ORBIT EQUIVALENCE AND CLASSIFICATION OF WEAK SOLENOIDS

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ABSTRACT. In this work, we study minimal equicontinuous actions which are locally quasi-analytic. The first main result shows that for minimal equicontinuous actions which are locally quasi-analytic, continuous orbit equivalence of the actions implies return equivalence. This generalizes results of Cortez and Medynets, and of Li. The second main result is that if G is a finitely-generated, virtually nilpotent group, then every minimal equicontinuous action by G is locally quasi-analytic. As an application, we show that the homeomorphism type of a nil-solenoid is determined by the virtual topological full group of its monodromy action.

1. INTRODUCTION AND MAIN RESULTS

The works of McCord [37] and Schori [46] in the 1960's, and the work of Fokkink and Oversteegen in [24], studied the self-homeomorphisms of weak solenoids. The authors' work in [15] studied the more general problem of the classification, up to homeomorphism, for weak solenoids, where it was shown that for certain classes of weak solenoids, two weak solenoids are homeomorphic if and only if their global monodromy actions are return equivalent.

The global monodromy of a weak solenoid is a minimal equicontinuous Cantor action of a finitely generated group, so the classification problem for weak solenoids motivates the study of invariants for such actions. The purpose of this work is to consider the topological full group as such an invariant, and to find conditions on the actions which imply that they are return equivalent, and hence provide a solution to the classification problem. For the class of nil-solenoids, Theorem 1.8 gives such a solution. Before stating our main results precisely, we first recall a few basic concepts.

Let G be a countably generated discrete group, and $\Phi\colon G \to \mathrm{Homeo}(\mathfrak{X})$ be an action of G on a topological space $\mathfrak{X}$. We also denote the action by $(\mathfrak{X}, G, \Phi)$, and write $g \cdot x$ for $\Phi(g)(x)$. We say that Φ is a *Cantor action* if $\mathfrak{X}$ is a Cantor space.

The orbit of a point $x \in \mathfrak{X}$ is the subset $\mathcal{O}(x) = \{g \cdot x \mid g \in G\}$. The action is *minimal* if for all $x \in \mathfrak{X}$, its orbit $\mathcal{O}(x)$ is dense in $\mathfrak{X}$. The *isotropy group* of $x \in \mathfrak{X}$ is the subgroup

$$G_x = \{g \in G \mid g \cdot x = x\} . \tag{1}$$

The action $(\mathfrak{X}, G, \Phi)$ is *effective* if the homomorphism Φ is assumed to have trivial kernel. It is *free* if for all $x \in \mathfrak{X}$ and $g \in G$, $g \cdot x = x$ implies that $g = e$, where $e \in G$ denotes the identity of the group. Introduce the *isotropy set*

$$\mathrm{Iso}(\Phi) = \{x \in \mathfrak{X} \mid \exists\, g \in G \, , \, g \neq id \, , \, g \cdot x = x\} = \bigcup_{e \neq g \in G} \mathrm{Fix}(g) \, , \tag{2}$$

where $\mathrm{Fix}(g) = \{x \in \mathfrak{X} \mid g \cdot x = x\}$. The action $(\mathfrak{X}, G, \Phi)$ is said to be *topologically free* [11, 35, 44] if the set $\mathrm{Iso}(\Phi)$ is meager in $\mathfrak{X}$. In particular, this means that $\mathrm{Iso}(\Phi)$ has empty interior. Note that if $e \neq g \in G$ and $\Phi(g)$ acts trivially on $\mathfrak{X}$, then $\mathrm{Iso}(\Phi) = \mathfrak{X}$, and thus a topologically free action must be effective.

The action $(\mathfrak{X}, G, \Phi)$ is *equicontinuous* with respect to a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, if for all $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in \mathfrak{X}$ and $g \in G$ we have

$$(3) \qquad\qquad d_X(x, y) < \delta \qquad \Longrightarrow \qquad d_X(g \cdot x, g \cdot y) < \varepsilon.$$

Note that the definition is independent of the choice of the metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$.

For a Cantor space $\mathfrak{X}$, let $\mathrm{CO}(\mathfrak{X})$ denote the collection of all clopen (compact open) subsets of $\mathfrak{X}$. Note that for $\phi \in \mathrm{Homeo}(\mathfrak{X})$ and $U \in \mathrm{CO}(\mathfrak{X})$, the image $\phi(U) \in \mathrm{CO}(\mathfrak{X})$.

We say that $U \subset \mathfrak{X}$ is *adapted* to the action $(\mathfrak{X}, G, \Phi)$ if U is a non-empty clopen subset, and for any $g \in G$, $\Phi(g)(U) \cap U \neq \emptyset$ implies that $\Phi(g)(U) = U$. It follows that

$$(4) \qquad\qquad G_U = \{g \in G \mid \varphi(g)(U) \cap U \neq \emptyset\}$$

is a subgroup of finite index in G, called the *stabilizer* of U.

Denote the restricted action on U by (U, G_U, Φ_U), and introduce the *restricted holonomy group*:

$$(5) \qquad\qquad \mathcal{H}_U = image \{\Phi_U \colon G_U \to \mathrm{Homeo}(U)\} .$$

DEFINITION 1.1. *For $i = 1, 2$, let $(\mathfrak{X}_i, G_i, \Phi_i)$ be minimal equicontinuous Cantor actions. We say the actions are* return equivalent *if there exists non-empty clopen subsets $U_i \subset \mathfrak{X}_i$ such that U_i is adapted to the action Φ_i, and there is a homeomorphism $h \colon U_1 \to U_2$ whose induced action $h_* \colon \mathrm{Homeo}(U_1) \to \mathrm{Homeo}(U_2)$ restricts to an isomorphism $h_* \colon \mathcal{H}_{U_1} \to \mathcal{H}_{U_2}$.*

Note that when $U_i = \mathfrak{X}_i$ and both actions are effective, then this definition reduces to the usual notion of conjugacy of the actions, up to the induced group isomorphism $h_* \colon G_1 \cong \mathcal{H}_{U_1} \to \mathcal{H}_{U_2} \cong G_2$. In the terminology of [27], this is called an *isomorphism* of the actions. For the general case, though, additional assumptions on the group or the action are required to induce an isomorphism of the actions of the groups G_1 and G_2 from an isomorphism $h_* \colon \mathcal{H}_{U_1} \to \mathcal{H}_{U_2}$.

PROBLEM 1.2. *Find invariants of minimal equicontinuous Cantor actions which are sufficient to classify the actions up to return equivalence.*

In this work, we are concerned with the following conjugacy invariant of a Cantor action $(\mathfrak{X}, G, \Phi)$.

DEFINITION 1.3. *The* topological full group *$[[\mathfrak{X}, G, \Phi]] \subset \mathrm{Homeo}(\mathfrak{X})$ is the subgroup consisting of homeomorphisms $\phi \colon \mathfrak{X} \to \mathfrak{X}$ such that for all $x \in \mathfrak{X}$, there exists $U \in \mathrm{CO}(\mathfrak{X})$ with $x \in U$, and $g \in G$ such that their restrictions to U satisfy $\phi|U = \Phi(g)|U$.*

This definition was introduced in the work by Glasner and Weiss [28, Section 2], and has been extensively studied for its relation to the classification problem of Cantor actions, and for the algebraic properties of the group itself, as discussed for example in the Séminaire Bourbaki survey [17], and also the works [8, 9, 25, 31]. The topological full group is closely related to the notion of continuous orbit equivalence of actions, as will be discussed in Section 4. This paper was motivated by the following result of Cortez and Medynets in [16]:

THEOREM 1.4. *For $i = 1, 2$, let G_i be a finitely-generated group, and let $(\mathfrak{X}, G_i, \Phi_i)$ be a free minimal equicontinuous Cantor action on a Cantor space $\mathfrak{X}$. If $[[\mathfrak{X}, G_1, \Phi_1]] = [[\mathfrak{X}, G_2, \Phi_2]]$ then the actions Φ_1 and Φ_2 are* structurally conjugate, *hence are return equivalent.*

We give a generalization of Theorem 1.4 for minimal equicontinuous Cantor actions that are not free, but satisfy a property called 'local quasi-analyticity' as given in Definition 2.1.

THEOREM 1.5. *For $i = 1, 2$, let G_i be a finitely-generated group, and $(\mathfrak{X}, G_i, \Phi_i)$ be a minimal equicontinuous Cantor action which is locally quasi-analytic. If $[[\mathfrak{X}, G_1, \Phi_1]] = [[\mathfrak{X}, G_2, \Phi_2]]$ then the actions Φ_1 and Φ_2 are return equivalent.*

Our second main result gives a sufficient condition for the group G which implies the LQA property.

THEOREM 1.6. *Let G be a Noetherian group. Then a minimal equicontinuous action $(\mathfrak{X}, G, \Phi)$ on a Cantor space $\mathfrak{X}$ is locally quasi-analytic.*

In particular, a finitely-generated nilpotent group is Noetherian, so Theorem 1.5 implies:

COROLLARY 1.7. *Let G be a finitely-generated nilpotent group. Then the topological full group is a complete invariant up to return equivalence for a minimal equicontinuous action $(\mathfrak{X}, G, \Phi)$. If two such actions have spatially isomorphic topological full groups, then the actions are return equivalent.*

This result can be considered as a localized version of [35, Theorem 1.3], but with the Noetherian hypothesis on G in place of the homological criterion of Li in [35, Section 5].

Finally, in Definition 4.5, we introduce the virtual isomorphism class $[[[\mathfrak{X}, G, \Phi]]]$ of the topological full group for a minimal equicontinuous action $(\mathfrak{X}, G, \Phi)$, and in Theorem 4.6 give an extension of Theorem 1.5 using this invariant of actions. As an application, in Section 5 we recall the notion of a nil-solenoid, which is a weak solenoid whose base space is a closed nil-manifold and whose global monodromy action is effective. Then we prove Theorem 5.5, which implies the following:

THEOREM 1.8. *The virtual isomorphism class $[[[\mathfrak{X}, G, \Phi]]]$ for the monodromy of a nil-solenoid is a complete invariant of the homeomorphism class of the solenoid.*

Appendix A contains a collection of examples of minimal Cantor actions which illustrate properties of actions as discussed in this work.

