Balancing the Lifting Values to Improve the Numerical Stability of Polyhedral Homotopy Continuation Methods

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Abstract

Polyhedral homotopy continuation methods exploit the sparsity of polynomial systems so that the number of solution curves to reach all isolated solutions is optimal for generic systems. The numerical stability of tracing solution curves of polyhedral homotopies is mainly determined by the height of the powers of the continuation parameter. To reduce this height we propose a procedure that operates as an intermediate stage between the mixed-volume computation and the tracing of solution curves. This procedure computes new lifting values of the support of a polynomial system. These values preserve the structure of the mixed-cell configuration obtained from the mixed-volume computation and produce better-balanced powers of the continuation parameter in the polyhedral homotopies.

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1 Introduction

During the last two decades, homotopy continuation methods have proven to be reliable and efficient to compute numerical approximations to all isolated zeros of polynomial systems. To exploit sparsity of polynomial systems, polyhedral homotopy continuation methods emerged in [7, 15]. For generic polynomial systems, the number of solution curves to be traced is optimal. We refer to [8] for a survey on recent advances in homotopy continuation methods for polynomial systems.

Polyhedral homotopies are nonlinear in the continuation parameter. Powers of the continuation parameter too close to zero can be scaled away from zero by a suitable scalar multiplication. After scaling, if very high powers exist in a polyhedral homotopy, small step sizes must be taken in order to successfully trace a solution curve of the polyhedral homotopy. Although the end of the solution curve can be reached as long as the required step size is not smaller than the available machine precision, the efficiency of the curve-tracing is greatly reduced. A more serious problem occurs when the continuation parameter, starting from 0, is not yet close enough to 1, some terms of the polyhedral homotopy with high powers of the continuation parameter have values smaller than the machine precision and some solution curves may come close to "valleys" where the values of the homotopy are numerically zero, but no solution curves exist inside the "valleys". This situation can easily cause the curve-tracings to be trapped in these "valleys" with no chance of reaching the ends of solution curves unless the curves are retraced with smaller step sizes.

Two known geometric approaches to control the numerical stability of polyhedral homotopy continuation methods are recursive liftings (as in Bernshtein's algorithm [2, 15]) and dynamic liftings [14]. However, because of using multiple liftings or flattenings, those approaches both require more expensive construction of subdivisions and create more homotopy curves which need to be traced than a random floating-point lifting.

In continuous complexity [3] one works with condition numbers which measure the reciprocal of the distance to the discriminant variety. Here, we have to deal with badly scaled powers. Tiny perturbations in the continuation parameter produce dramatic changes in the solutions. In our problem singularities are absent, but standard machine precision is insufficient. In that respect, the models of discrete complexity [4] where one bounds the bit size in order to achieve the required precision seem more appropriate. However, when the balancing method is not sufficient to remove the numerical difficulties, we prefer the geometric approaches mentioned above rather than to apply time-consuming software-driven multi-precision arithmetic.

To minimize the height of the powers of the continuation parameter in polyhedral homotopies, we search in the cone of all lifting vectors that induce the same mixed-cell configuration to obtain better-balanced powers of the continuation parameter of polyhedral homotopies. This idea can also be found in the proof of Proposition 1.11 in [13] and uses the concept of secondary fan [13, p. 71], [6, Chapter 7]. The organization of this paper is as follows. An introduction to polyhedral homotopies and the problem statement are given in the next section. The "sandwich model" to implement the balancing method is derived in section three and its efficient implementation is discussed in section four. In section five, numerical results are presented to show the efficiency of the balancing method.

2 Numerical Stability of Polyhedral Homotopies

For a system of polynomials $P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}))$ with $\mathbf{x} = (x_1, \ldots, x_n)$, we write

$$p_i(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{A}_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n,$$

where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $c_{i,\mathbf{a}} \in \mathbb{C}^* := \mathbb{C} \backslash \{0\}$ and $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. Here $\mathcal{A}_i$, a finite subset of $\mathbb{Z}^n$, is the *support* of $p_i(\mathbf{x})$, and we call $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ the *support* of $P(\mathbf{x})$. For notational convenience we do not exploit several supports being equal. The generalization of our method to this end is just a matter of notation.

