. Math.**, 114:29–52, 1981.
[39] K. Kuperberg, *A smooth counterexample to the Seifert conjecture*, **Ann. of Math. (2)**, 140:723–732, 1994.
[40] G. Kuperberg and K. Kuperberg, *Generalized counterexamples to the Seifert conjecture*, **Ann. of Math. (2)**, 144:239–268, 1996.
[41] G. Kuperberg, *A volume-preserving counterexample to the Seifert conjecture*, **Comment. Math. Helv.**, 71:70–97, 1996.
[42] K. Kuperberg, *Aperiodic dynamical systems*, **Notices Amer. Math. Soc.**, 46, no. 9:1035–1040, 1999.

[43] S. Mardešić and J. Segal, **Shape theory: The inverse system approach**, North-Holland Math. Library, Vol. 26, North-Holland Publishing Co., Amsterdam, 1982.
[44] S. Mardešić, *Absolute neighborhood retracts and shape theory*, in **History of topology**, pages 241–269, North-Holland Publishing Co., Amsterdam, 1999.
[45] S. Matsumoto, *K. M. Kuperberg's C^∞ counterexample to the Seifert conjecture*, **Sūgaku**, Mathematical Society of Japan, 47:38–45, 1995. Translation: **Sugaku Expositions**, 11:39–49, 1998, Amer. Math. Soc.
[46] D. Mauldin and M. Urbański, *Dimensions and measures in infinite iterated function systems*, **Proc. London. Math. Soc.**, 73(105–154, 496, 1996.
[47] V.S. Medvedev, *A new type of bifurcations on manifolds*, **Mat. Sb. (N.S.)**, 113(155):487–492, 496, 1980.
[48] K. Morita, *On shapes of topological spaces*, **Fund. Math.**, 86:251–259, 1975.
[49] K. Niederkrüger and A. Rechtman, *The Weinstein conjecture in the presence of submanifolds having a Legendrian foliation*, **J. Topol. Anal.**, 3:405–421, 2011.
[50] J. Palis and C. Pugh, *Fifty problems in dynamical systems*, in **Dynamical systems—Warwick 1974 (Proc. Sympos. Appl. Topology and Dynamical Systems, Univ. Warwick, Coventry, 1973/1974; presented to E. C. Zeeman on his fiftieth birthday)**, Lecture Notes in Math., Vol. 468, pages 345–353, Springer, Berlin, 1975
[51] J. Palis, *A global view of dynamics and a conjecture on the denseness of finitude of attractors*, **Astérisque**, Vol. 261:335–347, 2000.
[52] J. Palis, *Open questions leading to a global perspective in dynamics*, **Nonlinearity**, 21:T37–T43, 2008.
[53] P.B. Percell and F.W. Wilson, Jr., *Plugging flows*, **Trans. Amer. Math. Soc.**, 233:93–103, 1977.
[54] Ya.B. Pesin, *Characteristic Ljapunov exponents, and smooth ergodic theory*, **Uspehi Mat. Nauk**, 32:55–112, 1977. Transl. **Russian Math. Surveys**, 32(4):55–114, 1977.
[55] E. Pujals and M. Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms*, **Ann. of Math. (2)**, 151:961–1023, 2000.
[56] A. Rechtman, *Pièges dans la théorie des feuilletages: exemples et contre-exemples*, **Thèse**, l'Université de Lyon – École Normale Supérieure de Lyon, 2009.
[57] A. Rechtman, *Existence of periodic orbits for geodesible vector fields on closed 3-manifolds*, **Ergodic Theory Dynam. Systems**, 30:1817–1841, 2010.
[58] P.A. Schweitzer, *Counterexamples to the Seifert conjecture and opening closed leaves of foliations*, **Ann. of Math. (2)**, 100:386–400, 1974.
[59] H. Seifert, *Closed integral curves in* 3*-space and isotopic two-dimensional deformations*, **Proc. Amer. Math. Soc.**, 1:287–302, 1950.
[60] A. Shilnikov, L. Shilnikov and D. Turaev, *Blue-sky catastrophe in singularly perturbed systems*, **Mosc. Math. J.**, 5:269–282, 2005.
[61] M. Shub, *Dynamical systems, filtrations and entropy*, **Bull. Amer. Math. Soc.**, 80:27–41, 1974.
[62] P. Šindelářová, *An example on movable approximations of a minimal set in a continuous flow*, **Topology Appl.**, 154:1097–1106, 2007.
[63] C. Taubes, *The Seiberg-Witten equations and the Weinstein conjecture*, **Geom. Topol.**, 11:2117–2202, 2007.
[64] P. Walters, **An Introduction to Ergodic Theory**, Graduate Texts in Math. Vol. 79, Springer-Verlag, New York, 1982.
[65] R.F. Williams, *Expanding attractors*, **Inst. Hautes Études Sci. Publ. Math.**, 43:169–203, 1974.
[66] R.F. Williams, *The structure of Lorenz attractors*, **Inst. Hautes Études Sci. Publ. Math.**, 50:73–99, 1979.
[67] F.W. Wilson, Jr., *On the minimal sets of non-singular vector fields*, **Ann. of Math. (2)**, 84:529–536, 1966.
[68] Y. Yomdim, *Volume growth and entropy*, **Israel J. Math.**, 57:285–300, 1987.

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