2. Locally quasi-analytic and Hausdorff actions

In this section, we discuss notions of pointwise and uniform "regularity" for topological actions.

We first recall the notion of a locally quasi-analytic action. Haefliger used in [32] the notion of a quasi-analytic action on a topological space, in his study of pseudogroups of local isometries on locally connected spaces. The works [2, 3] by Álvarez-López, Candel and Moreira-Galicia reformulated Haefliger's definition for the case of topological actions on Cantor spaces as follows:

DEFINITION 2.1. [2, Definition 9.4] *A topological action $(\mathfrak{X}, G, \Phi)$ is* locally quasi-analytic, *or simply* LQA, *if there exists $\varepsilon > 0$ such that for any non-empty open set $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \varepsilon$, and for any non-empty open subset $V \subset U$, and elements $g_1, g_2 \in G$*

$$(6) \qquad \textit{if the restrictions } \Phi(g_1)|V = \Phi(g_2)|V, \quad \textit{then } \Phi(g_1)|U = \Phi(g_2)|U.$$

The action is said to be quasi-analytic *if (6) holds for $U = \mathfrak{X}$.*

Examples of equicontinuous Cantor actions $\mathfrak{X}$ which are locally quasi-analytic, but not quasi-analytic, are given in [21, 33], and also in Section A.

The idea of the proof for the following result appeared in the work [22] by Epstein, Millet and Tischler, in the context of pseudogroup actions. It has also appeared in the literature in various alternative formulations, for example as Proposition 3.6 in [44] and Lemma 2.2 in [35].

PROPOSITION 2.2. *An effective Cantor action $(\mathfrak{X}, G, \Phi)$ is quasi-analytic if and only if it is topologically free.*

Proof. Suppose that the action Φ is topologically free, then the isotropy set $\mathrm{Iso}(\Phi)$ is meager in $\mathfrak{X}$. In particular, this means that $\mathrm{Iso}(\Phi)$ has empty interior. Let $V \subset \mathfrak{X}$ be a non-empty open set, and suppose that $g_1, g_2 \in G$ satisfy $\Phi(g_1)|V = \Phi(g_2)|V$. Then $\Phi(g_2^{-1} g_1)|V = id|V$ so we must have that $g_2^{-1} g_1 = e \in G$. Thus, $g_1 = g_2$ and hence $\Phi(g_1) = \Phi(g_2)$, as was to be shown.

Conversely, suppose that the action Φ is quasi-analytic and effective, then for each $e \neq g \in G$ the complement of $\mathrm{Fix}(\Phi(g))$ is open and dense in $\mathfrak{X}$, so $\mathrm{Fix}(\Phi(g))$ is a closed nowhere dense set. Thus, $\mathrm{Iso}(\Phi)$ is the countable union of closed nowhere dense sets, hence by the Baire Category Theorem, $\mathrm{Iso}(\Phi)$ must be meager. $\square$

COROLLARY 2.3. *Let $(\mathfrak{X}, G, \Phi)$ be a minimal topological action. If G is a finitely-generated abelian group, and the action is effective, then it is quasi-analytic, hence is topologically free.*

Proof. Suppose that there exists a non-empty open set $V \subset \mathfrak{X}$ and let $g_1, g_2 \in G$ be such that the restrictions satisfy $\Phi(g_1)|V = \Phi(g_2)|V$. Let $g = g_2^{-1} g_1$ then the restriction $\Phi(g)|V$ is the identity map. Let $y \in \mathfrak{X}$ then there exists $g_y \in G$ such that $g_y \cdot y \in V$, so $y \in V_y \equiv g_y^{-1} \cdot V$. Since $\Phi(g) = \Phi(g_y^{-1} g g_y)$, the restriction $\Phi(g)|V_y$ is the identity map. Thus, $\Phi(g)$ is the identity on $\mathfrak{X}$, and as the action is effective, we have $g = id$. $\square$

The LQA property for a group action $(\mathfrak{X}, G, \Phi)$ can be interpreted in terms of the properties of the germinal groupoid $\mathcal{G}_\Phi$ associated to the action. This groupoid is fundamental for the study of the C^*-algebras these actions generate, as discussed for example by Renault in [43, 44]. Recall that for $g_1, g_2 \in G$, we say that $\Phi(g_1)$ and $\Phi(g_2)$ are *germinally equivalent* at $x \in \mathfrak{X}$ if $\Phi(g_1)(x) = \Phi(g_2)(x)$, and there exists an open neighborhood $x \in U \subset \mathfrak{X}$ such that the restrictions agree, $\Phi(g_1)|U = \Phi(g_2)|U$. We then write $\Phi(g_1) \sim_x \Phi(g_2)$. For $g \in G$, denote the equivalence class of $\Phi(g)$ at x by $[g]_x$. The collection of germs $\mathcal{G}(\mathfrak{X}, G, \Phi) = \{[g]_x \mid g \in G , x \in \mathfrak{X}\}$ is given the sheaf topology, and forms an *étale groupoid* modeled on $\mathfrak{X}$. We recall the following result from Winkelnkemper:

PROPOSITION 2.4. [49, Proposition 2.1] *The germinal groupoid $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is Hausdorff at $[g]_x$ if and only if, for all $[g']_x \in \mathcal{G}(\mathfrak{X}, G, \Phi)$ with $g \cdot x = g' \cdot x = y$, if there exists a sequence $\{x_i\} \subset \mathfrak{X}$ which converges to x such that $[g]_{x_i} = [g']_{x_i}$ for all i, then $[g]_x = [g']_x$.*

Winkelnkemper showed in [49, Proposition 2.3] that for a smooth foliation $\mathcal{F}$ of a connected manifold M for which the associated holonomy pseudogroup $\mathcal{G}_\mathcal{F}$ is generated by real analytic maps, then $\mathcal{G}_\mathcal{F}$ is a Hausdorff space. For Cantor actions, an analogous result holds for the LQA property.

PROPOSITION 2.5. *If an action $(\mathfrak{X}, G, \Phi)$ is locally quasi-analytic, then $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is Hausdorff.*

Proof. Assume that $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is not Hausdorff. Then there exists $g \in G$ and $x \in \mathfrak{X}$ such that $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is non-Hausdorff at $[g]_x$. By Proposition 2.4, there exists $[g']_x \in \mathcal{G}(\mathfrak{X}, G, \Phi)$ with $g \cdot x = g' \cdot x = y$, and a sequence $\{x_i\} \subset \mathfrak{X}$ which converges to x such that $[g]_{x_i} = [g']_{x_i}$ for all i, but $[g]_x \neq [g']_x$. Let $g'' = g^{-1}g' \in G$, then $g'' \cdot x = x$ and $[g'']_{x_i} = [id]_{x_i}$ for all i, but $[g'']_x \neq [id]_x$. In particular, the action of $\Phi(g'')$ is not the identity in any open neighborhood of x, but there exists a sequence of open sets $x_i \in U_i \subset \mathfrak{X}$ containing x in their limit for which the restriction $\Phi(g'')|U_i = id|U_i$. Hence, there does not exists $\varepsilon > 0$ such that $\Phi(g)|U$ is quasi-analytic for all open neighborhood $x \in U$ with diam $(U) < \varepsilon$. Thus, the action $(\mathfrak{X}, G, \Phi)$ is not locally quasi-analytic. $\square$

REMARK 2.6. Suppose that $(\mathfrak{X}, G, \Phi)$ is a Cantor action such that $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is Hausdorff. Then for each $x \in \mathfrak{X}$ and $g, g' \in G$ with $g \cdot x = g' \cdot x$ and $[g]_x \neq [g']_x$, there exists an open neighborhood $x \in U(x, g, g') \subset \mathfrak{X}$ such that the set $\{y \in U(x, g, g') \mid g \cdot y = g' \cdot y\}$ has no interior. If there exists $\varepsilon > 0$ such that any open set U with $x \in U$ and diam $(U) < \varepsilon$ then $U(x, g, g') = U$ for all $g, g' \in G$, this is just saying that $\mathcal{G}(\mathfrak{X}, G, \Phi)$ locally quasi-analytic. Thus, the locally quasi-analytic property can be viewed as a "uniform Hausdorff property" for $\mathcal{G}(\mathfrak{X}, G, \Phi)$.

We say that $x \in \mathfrak{X}$ is a *non-Hausdorff point* for the action $(\mathfrak{X}, G, \Phi)$ if there exists $g \in G$ such that the germ $[g]_x$ is not Hausdorff in $\mathcal{G}(\mathfrak{X}, G, \Phi)$. Examples of minimal equicontinuous Cantor actions with non-Hausdorff points are discussed in Section A. The existence of non-Hausdorff points has implications for the algebraic structure of the reduced C^*-algebra $C^*(\mathfrak{X}, G, \Phi)$ associated to the action, as discussed for example in [12, 23, 44].

3. EQUICONTINUOUS ACTIONS

In this section, we consider some basic properties of minimal equicontinuous Cantor actions. The main results of the section are Theorem 3.7, which gives an algebraic criterion for when a group action is locally quasi-analytic, and Corollary 3.8 which applies this criterion to nilpotent groups.

First, we recall the following folklore result and give a sketch of the proof, as the ideas involved are central to the study of equicontinuous actions and used in the discussions that follow.

PROPOSITION 3.1. *Let G be a finitely-generated group. Then a minimal Cantor action $(\mathfrak{X}, G, \Phi)$ is equicontinuous if and only if, for the induced action $\Phi_* \colon G \times \mathrm{CO}(\mathfrak{X}) \to \mathrm{CO}(\mathfrak{X})$, the G-orbit of every $U \in \mathrm{CO}(\mathfrak{X})$ is finite.*

Proof. Suppose that the action is equicontinuous, and let $U_1 \subset \mathfrak{X}$ be a non-empty proper clopen subset. Then $U_2 = \mathfrak{X} - U_1$ is also a non-empty proper clopen subset, and let $\varepsilon = d_{\mathfrak{X}}(U_1, U_2) > 0$, where $d_{\mathfrak{X}}$ is the choice of some metric on $\mathfrak{X}$. Let $\delta > 0$ be such that (3) holds for this choice of ε, for all $g \in G$. The iterates of the partition $\{U_1, U_2\}$ of $\mathfrak{X}$ define a closed partition of $\mathfrak{X}$,

$$\mathcal{C} = \bigcap \ \{g \cdot U_i \mid i = 1, 2, \ g \in G\} \ ,$$

which is invariant for the action Φ by construction. Let $K \in \mathcal{C}$ with $x \in K$, and suppose that $g \cdot x \in U_i$. Let $y \in \mathfrak{X}$ satisfy $d_{\mathfrak{X}}(x, y) < \delta$ then by (3) and the choice of ε we have that $g \cdot y \in U_i$ also. As this holds for all $g \in G$, the set K is open in $\mathfrak{X}$. Thus, $\mathcal{C}$ is a clopen partition, and by the compactness of $\mathfrak{X}$ it must be finite. Thus, the action Φ permutes this finite collection, hence there is a subgroup $G_{\mathcal{C}} \subset G$ of finite index which fixes every $K \in \mathcal{C}$. As U_1 is the union of sets in $\mathcal{C}$, the subgroup $G_{\mathcal{C}}$ also fixes U_1, and thus the G-orbit of U_1 is finite.