The first step of using polyhedral homotopy continuation methods to locate all isolated zeros of $P(\mathbf{x})$ is to solve a generic polynomial system $G(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) = \mathbf{0}$ with the same support $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ but with randomly chosen coefficients $c'_{i,\mathbf{a}}$, namely,

$$g_i(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{A}_i} c'_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n.$$

Secondly, the system $G(\mathbf{x}) = \mathbf{0}$ serves as the starting system in the linear homotopy

$$H(\mathbf{x}, t) = (1 - t) G(\mathbf{x}) + t P(\mathbf{x}) = \mathbf{0}$$

to solve $P(\mathbf{x}) = \mathbf{0}$. In this paper, we will focus on the problems that occur when we solve the system $G(\mathbf{x}) = \mathbf{0}$.

By Bernshtein's Theorem [2], the system $G(\mathbf{x}) = \mathbf{0}$, being generic, has exactly as many solutions in $(\mathbb{C}^*)^n$ as the mixed volume $\mathcal{M}(\mathcal{A})$ of the support $\mathcal{A}$. Since we focus on the homotopies we do not explicitly need the geometric constructions, for which we refer to [5, 7, 8, 9, 14] for precise definitions. To find all isolated solutions of $G(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) = \mathbf{0}$, we construct a nonlinear homotopy $\widehat{G}(\mathbf{x}, t) = (\widehat{g}_1(\mathbf{x}, t), \ldots, \widehat{g}_n(\mathbf{x}, t)) = \mathbf{0}$ by introducing a continuation parameter t as follows:

$$\widehat{g}_i(\mathbf{x}, t) = \sum_{\mathbf{a} \in \mathcal{A}_i} c'_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}} t^{\omega_i(\mathbf{a})}, \quad i = 1, \ldots, n, \tag{1}$$

where the powers $\omega_i(\mathbf{a})$ of t are determined by the *lifting function* $\omega_i : \mathcal{A}_i \to \mathbb{R}$ on the points $\mathbf{a}$ in the support $\mathcal{A}_i$, $i = 1, \ldots, n$.

For $t = 1$, (1) equals the system $G(\mathbf{x}) = \mathbf{0}$ that we want to solve. In order to trace the solution curves of $\widehat{G}(\mathbf{x}, t) = \mathbf{0}$ originating from $t = 0$, we set up polyhedral homotopies as in [7, 8] by using the subdivision S_ω of the support $\mathcal{A}$ induced by the lifting function $\omega = (\omega_1, \ldots, \omega_n)$. We briefly describe the procedure as follows. For any cell $C = (C_1, \ldots, C_n) \in S_\omega$, $\mathrm{conv}(C) := \mathrm{conv}(C_1) + \cdots + \mathrm{conv}(C_n)$ is a projection of a facet in the lower hull of the Minkowski sum of the Newton polytopes of the lifted support. The facet is characterized by a unique inner normal $(\alpha, 1) \in \mathbb{R}^{n+1}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. We will call $(\alpha, 1)$ the inner normal of C with respect to ω, or simply the inner normal of C when no ambiguity exists. For $C = (C_1, \ldots, C_n) \in S_\omega$ with inner normal $(\alpha, 1)$, the corresponding restriction of $G(\mathbf{x})$ to C, denoted by $G^\alpha(\mathbf{x}) = (g_1^\alpha(\mathbf{x}), \ldots, g_n^\alpha(\mathbf{x}))$, is given by

$$g_i^\alpha(\mathbf{x}) = \sum_{\mathbf{a} \in C_i} c'_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad i = 1, \ldots, n. \tag{2}$$

Only if $g_i^\alpha(\mathbf{x})$ has at least two monomials for each $i = 1, \ldots, n$, $G^\alpha(\mathbf{x}) = \mathbf{0}$ admits solutions in $(\mathbb{C}^*)^n$. For generic ω, only mixed cells $C = (C_1, \ldots, C_n)$ in the fine mixed subdivision S_ω for which $\#C_i = 2$ are of interest. We collect all such mixed cells of S_ω and call this set a *mixed-cell configuration* $\mathcal{M}_\omega$ of the support $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