Conversely, assume that the G-orbit of every $U \in \mathrm{CO}(\mathfrak{X})$ is finite. Fix a basepoint $x \in \mathfrak{X}$, and as $\mathfrak{X}$ is a Cantor space, one can choose a descending chain of clopen sets

$$\mathfrak{X} = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_\ell \supset \cdots \supset \{x\}$$

whose intersection is $\{x\}$. For each $\ell > 0$, the intersection of the finite collection $\{g \cdot V_\ell \mid g \in G\}$ has a clopen set containing the basepoint x, which we label U_ℓ, so that $x \in U_\ell \subset V_\ell$. Then U_ℓ is an adapted clopen subset with stabilizer group denoted by $G_\ell \subset G$. It then follows as in [19, Appendix A] or [16, Section 2], that the Cantor action $(\mathfrak{X}, G, \Phi)$ is conjugate to the odometer constructed from the group chain $\{G_\ell \mid \ell \geq 0\}$, hence $(\mathfrak{X}, G, \Phi)$ is a minimal equicontinuous action. $\qquad\square$

The above proof that each $U \in \mathrm{CO}(\mathfrak{X})$ has finite orbit is essentially the same as what was called the "coding method" used to study equicontinuous pseudogroup actions in [14], and discussed for group actions in [19, Appendix A].

We discuss in more detail another key idea in the above proof. Let $U \subset \mathfrak{X}$ be an adapted clopen set for the action Φ, with stabilizer subgroup $G_U \subset G$. Then for $g, g' \in G$ with $g \cdot U \cap g' \cdot U \neq \emptyset$ we have $g^{-1} g' \cdot U = U$, hence $g^{-1} g' \in G_U$. Thus, the translates $\{g \cdot U \mid g \in G\}$ form a finite clopen partition of $\mathfrak{X}$, and are in 1-1 correspondence with the quotient space $X_U = G/G_U$ so the stabilizer group $G_U \subset G$ has finite index. Note that the action of $g \in G_U$ on X_U is trivial precisely when $g \in C_U$, where $C_U \subset G_U$ is the largest normal subgroup of G contained in G_U. Thus, the action of the finite group G/C_U on X_U by permutations is a finite approximation of the action of G on $\mathfrak{X}$.

The restricted action of G_U on U defines a homomorphism $\Phi_U \colon G_U \to \mathrm{Homeo}(U)$, with kernel $\ker(\Phi_U) \subset G_U$. Suppose that $g \in \ker(\Phi_U)$ and the action Φ is quasi-analytic, then g acts trivially on $\mathfrak{X}$, which implies that g acts trivially on X_U hence $g \in C_U$. Thus, the quasi-analytic hypothesis is equivalent to the assumption that $\ker(\Phi_U)$ is a normal subgroup of G. Exactly the same reasoning yields the following criteria for an action to be locally quasi-analytic.

PROPOSITION 3.2. *Let $(\mathfrak{X}, G, \Phi)$ be a minimal equicontinuous action on a Cantor space $\mathfrak{X}$. Suppose that the action Φ is locally quasi-analytic. Then there exists $\varepsilon > 0$ such that for each pair of non-empty adapted clopen sets $V \subset U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \varepsilon$, the subgroup $\ker(\Phi_V) \subset G_V \subset G_U$ is normal in G_U.*

The action of the Grigorchuk group on the tree boundary (as discussed in Example A.6) admits pairs of clopen subsets $V \subset U$ with arbitrarily small diameters, for which $\ker(\Phi_V)$ is not normal in G_U, and thus its action is not locally quasi-analytic.

We next develop the idea behind the proof of Proposition 3.2 to develop an effective criteria, given in Theorem 3.7, for showing that an action must be locally quasi-analytic.

For a choice of basepoint $x \in \mathfrak{X}$ and scale $\varepsilon > 0$, there exists an adapted clopen set $U \in \mathrm{CO}(\mathfrak{X})$ with $x \in U$ and $\mathrm{diam}(U) < \varepsilon$. Iterating this construction, for a given basepoint x, one can always construct the following:

DEFINITION 3.3. *Let $(\mathfrak{X}, G, \Phi)$ be a minimal equicontinuous action on a Cantor space $\mathfrak{X}$. A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ is said to be an* adapted neighborhood *basis at $x \in \mathfrak{X}$ for the action Φ if $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 1$ with $\cap U_\ell = \{x\}$, and each U_ℓ is adapted to the action Φ.*

For such a collection, set $G_\ell = G_{U_\ell}$ we obtain a descending chain of finite index subgroups

$$\mathcal{G}_{\mathcal{U}} = \{G = G_0 \supset G_1 \supset G_2 \supset \cdots\} \, .$$

The intersection $K(\mathcal{G}_{\mathcal{U}}) = \bigcap_{\ell \geq 0} G_\ell$ is called the kernel of $\mathcal{G}_{\mathcal{U}}$.

Next, set $X_\ell = G/G_\ell$ and note that G acts transitively on the left on X_ℓ. The inclusion $G_{\ell+1} \subset G_\ell$ induces a natural G-invariant quotient map $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$. Introduce the inverse limit

$$(7) \qquad\qquad\qquad X_\infty \equiv \varprojlim \; \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell > 0\}$$

which is a Cantor space with the Tychonoff topology, and the action on the factors X_ℓ induces a minimal equicontinuous action $\Phi_x \colon G \times X_\infty \to X_\infty$. Note that for $g \in K(\mathcal{G}_{\mathcal{U}})$, the left action of g on X_ℓ fixes the coset $e_\ell \in X_\ell$ and hence fixes the limiting point $e_\infty \in X_\infty$.

For each $\ell \geq 0$, we have the "partition coding map" $\sigma_\ell \colon \mathfrak{X} \to X_\ell$ which is G-equivariant. The maps $\{\sigma_\ell\}$ are compatible with the quotient maps in (7), and so define a limit map $\sigma_\infty \colon \mathfrak{X} \to X_\infty$. The fact that the diameters of the clopen sets $\{U_\ell\}$ tend to zero, implies that σ_∞ is a homeomorphism. Let $\tau_x \colon X_\infty \to \mathfrak{X}$ denote the inverse map, which commutes with the G actions on both spaces, and satisfies $\tau_x(e_\infty) = x$. The minimal equicontinuous action (X_∞, G, Φ_x) is called the *odometer representation* centered at x for the action $(\mathfrak{X}, G, \Phi)$.

Suppose that $\mathcal{U}$ and $\mathcal{U}'$ are two choices of adapted neighborhood bases at x, then the corresponding group chains $\mathcal{G}_{\mathcal{U}}$ and $\mathcal{G}_{\mathcal{U}'}$ are equivalent as descending chains, as shown in [24, 19]. For two choices of basepoints $x, x' \in \mathfrak{X}$, with adapted neighborhood basis $\mathcal{U}$ at x and $\mathcal{U}'$ at x', the corresponding group chains $\mathcal{G}_{\mathcal{U}}$ for $\mathcal{U}$ and $\mathcal{G}_{\mathcal{U}'}$ for $\mathcal{U}'$ need not be equivalent, but are *conjugate equivalent* as descending chains. The conjugacy relation on group chains, as introduced by Fokkink and Oversteegen in [24], is in essence the relation on group chains which results in conjugating them by elements of the closure of the group action. It is key for showing in [19] that the invariants of the action $(\mathfrak{X}, G, \Phi)$ defined in [21, 33] are independent of the choices of basepoints and group chain models.

We recall a basic property of the odometer models:

LEMMA 3.4. *A minimal equicontinuous Cantor action $(\mathfrak{X}, G, \Phi)$ is free if and only if for all $x \in \mathfrak{X}$ and any adapted neighborhood basis $\mathcal{U}$ at x, the kernel $K(\mathcal{G}_{\mathcal{U}})$ is the trivial group.*

Let $\mathcal{G}_U$ be the group chain associated to adapted neighborhood system at x, and suppose that $y = g \cdot x$, then set

$$\mathcal{G}_U^y = \{G = G_0 \supset g\, G_1\, g^{-1} \supset g\, G_2\, g^{-1} \supset \cdots\} \, ,$$

which is the group chain associated to the adapted neighborhood basis $\mathcal{U}^g$ at y. Clearly, $K(\mathcal{G}_{\mathcal{U}}^y) = g\, K(\mathcal{G}_{\mathcal{U}}^x)\, g^{-1}$, so the property that $K(\mathcal{G}_{\mathcal{U}}^y)$ is trivial is independent of the choice of $y \in \mathcal{O}(x)$. On the other hand, if $y \notin \mathcal{O}(x)$ then whether $K(\mathcal{G}_{\mathcal{U}}^y)$ is trivial or not may depend on the choice of y.

We next prove a criterion for when a minimal equicontinuous Cantor action $(\mathfrak{X}, G, \Phi)$ is locally quasi-analytic. Recall the following property of groups.

DEFINITION 3.5. [5] *A group Γ is said to be* Noetherian *if every increasing chain of closed subgroups $\{H_i \mid i \geq 1\}$ of Γ has a maximal element H_N.*

An equivalent definition of the Noetherian property is that every subgroup of Γ is finitely generated. Hence, Γ has at most countably many subgroups.

Recall that a group Γ is *polycyclic* if there exists an integer $k > 0$ and a chain of subgroups

$$\{e\} = \Gamma_{k+1} \subset \Gamma_k \subset \cdots \subset \Gamma_0 = \Gamma \tag{8}$$

such that each $\Gamma_{\ell+1}$ is normal in Γ_ℓ and the quotient $\Gamma_\ell/\Gamma_{\ell+1}$ is a cyclic group for $0 \leq \ell < k$. For example, a finitely-generated nilpotent group is polycyclic. A group Γ is *virtually polycyclic* if there exists a subgroup $\Gamma_0 \subset \Gamma$ of finite index such that Γ_0 is polycyclic. The following result is folklore.

PROPOSITION 3.6. *Let Γ be a virtually polycyclic group, then Γ is Noetherian.*

The following result is a consequence of the Noetherian property for Cantor actions.

THEOREM 3.7. *Let G be a Noetherian group. Then a minimal equicontinuous Cantor action $(\mathfrak{X}, G, \Phi)$ is locally quasi-analytic.*

Proof. We assume that the action $(\mathfrak{X}, G, \Phi)$ is not locally quasi-analytic, and construct an increasing chain of subgroups in G with no maximal element, contradicting that G is Noetherian.

Fix $x \in \mathfrak{X}$ and let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ be an adapted neighborhood basis at x for the action Φ. For $\ell \geq 1$, the collection of translates $\{g \cdot U_\ell \mid g \in G\}$ is a disjoint clopen covering of $\mathfrak{X}$. Let $\lambda_\ell > 0$ be a Lebesgue number for this covering. That is, if $U \subset \mathfrak{X}$ is an open set with $\mathrm{diam}(U) < \lambda_\ell$ then there exists $g \in G$ such that $U \subset g \cdot U_\ell$.

The assumption that Definition 2.1 does not hold implies that for each $\varepsilon > 0$, there exists non-empty clopen subsets $V \subset U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \varepsilon$ such that (6) fails for this pair. That is, there exists $h_1, h_2 \in G$ such that $\Phi(h_1)|V = \Phi(h_2)|V$ but $\Phi(h_1)|U \neq \Phi(h_2)|U$. Set $h = h_2^{-1} h_1$ then $\Phi(h)|V$ is the identity map, but $\Phi(h)|U$ is not. Note that for this h we have $h \cdot U \cap U \neq \emptyset$.

We next construct an increasing chain of subgroups $\{H_i \mid i \geq 1\}$ by induction. To begin, set $\ell_1 = 1$. Let $W_1 \subset \mathfrak{X}$ be a non-empty open set with $\mathrm{diam}(W_1) < \lambda_1$ such that there exists a non-empty open subset $V_1 \subset W_1$ and $h_1 \in G$ so that $\Phi(h_1)|V_1$ is the identity, and $\Phi(h_1)|W_1$ is not, with $h_1 \cdot W_1 \cap W_1 \neq \emptyset$. Then there exists $g_1 \in G$ such that $W_1 \subset g_1 \cdot U_1$, so $g_1^{-1} \cdot W_1 \subset U_1$. Set $\xi_1 = g_1^{-1} h_1 g_1$ and note that $\xi_1 \cdot U_1 \cap U_1 \neq \emptyset$, hence $\xi_1 \in G_{U_1}$.

Note that $g_1^{-1} \cdot V_1 \subset U_1$. As the action of G_{U_1} on U_1 is minimal, there exists $\gamma_1 \in G_{U_1}$ such that $\gamma_1 \cdot x \in g_1^{-1} \cdot V_1$. Set $V_1' = \gamma_1^{-1} g_1^{-1} \cdot V_1$ so that $x \in V_1' \subset U_1$. Then for

$$k_1 = \gamma_1^{-1} \xi_1 \gamma_1 = \gamma_1^{-1} g_1^{-1} h_1 g_1 \gamma_1 \ ,$$

we have that $\Phi(k_1)|U_1$ is not the identity, but $\Phi(k_1)|V_1'$ is the identity. Choose $\ell_2 > \ell_1 = 1$ such that $U_{\ell_2} \subset V_1'$ and thus $\Phi(k_1)|U_{\ell_2}$ is also the identity.

Assume that for $i > 1$, and increasing sequence of integers $1 = \ell_1 < \ell_2 < \cdots < \ell_i$ have been chosen as above. Let $W_i \subset \mathfrak{X}$ be a non-empty open set with $\mathrm{diam}(W_i) < \lambda_{\ell_i}$ such that there exists a non-empty open subset $V_i \subset W_i$ and $h_i \in G$ so that $\Phi(h_i)|V_i$ is the identity, and $\Phi(h_i)|W_i$ is not the identity, with $h_i \cdot W_i \cap W_i \neq \emptyset$. Then there exists $g_i \in G$ such that $W_i \subset g_i \cdot U_{\ell_i}$ and so $g_i^{-1} \cdot W_i \subset U_{\ell_i}$. Set $\xi_i = g_i^{-1} h_i g_i$ and note that $\xi_i \cdot U_{\ell_i} \cap U_{\ell_i} \neq \emptyset$, hence $\xi_i \in G_{U_{\ell_i}}$.

Note that $g_i^{-1} \cdot V_i \subset U_{\ell_i}$. As the action of $G_{U_{\ell_i}}$ on U_{ℓ_i} is minimal, there exists $\gamma_i \in G_{U_{\ell_i}}$ such that $\gamma_i \cdot x \in g_i^{-1} \cdot V_i$. Set $V_i' = \gamma_i^{-1} g_i^{-1} \cdot V_i$ so that $x \in V_i' \subset U_{\ell_i}$. Then for

$$k_i = \gamma_i^{-1} \xi_i \gamma_i = \gamma_i^{-1} g_i^{-1} h_i g_i \gamma_i \ ,$$

we have that $\Phi(k_i)|U_{\ell_i}$ is not the identity, but $\Phi(k_i)|V_i'$ is the identity. Choose $\ell_{i+1} > \ell_i$ such that $U_{\ell_{i+1}} \subset V_i'$ and thus $\Phi(k_i)|U_{\ell_{i+1}}$ is also the identity.

Now assume that indices $\{\ell_i \mid i \geq 1\}$ and elements $\{k_i \mid i \geq 1\}$ have been chose as above. Define:

$$H_i = \{g \in G_{\ell_i} \mid \Phi(g)|U_{\ell_i} = id\} \ . \tag{9}$$

Observe that for each $g \in H_i$ and $j > i$ we have $U_{\ell_j} \subset U_{\ell_i}$ hence $g \cdot U_{\ell_j} = U_{\ell_j}$ thus $H_i \subset G_{\ell_j}$. Moreover, $\Phi(g)|U_{\ell_j}$ is the identity, hence $H_i \subset H_j$.

Now let $i \geq 1$, then by choice, $\Phi(h_i)|W_i$ is not the identity, hence $\Phi(k_i)|U_{\ell_i}$ is not the identity, so $k_i \notin H_i$. On the other hand, for $j > i$ the restriction $\Phi(k_i)|U_{\ell_j}$ is the identity, hence $k_i \in H_j$.

It follows that the collection $\{H_i \mid i \geq 1\}$ forms a strictly increasing chain of subgroups in G, which therefore is not Noetherian. $\square$

Theorem 3.7 implies the following extension of Corollary 2.3.

COROLLARY 3.8. *Let G be a finitely generated nilpotent group. Then a minimal equicontinuous Cantor action $(\mathfrak{X}, G, \Phi)$ is locally quasi-analytic.*

The proof of Theorem 3.7 yields the following result.

COROLLARY 3.9. *Let $(\mathfrak{X}, G, \Phi)$ be a minimal equicontinuous Cantor action which is not locally quasi-analytic. Then for any $x \in \mathfrak{X}$, the isotropy subgroup $G_x \subset G$ contains an infinite strictly increasing chain of subgroups.*

Proof. For given x, choose an adapted neighborhood basis at x. Then for H_i as constructed in the proof above, the action of each $h \in H_i$ fixes the set $U_{\ell_{i+1}}$ so in particular $h \in G_x$. Thus $H_i \subset G_x$. $\square$

It is remarkable that the first construction of a non-homogenous weak solenoid by Schori in [46], see also [13], has this ascending chain property for the isotropy subgroups of its monodromy action. The Grigorchuk groups [29], and more generally branch groups [6, 30], provide a large class of examples of groups acting on trees which give rise to Cantor actions that are not locally quasi-analytic. This is discussed further in Example A.6.

The authors, in joint work with Jessica Dyer, gave in [19, Example 8.5] examples of minimal equicontinuous Cantor actions $(\mathfrak{X}, G, \Phi)$ where G is a torsion-free 2-step nilpotent group, its discriminant group $\mathcal{D}_x$ is a Cantor space, and the action is locally quasi-analytic. It was asked in [21] whether it is possible to construct examples of minimal equicontinuous Cantor actions $(\mathfrak{X}, G, \Phi)$ where G is a finitely-generated nilpotent group, and the action is not locally quasi-analytic. Corollary 3.8 shows this is impossible.

4. CONTINUOUS ORBIT EQUIVALENCE AND RIGIDITY

The concept of *continuous orbit equivalence* between Cantor actions was introduced by Mike Boyle in his thesis [10], and has played a fundamental role in the classification of Cantor actions in many subsequent works [9, 25, 26]. The related notion of the topological full group of an action in Definition 1.3 has provided a rich source of examples of finitely generated groups with exceptional properties, as discussed for example in [17]. In this section, we show how these notions can be used to show return equivalence for minimal equicontinuous Cantor actions.

DEFINITION 4.1. *For $i = 1, 2$, let $(\mathfrak{X}_i, G_i, \Phi_i)$ be an action on a Cantor space $\mathfrak{X}_i$. The actions are said to be* continuously orbit equivalent *if there exists a homeomorphism $h\colon \mathfrak{X}_1 \to \mathfrak{X}_2$ and continuous functions*

(1) $\alpha\colon G_1 \times \mathfrak{X}_1 \to G_2$, $h(\Phi_1(g_1)(x_1)) = \Phi_2(\alpha(g_1, x_1), h(x_1))$ for all $g_1 \in G_1$ and $x_1 \in \mathfrak{X}_1$;
(2) $\beta\colon G_2 \times \mathfrak{X}_2 \to G_1$, $h^{-1}(\Phi_2(g_2, x_2)) = \Phi_1(\beta(g_2, x_2), h^{-1}(x_2))$ for all $g_2 \in G_2$ and $x_2 \in \mathfrak{X}_2$.