For the cell $C = (C_1, \ldots, C_n) \in \mathcal{M}_\omega$ with inner normal $(\alpha, 1)$, we have

$$\begin{cases} \langle \mathbf{b}, \alpha \rangle + \omega_i(\mathbf{b}) = \langle \mathbf{a}, \alpha \rangle + \omega_i(\mathbf{a}), & \forall\ \mathbf{a}, \mathbf{b} \in C_i, \\ \langle \mathbf{c}, \alpha \rangle + \omega_i(\mathbf{c}) > \langle \mathbf{a}, \alpha \rangle + \omega_i(\mathbf{a}), & \text{for } \mathbf{a} \in C_i,\ \forall\ \mathbf{c} \in \mathcal{A}_i \backslash C_i, \end{cases} \qquad i = 1, \ldots, n, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. Substituting the coordinate transformation

$$x_i = y_i t^{\alpha_i}, \quad i = 1, \ldots, n, \tag{4}$$

into the system (1), we obtain $\widehat{G}^\alpha(\mathbf{y}, t) = (\widehat{g}_1^\alpha(\mathbf{y}, t), \ldots, \widehat{g}_n^\alpha(\mathbf{y}, t))$ with $\mathbf{y} = (y_1, \ldots, y_n)$ and

$$\widehat{g}_i^\alpha(\mathbf{y}, t) = \sum_{\mathbf{a} \in C_i} c'_{i,\mathbf{a}} \mathbf{y}^{\mathbf{a}} t^{\langle \mathbf{a}, \alpha \rangle + \omega_i(\mathbf{a})} + \sum_{\mathbf{c} \in \mathcal{A}_i \backslash C_i} c'_{i,\mathbf{c}} \mathbf{y}^{\mathbf{c}} t^{\langle \mathbf{c}, \alpha \rangle + \omega_i(\mathbf{c})}, \quad i = 1, \ldots, n. \tag{5}$$

By (3), the lowest powers of t occur for $\mathbf{a} \in C_i$. We can factor those out to obtain the *polyhedral homotopy* $\bar{G}^\alpha(\mathbf{y}, t) = (\bar{g}_1^\alpha(\mathbf{y}, t), \ldots, \bar{g}_n^\alpha(\mathbf{y}, t)) = \mathbf{0}$ with

$$\bar{g}_i^\alpha(\mathbf{y}, t) = \sum_{\mathbf{a} \in C_i} c'_{i,\mathbf{a}} \mathbf{y}^{\mathbf{a}} + \sum_{\mathbf{c} \in \mathcal{A}_i \backslash C_i} c'_{i,\mathbf{c}} \mathbf{y}^{\mathbf{c}} t^{\langle \mathbf{c}, \alpha \rangle - \langle \mathbf{a}, \alpha \rangle + \omega_i(\mathbf{c}) - \omega_i(\mathbf{a})} = 0, \quad i = 1, \ldots, n. \tag{6}$$

By (3), the powers of t in (6) are all positive. At $t = 0$, the starting system is $\bar{G}^\alpha(\mathbf{y}, 0) = G^\alpha(\mathbf{y}) = \mathbf{0}$ in (2). We can solve this binomial system by consecutive coordinate transformations. When $t = 1$, $y_i = x_i$ in (4), so the homotopy $\bar{G}^\alpha(\mathbf{y}, t) = (\bar{g}_1^\alpha(\mathbf{y}, t), \ldots, \bar{g}_n^\alpha(\mathbf{y}, t)) = \mathbf{0}$ leads to as many solutions of $G(\mathbf{x}) = \mathbf{0}$ as the mixed volume of the cell C. By application of the same procedure to all cells $C \in \mathcal{M}_\omega$, all solutions to $G(\mathbf{x}) = \mathbf{0}$ are reached.

Now, when the powers of t in the polyhedral homotopy $\bar{G}^\alpha(\mathbf{y}, t) = \mathbf{0}$ in (6) are too high or too close to zero, instability might occur in tracing solution curves of $\bar{G}^\alpha(\mathbf{y}, t) = \mathbf{0}$ unless very small step sizes are taken. For the standard prediction-correction method [1] in tracing a curve, the number of steps required for homotopy curves with high powers of t is usually much greater than those for other curves. This may severely reduce the efficiency of the algorithm. A more serious problem can occur when curve-tracings are trapped into "valleys" where the values of the polyhedral homotopy are numerically zeros, but no solution curves exist inside the "valleys" as the following example shows.

Example 1 Consider the polynomial system $P(\mathbf{x}) = (p_1(\mathbf{x}), p_2(\mathbf{x})) = \mathbf{0}$ with $\mathbf{x} = (x_1, x_2)$, where

$$\begin{aligned} p_1(\mathbf{x}) &= c_1 x_1^2 + c_2 x_2^2 + c_3 = 0, \\ p_2(\mathbf{x}) &= c_4 x_1^3 x_2^3 + c_5 x_1 + c_6 x_2 = 0 \end{aligned}$$

with support $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$, where $\mathcal{A}_1 = \{(2,0), (0,2), (0,0)\}$ and $\mathcal{A}_2 = \{(3,3), (1,0), (0,1)\}$. This system has mixed volume $\mathcal{M}(\mathcal{A}) = 12$.