The actions are said to be virtually continuously orbit equivalent *if there exists adapted clopen subsets $U_i \subset \mathfrak{X}_i$ such that the restricted actions (U_i, G_{U_i}, Φ_i) are continuously orbit equivalent.*

The homeomorphism h is called a *continuous orbit equivalence* between the two actions. Note that the functions α and β are not assumed to satisfy the cocycle property.

Given an action $(\mathfrak{X}_2, G_2, \Phi_2)$ and a homeomorphism $h\colon \mathfrak{X}_1 \to \mathfrak{X}_2$, define the conjugate action $(\mathfrak{X}_1, G_2, \Phi_2^h)$ by setting

$$\Phi_2^h(g_2, x_1) = h^{-1}(\Phi_2(g_2, h(x_1))) \text{ for } g_2 \in G_2, \ x_1 \in \mathfrak{X}_1 \ .$$

We recall an observation from [28, Section 2] about the topological full group:

PROPOSITION 4.2. *For $i = 1, 2$, let $(\mathfrak{X}_i, G_i, \Phi_i)$ be an action on a Cantor space $\mathfrak{X}_i$. Then a homeomorphism $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ is a continuous orbit equivalence between the actions, if and only if the following two conditions hold:*

(1) $\Phi_1(g_1, \cdot) \in [[\mathfrak{X}_1, G_2, \Phi_2^h]]$ for all $g_1 \in G_1$;
(2) $\Phi_2^h(g_2, \cdot) \in [[\mathfrak{X}_1, G_1, \Phi_1]]$ for all $g_2 \in G_2$.

That is, the actions are continuously orbit equivalent if and only if the images $\Phi_1(G_1) \subset \mathrm{Homeo}(\mathfrak{X}_1)$ and $\Phi_2^h(G_2) = h^{-1} \circ \Phi_2(G_2) \circ h \subset \mathrm{Homeo}(\mathfrak{X}_1)$ generate the same topological full group.

In the works [16, 35], the following is called the "rigidity problem" for Cantor actions.

PROBLEM 4.3. *For $i = 1, 2$, let $(\mathfrak{X}, G_i, \Phi_i)$ be an action on a fixed Cantor space $\mathfrak{X}$, and suppose that $[[\mathfrak{X}, G_1, \Phi_1]] = [[\mathfrak{X}, G_2, \Phi_2]]$. Give conditions on the actions Φ_1 and Φ_2 which are sufficient to imply that they are return equivalent.*

Given a Cantor action $(\mathfrak{X}, G_1, \Phi_1)$, one can construct a new Cantor action $(\mathfrak{X}, G_2, \Phi_2)$ with the same topological full group by simply adjoining elements of $[[\mathfrak{X}_1, G_1, \Phi_1]]$ to the action $\Phi_1(G_1)$. We discuss this construction further in Example A.4. The construction of such examples shows that for a solution of the rigidity problem, there must be hypotheses imposed which rule them out.

Let $(\mathfrak{X}_i, G_i, \Phi_i)$ be a minimal equicontinuous action on a Cantor space $\mathfrak{X}_i$ for $i = 1, 2$. Assume there exists a continuous orbit equivalence $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ between two actions, then by the above remarks, we can assume without loss of generality that $\mathfrak{X}_1 = \mathfrak{X}_2 = \mathfrak{X}$, and there exists functions α, β as in Definition 4.1. If the both actions are free, then the functions α and β are uniquely determined by the actions and satisfy a cocycle property. This is a key point in the works [16, 27]. It was observed in [11, 35], and also implicitly in [44], that if the actions are topologically free on $\mathfrak{X}$, then the orbit functions α and β still satisfy the cocycle identities. Li showed that if, in addition, the actions satisfy the *continuous cocycle rigidity* property of [35, Section 4], then the actions are conjugate.

The work by Cortez and Medynets [16] showed that for *free* minimal equicontinuous Cantor actions, the orbit cocycles α and β induce a return equivalence between the two actions. The observation behind the proof of Theorem 1.5 is that if both actions are assumed to be locally quasi-analytic, then the method of proof for [16, Theorem 3.3] can be used to show the actions are return equivalent.

Proof of Theorem 1.5. Medynets states in [38, Remark 3] (and also in Theorem 4.2 in [16]) that if the topological full groups of two minimal Cantor actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}, H, \Psi)$ are isomorphic as abstract groups, then there is a homeomorphism $h \colon \mathfrak{X} \to \mathfrak{X}$ which induces the isomorphism. That is, every such abstract group isomorphism is implemented by a spatial isomorphism, which by Proposition 4.2 yields a topological orbit equivalence between the actions.

Thus, we may assume that $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}, H, \Psi)$ are minimal equicontinuous actions on a Cantor space $\mathfrak{X}$ which have the same orbits. Let $\varepsilon > 0$ be chosen so that both actions are locally quasi-analytic for clopen sets $U \subset \mathfrak{X}$ satisfying $\mathrm{diam}(U) < \varepsilon$. Choose a basepoint $x \in \mathfrak{X}$.

Let $V \subset \mathfrak{X}$ be an adapted clopen set for the action Ψ with $x \in V$ and $\mathrm{diam}(V) < \varepsilon$, with stabilizer group $H_V \subset H$. Then by Proposition 2.2, the action of $\mathcal{H}_V^\Psi = \Psi_V(H_V) \subset \mathrm{Homeo}(V)$ on V is topologically free.

Let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ be an adapted neighborhood basis at x for the action $(\mathfrak{X}, G, \Phi)$, with $U_1 \subset V$. Let $\mathcal{G}_\mathcal{U} = \{G_\ell \mid \ell \geq 0\}$ be the associated group chain with $G_0 = G$ and $G_1 = G_{U_1}$. Then the action of $\mathcal{H}_{U_1}^\Phi = \Phi_{U_1}(G_1) \subset \mathrm{Homeo}(U_1)$ on U_1 is also topologically free.

Let $\alpha \colon G \times \mathfrak{X} \to H$ be the continuous function in Definition 4.1. The subgroup $G_1 \subset G$ has finite index, and G is finitely-generated, so there exists a finite generating set $\{g_1, \ldots, g_k\} \subset G_1$. By the continuity of α, there exists $\delta_1 > 0$ such that

(10) $$\alpha(g_i, x) = \alpha(g_i, y) \text{ for } 1 \leq i \leq k, \ x, y \in \mathfrak{X} \text{ with } d_{\mathfrak{X}}(x, y) < \delta_1 .$$

Let $\ell_2 \geq 2$ be such that $\mathrm{diam}(g \cdot U_{\ell_2}) < \delta_1$ for all $g \in G_1$. The collection $\{g \cdot U_{\ell_2} \mid g \in G_1\}$ is a finite clopen partition of U_1 so there exists $0 < \delta_2 < \delta_1$ such that for any $y \in U_1$ there exists $g \in G_1$ such that $B_{d_{\mathfrak{X}}}(y, \delta_2) \subset g \cdot U_{\ell_2}$. Then δ_2 is a Lebesgue number for the open covering of U_1 by translates of U_{ℓ_2}.

By the uniform continuity of the action Φ, there exists $0 < \delta_3 \leq \delta_2$ such that, for all $g \in G$,

$$(11) \qquad d_{\mathfrak{X}}(\Phi(g)(x), \Phi(g)(y)) < \delta_2 \text{ for all } x, y \in \mathfrak{X} \text{ with } d_{\mathfrak{X}}(x, y) < \delta_3 .$$

Let $\ell_3 > \ell_2$ be such that $\mathrm{diam}(g \cdot U_{\ell_3}) < \delta_3$ for all $g \in G_1$. Note that $U_{\ell_3} \subset U_{\ell_2} \subset U_1$.

Now consider the restriction $\alpha \colon G_1 \times U_1 \to H$. For each $g \in G_1$ and $x \in U_1$ we have $\Phi(g)(x) \in U_1$. Let $h \in H$ be such that $\Psi(h)(x) = \Phi(g)(x)$, then $U_1 \subset V$ implies that $\Psi(h)(V) \cap V \neq \emptyset$ hence $h \in H_V$. Recall that $\mathcal{H}_V^{\Psi} = \Psi_V(H_V) \subset \mathrm{Homeo}(V)$, thus the restriction of α to $G_1 \times U_1$ induces a map $\widehat{\alpha}_1 = \Psi_V \circ \alpha \colon G_1 \times U_1 \to \mathcal{H}_V^{\Psi}$.

LEMMA 4.4. *The map $\widehat{\alpha}_1 \colon G_1 \times U_1 \to \mathcal{H}_V^{\Psi}$ is a cocycle over the action of Φ_{U_1} on U_1.*

Proof. Let $g, g' \in G_1$ and let $x \in U_1$. Then $\Phi_{U_1}(g' g)(x) = \Phi_{U_1}(g')(x')$ for $x' = \Phi_{U_1}(g)(x)$.

The action of $\mathcal{H}_V^{\Psi}$ is topologically free, so there exists a dense $\mathcal{H}_V^{\Psi}$-invariant subset $Z \subset V$ such that the action of $\mathcal{H}_V^{\Psi}$ on Z is free. That is, for $\psi, \psi' \in \mathcal{H}_V^{\Psi}$ and $z \in Z$, if $\psi(z) = \psi'(z)$ then $\psi = \psi' \in \mathrm{Homeo}(V)$.

Let $x \in Z$ and $h = \alpha_1(g, x)$, then $\psi = \Psi_V(h)$ is the unique map in $\mathcal{H}_V^{\Psi}$ such that $\psi(x) = x'$. Note that $x' \in Z$ so there is a unique $\psi' \in \mathcal{H}_V^{\Psi}$ such that $x'' = \psi'(x') = \Phi_{U_1}(g')(x')$. Then $x'' = \psi' \circ \psi(x)$.