Choose a random lifting $\omega = (\omega_1, \omega_2)$ where $\omega_1 : \mathcal{A}_1 \to \mathbb{R}$ and $\omega_2 : \mathcal{A}_2 \to \mathbb{R}$ with

$$\omega_1((2,0)) = 0.655416, \quad \omega_1((0,2)) = 0.200995, \quad \omega_1((0,0)) = 0.893622$$

and

$$\omega_2((3,3)) = 0.281886, \quad \omega_2((1,0)) = 0.525000, \quad \omega_2((0,1)) = 0.314127.$$

Then the mixed-cell configuration $\mathcal{M}_\omega$ in the fine mixed subdivision of $\mathcal{A}$ induced by ω consists of two mixed cells:

$$C^{(1)} = (\{(2,0), (0,2)\}, \{(3,3), (1,0)\})$$

and

$$C^{(2)} = (\{(0,2), (0,0)\}, \{(1,0), (0,1)\}).$$

To construct the system $G(\mathbf{x}) = \mathbf{0}$, we choose the following set of randomly generated coefficients,

$$\begin{aligned} c_1' &= -0.434847 - 0.169859i, & c_2' &= 0.505911 + 0.405431i, \\ c_3' &= 0.0738596 + 0.177798i, & c_4' &= -0.0906755 + 0.208825i, \\ c_5' &= 0.175313 - 0.163549i, & c_6' &= 0.527922 - 0.364841i. \end{aligned}$$

The polyhedral homotopy induced by the cell $C^{(1)}$ and ω is $\bar{G}(\mathbf{y},t) = (\bar{g}_1(\mathbf{y},t), \bar{g}_2(\mathbf{y},t)) = \mathbf{0}$, where

$$\begin{aligned}\bar{g}_1(\mathbf{y},t) &= c_1' y_1^2 + c_2' y_2^2 + c_3' t^{50.63523},\\ \bar{g}_2(\mathbf{y},t) &= c_4' y_1^3 y_2^3 + c_5' y_1 + c_6' y_2 t^2.\end{aligned}$$

There are ten solution curves of $\bar{G}(\mathbf{y},t) = \mathbf{0}$ emanating from ten solutions of $\bar{G}(\mathbf{y},0) = \mathbf{0}$. At $t = 0.65$, five of those ten curves have phase space tangent vectors $(\frac{dy_1}{dt}, \frac{dy_2}{dt})$ all roughly pointing to $\mathbf{y} = (0,0)$ at the points on the curves. For the standard prediction-correction method, starting at these points, the prediction step with step size 0.025 will give the predicted points close to $(\mathbf{y},t) = (0,0,0.675)$ for all those five curves. Since $t^{50.63523}$ is about 10^{-9} for $t = 0.675$, the function values of $\bar{G}(\mathbf{y},t)$ in a very small neighborhood of $(0,0,0.675)$ (the "valley") are almost zero. Starting from these predicted points at $t = 0.675$, Newton iterations for the correction step will converge to the "valley" rather than the points on the curves at $t = 0.675$. But there are no solution curves of $\bar{G}(\mathbf{y},t) = \mathbf{0}$ passing through the "valley".

The problem described in the above example occur very rarely in low dimensions, but is representative for large-scale applications. Due to "large" powers of t we observe two problems: the solutions are very sensitive to small perturbations of t, especially as $t \approx 1$, and can be attracted towards spurious zero-component solutions. Deciding about what is too "large" theoretically may lead to too pessimistic results, whereas in practice this decision depends on a lot of factors, such as the settings of the path tracker, choice of coefficients, shape of Newton polytopes, etc, which are hard to capture. In any case, making the powers of t lower by balancing reduces the chance of running into problems. Setting up a probability model to measure this chance lies outside the scope of this paper.