By the defining identity (1) in Definition 4.1 with h the identity map, for $x \in Z \cap U_1$ we have that

$$\widehat{\alpha}_1(g, x) = \psi , \ \ \widehat{\alpha}_1(g', x') = \psi' , \ \ \widehat{\alpha}_1(g' g, x) = \psi' \psi .$$

The continuity of the actions Φ and Ψ imply that this holds for all $x \in U_1$ as $Z \cap U_1$ is dense in U_1. Thus, we have the cocycle identity

$$(12) \qquad \widehat{\alpha}_1(g' g, x) = \widehat{\alpha}_1(g', \Phi_{U_1}(x)) \, \widehat{\alpha}_1(g, x)$$

for all $g', g \in G_{U_1}$ and $x \in U_1$, as was to be shown. $\qquad\qquad\qquad\qquad\qquad\square$

Next, we show that the cocycle $\widehat{\alpha}_1$ is defined by a group homomorphism when restricted to the clopen subset $U_{\ell_3} \subset U_1$. The method of proof is the same as for the proof of [16, Theorem 3.3].

Recall that $\{g_1, \ldots, g_k\} \subset G_1$ is a set of generators, then for each $1 \leq i \leq k$, by the choice of U_{ℓ_2} and (10), for any $g' \in G_1$ the value of $\widehat{\alpha}_1(g_i, x)$ is constant for $x \in g' \cdot U_{\ell_2}$.

By the choice of δ_1, for $d_{\mathfrak{X}}(x, y) < \delta_1$ and $1 \leq i \leq k$, we have $\widehat{\alpha}_1(g_i, x) = \widehat{\alpha}_1(g_i, y)$. Then by the choice of δ_3 and U_{ℓ_3}, for any $1 \leq i \leq k$, $g' \in G_{U_1}$ and $x, y \in U_{\ell_3}$ then

$$\widehat{\alpha}_1(g_i, \Phi_{U_1}(g')(x)) = \widehat{\alpha}_1(g_i, \Phi_{U_1}(g')(y)) .$$

We apply this to the case where $g = g_{i_1} \cdots g_{i_\nu} \in G_{U_{\ell_3}}$. Set $g' = g_{i_2} \cdots g_{i_\nu}$ then for $x, y \in Z \cap U_{\ell_3}$,

$$(13) \qquad \begin{aligned} \widehat{\alpha}_1(g, x) &= \widehat{\alpha}_1(g_{i_1} g', x) = \widehat{\alpha}_1(g_{i_1}, \Phi_{U_1}(g')(x)) \circ \widehat{\alpha}_1(g', x) \\ &= \widehat{\alpha}_1(g_{i_1}, \Phi_{U_1}(g')(y)) \circ \widehat{\alpha}_1(g', y) = \widehat{\alpha}_1(g_{i_1} g', y) = \widehat{\alpha}_1(g, y). \end{aligned}$$

As the values of $\widehat{\alpha}_1(g, x), \widehat{\alpha}_1(g, y) \in \mathcal{H}_V^{\Psi}$ are defined using the identity (1) in Definition 4.1, the identity (13) holds on the closure of $Z \cap U_{\ell_3}$ which is all of U_{ℓ_3}. Thus, the calculation (13) shows that the restricted cocycle $\widehat{\alpha}_3 \colon G_{U_{\ell_3}} \times U_{\ell_3} \to \mathcal{H}_V^{\Psi}$ is independent of the point in the second factor U_{ℓ_3}, hence induces a group homomorphism denoted by $\widetilde{\alpha}_3 \colon G_{U_{\ell_3}} \to \mathcal{H}_V^{\Psi}$.

Suppose that $g, g' \in G_{U_{\ell_3}}$ satisfy $\Phi_{U_{\ell_3}}(g) = \Phi_{U_{\ell_3}}(g') \in \mathrm{Homeo}(U_{\ell_3})$, then by the defining identity (1) in Definition 4.1 with h the identity map, for $x \in Z \cap U_1$ we have $\widehat{\alpha}_1(g, x) = \widehat{\alpha}_1(g', x) \in \mathcal{H}_V^{\Psi}$.

It follows that $\widehat{\alpha}_3$ induces a homomorphism $\widehat{\alpha}_3 \colon \mathcal{H}^{\Phi}_{U_{\ell_3}} \to \mathcal{H}^{\Psi}_V$. The homomorphism $\widehat{\alpha}_3$ is in fact an isomorphism. The proof is omitted, as it follows by the same arguments as in the works [35, 16].

Set $\mathbf{G} = \mathcal{H}^{\Phi}_{U_{\ell_3}}$ and $\mathbf{H} = \widehat{\alpha}_3(\mathbf{G}) \subset \mathcal{H}^{\Psi}_V$, and denote by $\mathbf{A} \equiv \widehat{\alpha}_3 \colon \mathbf{G} \to \mathbf{H}$.

Set $W = U_{\ell_3}$. Note that the group $\mathbf{G}$ is the stabilizer of W, and the orbits of $\mathbf{H}$ on points in W equal the orbits of $\mathbf{G}$, hence $\mathbf{H}$ also stabilizes W. Thus, $\mathbf{A}$ conjugates the action of $\mathbf{G}$ and $\mathbf{H}$ on W. This completes the proof of Theorem 1.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

The conclusion of Theorem 1.5 cannot be improved to obtain a conjugacy of the actions Φ and Ψ, as illustrated in Examples A.2 and A.3. However, one can show that $\mathbf{G}$ and $\mathbf{H}$ determine subgroups of the same index in G and H, respectively, using the same methods as in the proof of [16, Theorem 3.3], so the actions Φ and Ψ are structurally conjugate in the sense of [16].

Recall that in Definition 4.1, the notion of virtually continuously orbit equivalent actions was introduced, defined in terms of the induced actions on adapted clopen sets for the actions. There is a corresponding extension to the full groups for a Cantor action:

DEFINITION 4.5. *Let $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{Y}, H, \Psi)$ be minimal Cantor actions. Their full groups are said to be* virtually isomorphic *if there exists adapted clopen subsets $U \subset \mathfrak{X}$ and $V \subset \mathfrak{Y}$ such that their topological full groups $[[U, G_U, \Phi_U]]$ and $[[V, H_V, \Psi_V]]$ of their restricted actions are isomorphic, and we write $[[[\mathfrak{X}, G, \Phi]]] \sim [[[\mathfrak{Y}, H, \Psi]]]$.*

Note that if $\phi \in [[U, G_U, \Phi_U]]$ then as $\mathfrak{X} - U$ is a clopen set, we can extend the map ϕ as the identity on $\mathfrak{X} - U$ to obtain $\widetilde{\phi} \in [[\mathfrak{X}, G, \Phi]]$. This yields an inclusion $[[U, G_U, \Phi_U]] \subset [[\mathfrak{X}, G, \Phi]]$ of groups. The image subgroup will not have finite index unless $U = \mathfrak{X}$, so the terminology "virtual" is an abuse of the usual sense of this terminology, but is chosen as the subgroup $G_U \subset G$ does have finite index.

As for the case of the topological full groups of actions, [38, Remark 3] implies that if $[[U, G_U, \Phi_U]]$ and $[[V, H_V, \Psi_V]]$ are isomorphic as groups, then there is a homeomorphism $h \colon U \to V$ which induces the isomorphism of the full groups. Thus, if the actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{Y}, H, \Psi)$ have virtually isomorphic topological full groups, then the restricted actions (U, G_U, Φ_U) and (V, H_V, Ψ_V) are topologically orbit equivalent. Then by Theorem 1.5 applied to these restricted actions, they are return equivalent. Moreover, if each action is locally quasi-analytic, then the restricted actions are as well. Furthermore, if $W \subset U$ is an adapted set for the restricted action Φ_U then it is also adapted for the action Φ, and similarly for the action Ψ_V. We thus obtain:

THEOREM 4.6. *Let $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{Y}, H, \Psi)$ be minimal Cantor actions which are locally quasi-analytic, and such that $[[[\mathfrak{X}, G, \Phi]]] \sim [[[\mathfrak{Y}, H, \Psi]]]$. Then the actions are return equivalent.*

REMARK 4.7. The virtual isomorphism class $[[[\mathfrak{X}, G, \Phi]]]$ of the full group of a minimal Cantor action $(\mathfrak{X}, G, \Phi)$ is a "local invariant" of the action, as it is determined by the action restricted to arbitrarily small adapted sets for the action. Another such local invariant is the asymptotic discriminant introduced in [33], which distinguishes large classes of minimal equicontinuous actions. It seems an interesting problem to determine the "asymptotic invariants" of return equivalence directly from group invariants of the virtual isomorphism class $[[[\mathfrak{X}, G, \Phi]]]$.

5. CLASSIFICATION OF NIL-SOLENOIDS

In this section, we recall the construction and a few basic properties of weak solenoids, which are a special class continua introduced by McCord [37]. Schori gave an example in [46] of a weak solenoid whose Cantor fiber was not a Cantor group, which motivated the study of the self-homeomorphism group of weak solenoids. This was further investigated by Fokkink and Oversteegen in [24], who introduced the methods that were further developed in the works by the authors [19, 20, 21].

The classification of weak solenoids was studied in the works [1, 15, 33], where it is shown in particular that a homeomorphism between weak solenoids induces a return equivalence between their global monodromy Cantor actions. This result is recalled as Theorem 5.2 below. We introduce the class

of nil-solenoids, then give the proof of Theorem 1.8, which represents a broad generalization of the classification result for 1-dimensional solenoids by Aarts and Fokkink in [1].

We first recall some basic concepts of weak solenoids. Let M_0 be a closed connected manifold, and $x_0 \in M_0$ a choice of basepoint. Let $G = \pi_1(M_0, x_0)$ denote its fundamental group. Let $\mathcal{G}$ be a properly descending chain of finite index subgroups,

$$(14) \qquad \mathcal{G} = \{G = G_0 \supset G_1 \supset G_2 \supset \cdots\} .$$

For $\ell \geq 0$, each subgroup G_ℓ determines a finite covering $\pi_\ell \colon M_\ell \to M_0$, where M_ℓ is a closed manifold. The inclusions $G_{\ell+1} \subset G_\ell$ induce non-trivial proper covering maps $p_{\ell+1} \colon M_{\ell+1} \to M_\ell$. The collection of these maps, $\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$, is called a *presentation*.

Associated to $\mathcal{P}$ is the *weak solenoid* $\mathcal{S}_\mathcal{P}$ which is the inverse limit space

$$(15) \qquad \mathcal{S}_\mathcal{P} \equiv \varprojlim \, \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \ \subset \prod_{\ell \geq 0} M_\ell .$$

By definition, for a sequence $\{x_\ell \in M_\ell \mid \ell \geq 0\}$, we have

$$(16) \qquad x = (x_0, x_1, \ldots) \in \mathcal{S}_\mathcal{P} \iff p_\ell(x_\ell) = x_{\ell-1} \text{ for all } \ell \geq 1 .$$

The set $\mathcal{S}_\mathcal{P}$ is given the relative topology, induced from the product topology, so that $\mathcal{S}_\mathcal{P}$ is compact and connected. The map $\Pi_\ell \colon \mathcal{S}_\mathcal{P} \to M_\ell$ is given by projection onto the ℓ-th factor in (15).

For example, if $M_\ell = \mathbb{S}^1$ for each $\ell \geq 0$, and the map p_ℓ is a proper covering map of degree $m_\ell > 1$ for $\ell \geq 1$, then $\mathcal{S}_\mathcal{P}$ is an example of a classic solenoid. The generalization of 1-dimensional solenoids to the class of weak solenoids was introduced by McCord in [37]. In particular, McCord showed that $\mathcal{S}_\mathcal{P}$ has a uniform local product structure.

PROPOSITION 5.1. [37, 14] *Let $\mathcal{S}_\mathcal{P}$ be a weak solenoid, whose base space M_0 is a compact manifold of dimension $n \geq 1$. Then $\mathcal{S}_\mathcal{P}$ is a foliated space, with foliation $\mathcal{F}_\mathcal{P}$ whose leaves have dimension n. That is, for each $x \in \mathcal{S}_\mathcal{P}$ there is an open neighborhood $x \in U_x \subset \mathcal{S}_\mathcal{P}$ homeomorphic to the product space $(-1,1)^n \times K_x$ where K_x is a Cantor space, which is a foliation chart for $\mathcal{F}_\mathcal{P}$.*

A weak solenoid is a *matchbox manifold* of dimension n in the terminology of [14], or a *solenoidal manifold* in the terminology of [47, 48].

Let $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$ denote the fiber over the basepoint $x_0 \in M_0$. The global monodromy of the foliation $\mathcal{F}_\mathcal{P}$ on $\mathcal{S}_\mathcal{P}$ is the action $\Phi_0 \colon G \times \mathfrak{X}_0 \to \mathfrak{X}_0$ defined by the holonomy transport along the leaves of the foliation $\mathcal{F}_\mathcal{P}$. This action is minimal and equicontinuous. A choice of basepoint $x = (x_0, x_1, x_2, \ldots) \in \mathfrak{X}_0$ defines basepoints in each covering space M_ℓ.

Let X_∞ be the inverse limit sequence associated to the group chain $\mathcal{G}$ in (14), as defined by (7), and let $\Phi \colon G \times X_\infty \to X_\infty$ be the associated minimal equivariant Cantor action. Then the homeomorphism $\tau_x \colon X_\infty \to \mathfrak{X}_0$ defined in Section 3 conjugates the actions $(\mathfrak{X}_0, G, \Phi_0)$ and (X_∞, G, Φ). Moreover, the kernel $K(\mathcal{G})$ is isomorphic to the fundamental group $\pi_1(L_x, x)$ of the leaf $L_x \subset \mathcal{S}_\mathcal{P}$ containing x.

THEOREM 5.2. [15, Theorem 1.1] *Let $\mathcal{P}$ be a presentation of a weak solenoid $\mathcal{S}_\mathcal{P}$ over a closed manifold M_0 of dimension n with fundamental group G, and $\mathcal{P}'$ be a presentation of a weak solenoid $\mathcal{S}_{\mathcal{P}'}$ over a closed manifold M_0' of dimension n' with fundamental group G'. Suppose that $\mathcal{S}_\mathcal{P}$ is homeomorphic to $\mathcal{S}_{\mathcal{P}'}$, then the global holonomy action (X_∞, G, Φ) of $\mathcal{F}_\mathcal{P}$ is return equivalent to the global holonomy action (X_∞', G', Φ') of $\mathcal{F}_{\mathcal{P}'}$.*

We introduce the following terminology, which is well-defined by Theorem 5.2.

DEFINITION 5.3. *Let $[[[\mathcal{F}_\mathcal{P}]]]$ denote the virtual isomorphism class of the full group of the monodromy action $(\mathfrak{X}_0, G, \Phi_0)$ for the weak solenoid $\mathcal{S}_\mathcal{P}$ with foliation $\mathcal{F}_\mathcal{P}$.*

COROLLARY 5.4. *Let $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_{\mathcal{P}'}$ be homeomorphic weak solenoids, then $[[[\mathcal{F}_\mathcal{P}]]] \sim [[[\mathcal{F}_{\mathcal{P}'}]]]$.*

We can now give an application of Theorem 1.5 which gives a partial converse to Corollary 5.4.

A closed manifold M_0 is said to be a *nilmanifold* if its fundamental group $G = \pi_1(M_0, x_0)$ is a nilpotent group, for some choice of basepoint $x_0 \in M_0$, and its universal covering $\widetilde{M_0}$ is a contractible space. In particular, this implies that the fundamental group G is torsion free. Let $\mathcal{G}$ be a group chain as in (14), such that the kernel $K(\mathcal{G})$ is the trivial group. The inverse limit space $\mathcal{S}_{\mathcal{P}}$ is called a *nil-solenoid*.

THEOREM 5.5. *Let $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{P}'}$ be nil-solenoids, and suppose that $[[[\mathcal{F}_{\mathcal{P}}]]] \sim [[[\mathcal{F}_{\mathcal{P}'}]]]$. Then $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{P}'}$ are homeomorphic continua.*

Proof. Let G_0 denote the fundamental group for the closed manifold M_0 in the presentation $\mathcal{P}$, and G_0' the fundamental group for the closed manifold M_0' in the presentation $\mathcal{P}'$.

The assumption that $[[[\mathcal{F}_{\mathcal{P}}]]] \sim [[[\mathcal{F}_{\mathcal{P}'}]]]$ implies there exists adapted clopen sets $U \subset X_\infty$ and $U' \subset X_\infty'$ and a homeomorphism $h \colon U \to U'$ that induces an isomorphism between the topological full groups $[[U, G_U, \Phi_U]]$ and $[[U', G_{U'}', \Phi_{U'}']]$ for the minimal equicontinuous Cantor actions (U, G_U, Φ_U) and $(U', G_{U'}', \Phi_{U'}')$.

As G_0 and G_0' are torsion-free, nilpotent, and finitely generated groups, the same also holds for the subgroup G_U of finite index, and similarly for $G_{U'}'$. It then follows from Corollary 3.8 that both restricted actions are locally quasi-analytic. It then follows from Theorem 1.5 that the restricted actions are return equivalent, and thus the actions (X_∞, G, Φ) and (X_∞', G', Φ') are return equivalent.

Note that the assumption in the definition of a nil-solenoid $\mathcal{S}_{\mathcal{P}}$ that the foliation $\mathcal{F}_{\mathcal{P}}$ has a simply connected leaf $L_0 \subset \mathcal{S}_{\mathcal{P}}$ implies that the kernel $K(\mathcal{G})$ for its defining group chain $\mathcal{G}$ is trivial. The simply connected leaf L_0 is homeomorphic to the universal covering of the base manifold M_0 hence is contractible. Analogous comments hold for the nil-solenoid $\mathcal{S}_{\mathcal{P}'}$.

Return equivalence implies that the fundamental group $G_0 = \pi_1(M, x_0)$ contains a subgroup of finite-index which is isomorphic to a subgroup of finite index in $G_0' = \pi_1(M', x_0')$. For a nil-manifold, its dimension is determined by the cohomological dimension of its fundamental group, so M_0 and M_0' have finite coverings of the same dimension, hence must have equal dimensions.

Then by Theorem 1.5 of [15], the continua $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{P}'}$ are homeomorphic. $\qquad\square$

APPENDIX A. EXAMPLES

In this appendix, we give a collection of examples of minimal Cantor actions, which illustrate the complexities that these actions can exhibit. The first examples are elementary, then increase in the subtleness of their properties. The presentations of the examples are brief, with references to their detailed constructions and study in the cited literature, as the interest is in showing the possibilities.

EXAMPLE A.1 (Abelian actions)**.** By Corollary 2.3, for G a finitely-generated abelian group, every effective minimal action of G on a Cantor space $\mathfrak{X}$ is quasi-analytic, hence topologically free. The work of Li in [35] studies the relation between continuous orbit equivalence and conjugation of minimal equicontinuous Cantor actions. Example 3.5 in [35] constructs two abelian actions of $\mathbb{Z}^n$ which are continuously orbit equivalent but not conjugate. By Theorem 1.2 in [16], or Theorem 1.5 above, the actions must be return equivalent. The results of Giordano, Putman and Skau in [27] classifies such actions in terms of their Pontrjagin duals.

EXAMPLE A.2 (Direct products)**.** Consider first a "toy" example. Let G and G' be non-isomorphic finite groups of the same order, $N = |G| = |G'| \geq 4$. Let $X = \{1, 2, \ldots, N\}$ be a finite set, so obviously not a Cantor space. Choose an isomorphism $\varphi \colon G \to X$ and define the action (X, G, Φ) by using φ to conjugate the left action of G on itself to an action on X. Likewise, choose an isomorphism $\varphi' \colon G' \to X$ and define the action (X, G', Φ') by using φ' to conjugate the left action of G' on itself to an action on X. Note that for both of these actions, there is only one orbit of the action. Then both actions are minimal and equicontinuous, and the identity map $X \to X$ is a continuous orbit equivalence. However, the actions Φ and Φ' cannot be conjugate as there is no group isomorphism $h_* \colon G \to G'$.

This simple example can then be modified to obtain examples which show that the conclusion of Theorem 1.5 cannot be improved to obtain a conjugacy of the actions. Let $(\mathfrak{Y}, H, \Psi)$ be any minimal equicontinuous Cantor action of a free abelian group H. Set $\mathfrak{X} = X \times \mathfrak{Y}$ which is a Cantor space. Then the product actions $(\mathfrak{X}, G \times H, \Phi \times \Psi)$ and $(\mathfrak{X}, G' \times H, \Phi' \times \Psi)$ are locally quasi-analytic and continuously orbit equivalent, so satisfy the hypotheses of Theorem 1.5, but cannot be conjugate.

Example 7.6 in [19] gives another variation on a product action, where the finite group G is assumed to be simple, and one chooses a non-trivial proper subgroup $K \subset G$. The space $X \cong G/K$ is given the left action by G, then using a product action as above, we obtain a minimal equicontinuous Cantor action which is locally quasi-analytic and has discriminant $\mathcal{D}_x \cong K$. If $K, K' \subset G$ are non-isomorphic proper subgroups with the same order, then for product actions with the actions of G and G/K and G/K' as above, we obtain minimal equicontinuous Cantor actions which are locally quasi-analytic and continuously orbit equivalent, but cannot be conjugate as their discriminants are not isomorphic.

EXAMPLE A.3 (Semi-direct products). The direct product construction above can be made somewhat more interesting by using a semi-direct product construction, as in:

● Example 7.5 in [19], where G is the infinite dihedral group, and the discriminant $\mathcal{D}_x \cong \mathbb{Z}/2\mathbb{Z}$. This action is non-homogeneous but is stable.

● Examples 8.8 and 8.9 in [19], where G is a generalized dihedral group, and the discriminant of the action is a Cantor subgroup.

● Example 7.1 in [20], where G is a semi-direct product of $\mathbb{Z}^n$ with a finite subgroup $H \subset \Sigma(n)$ of the permutation group on n letters.

Note that in all of these cited examples, the semi-direct product construction results in an extension of an infinite group by a finite normal subgroup.

EXAMPLE A.4 (Full group modifications). Let $(\mathfrak{X}, G, \Phi)$ be a minimal equicontinuous Cantor action. Assume the action Φ is topologically free, so that we can identify G with its image $\Phi(G) \subset \mathrm{Homeo}(\mathfrak{X})$. The idea of this construction is to modify the group G by the addition of a finite set of elements from its topological full group $[[\mathfrak{X}, G, \Phi]]$.

Let $U \subset \mathfrak{X}$ be a proper adapted clopen subset, with stabilizer group $G_U \subset G$. Let $A: G_U \to G_U$ be a non-trivial automorphism. Define a new action $\Phi_U^A: G_U \to \mathrm{Homeo}(\mathfrak{X})$ by setting, for $g \in G_U$:

$$(17) \qquad \Phi_U^A(g)(x) = A(g) \cdot x \text{ for } x \in U, \text{ and } \Phi_U^A(g)(x) = x \text{ for } x \in \mathfrak{X} - U .$$

Let $H \subset \mathrm{Homeo}(\mathfrak{X})$ denote the group generated by the images $\Phi(G)$ and $\Phi_U^A(G_U)$. Then H is finitely generated, and by construction, the action $\Psi: H \times \mathfrak{X} \to \mathfrak{X}$ is continuously orbit equivalent to the given action Φ. Note that the action of H is locally quasi-analytic but not topologically free.

The clopen set U is adapted to both actions, and the restriction of Ψ to U is just the conjugated action $\Phi_U^A: G_U \to \mathrm{Homeo}(\mathfrak{X})$. Thus, the images $\Phi_U(G_U) \subset \mathrm{Homeo}(U)$ and $\Phi_U^A(G_U) \subset \mathrm{Homeo}(U)$ are equal. It follows that the actions are return equivalent. However, H is almost surely not isomorphic to G, so the actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}, H, \Psi)$ cannot be conjugate.

EXAMPLE A.5 (Heisenberg). In her thesis [18], Dyer constructed examples of subgroup chains in the 2-dimensional Heisenberg group, so that the discriminant of the Cantor action constructed from the chain is a Cantor group. These are probably the simplest examples of Cantor actions by torsion-free finitely generated nilpotent groups. We recall Example 8.5 from [19].

Let $\mathcal{H}$ be the discrete Heisenberg group, presented in the form $\mathcal{H} = (\mathbb{Z}^3, *)$ with the group operation $*$ given by $(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy')$. Let $A_n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix}$, where p and q are distinct primes, and consider the action represented by a group chain

$$G_0 = \mathcal{H} , \ \{G_n \mid n \geq 1\} \text{ where } G_n \equiv A_n \mathbb{Z}^2 \times p^n \mathbb{Z} .$$

Then the discriminant $\mathcal{D}_x$ is a Cantor group, by explicit calculation [18, Example 5.14]. It would be interesting to have a procedure for constructing nilpotent actions of higher dimension whose discriminants are Cantor groups. All such examples must be locally quasi-analytic by Corollary 3.8.

EXAMPLE A.6 (Actions on rooted trees). Every minimal equicontinuous Cantor action can be interpreted as an action on a rooted spherically homogeneous tree. For actions as discussed in the previous examples, this fact is just one facet of their study, but for other actions it is a part of their definition. We briefly recall some basic facts about trees, then describe two classes of "arboreal actions" currently under investigation.

Let T be a tree. That is, T consists of the set of vertices $V = \cup_{\ell \geq 0} V_\ell$, where each V_ℓ is a finite set, and of the set of edges E, where an edge $t = [a, b]$ can join two vertices a and b only if $a \in V_\ell$ and $b \in V_{\ell+1}$, and every vertex $b \in V_{\ell+1}$ is joined to precisely one vertex in V_ℓ. We assume that V_0 is a singleton, so T is *rooted* with root vertex $V_0 = \{v_0\}$. The tree T is *spherically homogeneous*, if there is a sequence of positive integers $(n_1, n_2, \ldots)$ such that for every $\ell \geq 1$, every vertex $v \in V_{\ell-1}$ is joined by an edge to precisely n_ℓ vertices in V_ℓ. We assume that $n_\ell > 1$ for all $\ell \geq 1$.

An automorphism $g \in \mathrm{Aut}(T)$ of the rooted tree T permutes the vertices within each level V_ℓ, while preserving the connectedness of the tree. That is, for all $\ell \geq 1$ the vertices $v_{\ell-1} \in V_{\ell-1}$ and $v_\ell \in V_\ell$ are joined by an edge if and only if $g \cdot v_{\ell-1} \in V_{\ell-1}$ and $g \cdot v_\ell \in V_\ell$ are joined by an edge. Thus $\mathrm{Aut}(T)$ acts on infinite paths in the tree. Here, a path in T is an ordered infinite sequence of vertices $(v_\ell)_{\ell \geq 0}$ such that for all $\ell \geq 0$ the vertex v_ℓ is at level V_ℓ, and the consecutive vertices v_ℓ and $v_{\ell+1}$ are joined by an edge. Denote by $\mathcal{P}$ the space of all infinite paths in T with cylinder set topology. As $n_\ell > 1$ for $\ell \geq 1$, the space of paths $\mathcal{P}$ is a Cantor space.

Let $(\mathfrak{X}, G, \Phi)$ be an effective minimal equicontinuous Cantor actions. The "tree model" for this action is constructed as follows. Choose a basepoint $x \in \mathfrak{X}$, and choose an adapted neighborhood system $\mathcal{U} = \{U_\ell \mid \ell \geq 1\}$ at x, with stabilizer groups $G_\ell = G_{U_\ell}$. Set the root vertex $v_0 = \{\mathfrak{X}\}$, and let the vertices at level $\ell \geq 1$ be the collection of clopen sets $V_\ell \equiv \{g \cdot U_\ell \mid g \in G/G_\ell\}$. For $g, g' \in G$, there is an edge $t = [g \cdot U_\ell, g' \cdot U_{\ell+1}]$ between $g \cdot U_\ell$ and $g' \cdot U_{\ell+1}$ exactly when $g' \cdot U_{\ell+1} \subset g \cdot U_\ell$. This defines a spherically homogeneous tree $T_\mathcal{U}$ where the degree $n_\ell = [G_{\ell-1} : G_\ell]$. The map $\tau_\infty \colon X_\infty \to \mathfrak{X}$ introduced in Section 3 can be interpreted as defining a homeomorphism $\tau_\mathcal{U} \colon \mathcal{P}_\mathcal{U} \to \mathfrak{X}$ which commutes with the action of G.

Now let T be a rooted spherically homogeneous tree. It was shown in the work [7] that $\mathrm{Aut}(T)$ is the infinite wreath product of the symmetric groups $\mathfrak{S}(n_\ell)$, and so has a dense countably generated subgroup $G_0 \subset \mathrm{Aut}(T)$. This is described in [36, Section 2.5], and see also Nekrashevych [39]. Moreover, it is shown in [36] that the action of G_0 on the path space $\mathcal{P}$ is not locally quasi-analytic. Here is a basic question:

PROBLEM A.7. *Let T be a rooted spherically homogeneous tree. Construct finitely-generated subgroups $G_0 \subset \mathrm{Aut}(T)$ such that the induced action on the Cantor space of paths $\mathcal{P}$ is not locally quasi-analytic.*

One interest in constructing finitely-generated groups acting on Cantor spaces is that each such action can be realized as the global monodromy action of a suspension foliation as described in Section 8.1 of [33]. If the action is a minimal equicontinuous Cantor action, then the resulting foliated space is homeomorphic to a weak solenoid as described in Section 5 above.

The non-Hausdorff examples of Grigorchuk and Nekrashevych provide one class of solutions to Problem A.7. The construction by Grigorchuk of his celebrated groups with intermediate growth in [29] introduced the notion of groups acting on rooted spherically homogeneous trees which are generated by automata. The branch groups of [6, 30] are a far reaching extension of Grigorchuk's original construction. It is easy to see that the action of the standard Grigorchuk group on the Cantor set of paths is not locally quasi-analytic. Nekrashevych studied in [40, 41] the dynamical properties of these actions, and showed that they often contain non-Hausdorff elements in their action groupoids. An action containing a non-Hausdorff element cannot be locally quasi-analytic by Proposition 2.5.

EXAMPLE A.8 (Absolute Galois actions). Odoni initiated in [42] the study of arboreal representations of the absolute Galois group. Such representations are given by the action of a profinite group on a spherically homogeneous rooted tree. This field of study is surveyed by Jones in [34]. The second author developed in [36] a method of associating to an arboreal representation an infinite chain of discrete groups, and gave examples of arboreal representations, some of which are locally quasi-analytic, and some of which are not. The examples of not locally quasi-analytic actions in [36] are by infinitely-generated groups, and it is an open problem to construct arboreal representations which are finitely generated and not locally quasi-analytic.

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