3 Balancing the Powers by the Sandwich Model

As indicated in the last section, generic random liftings can induce highly nonlinear polyhedral homotopies which may produce numerical instabilities. To overcome this problem, our strategy is to find a new lifting function $\nu = (\nu_1, \ldots, \nu_n)$ where $\nu_i : \mathcal{A}_i \to \mathbb{R}$ for $i = 1, \ldots, n$, *based on the already computed mixed-cell configuration* $\mathcal{M}_\omega$. The mixed-cell configuration $\mathcal{M}_\nu$ induced by the new lifting ν will be the same as $\mathcal{M}_\omega$, but the highest power of the continuation parameter t in the polyhedral homotopies induced by ν is the smallest among all those polyhedral homotopies induced by liftings which keep the mixed-cell configuration $\mathcal{M}_\omega$ invariant. In this way, reidentifying the mixed-cell configuration $\mathcal{M}_\nu$, which is very time consuming, becomes unnecessary.

Suppose the collection of cells in $\mathcal{M}_\omega$ is given. To find a new lifting function ν with $\mathcal{M}_\omega = \mathcal{M}_\nu$, for every $C = (C_1, \ldots, C_n) \in \mathcal{M}_\omega$ where $C_i = \{\mathbf{a}_i, \mathbf{b}_i\} \subset \mathcal{A}_i$, $i = 1, 2, \ldots, n$, we impose on the lifting function $\nu = (\nu_1, \ldots, \nu_n)$ the conditions:

$$\left\{ \begin{array}{ll} \langle(\mathbf{b}_i, \nu_i(\mathbf{b}_i)), (\gamma, 1)\rangle = \langle(\mathbf{a}_i, \nu_i(\mathbf{a}_i)), (\gamma, 1)\rangle, & \\ \langle(\mathbf{c}, \nu_i(\mathbf{c})), (\gamma, 1)\rangle > \langle(\mathbf{a}_i, \nu_i(\mathbf{a}_i)), (\gamma, 1)\rangle, & \forall\ \mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\}, \end{array} \right. \quad i = 1, \ldots, n,$$

or,

$$\begin{aligned} \langle \mathbf{b}_i, \gamma\rangle + \nu_i(\mathbf{b}_i) - \langle \mathbf{a}_i, \gamma\rangle - \nu_i(\mathbf{a}_i) &= 0 && (7) \\ & && i = 1, \ldots, n, \\ \langle \mathbf{c}, \gamma\rangle + \nu_i(\mathbf{c}) - \langle \mathbf{a}_i, \gamma\rangle - \nu_i(\mathbf{a}_i) &> 0, \quad \forall\ \mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\}, && (8) \end{aligned}$$

where $(\gamma, 1)$ is the inner normal of C in $\mathcal{M}_\nu$.

From (7), we have $\langle \mathbf{a}_i - \mathbf{b}_i, \gamma\rangle = \nu_i(\mathbf{b}_i) - \nu_i(\mathbf{a}_i)$ for $i = 1, \ldots, n$. Since $C = (C_1, \ldots, C_n)$ is a mixed cell in the fine mixed subdivision S_ω, the edges spanned by $\{\mathbf{a}_i, \mathbf{b}_i\}$, $i = 1, \ldots, n$, determine a full-dimensional parallelepiped in $\mathbb{R}^n$. Equivalently, the matrix

$$A = \begin{bmatrix} \mathbf{a}_1 - \mathbf{b}_1 \\ \vdots \\ \mathbf{a}_n - \mathbf{b}_n \end{bmatrix}$$

is nonsingular, and so, γ can be expressed uniquely as a linear combination of the lifting values $\nu_i(\mathbf{a}_i)$ and $\nu_i(\mathbf{b}_i)$, $i = 1, \ldots, n$. Namely,

$$\gamma^T = A^{-1} \begin{bmatrix} \nu_1(\mathbf{b}_1) - \nu_1(\mathbf{a}_1) \\ \vdots \\ \nu_n(\mathbf{b}_n) - \nu_n(\mathbf{a}_n) \end{bmatrix}. \qquad (9)$$

As in (6), the polyhedral homotopy induced by lifting $\nu = (\nu_1, \ldots, \nu_n)$ and mixed cell $C = (\{\mathbf{a}_1, \mathbf{b}_1\}, \ldots, \{\mathbf{a}_n, \mathbf{b}_n\})$ with inner normal $(\gamma, 1)$ is

$$\bar{g}_i^\gamma(\mathbf{y}, t) = c'_{i,\mathbf{a}_i}\mathbf{y}^{\mathbf{a}_i} + c'_{i,\mathbf{b}_i}\mathbf{y}^{\mathbf{b}_i} + \sum_{\mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\}} c'_{i,\mathbf{c}}\mathbf{y}^{\mathbf{c}} t^{e_{\mathbf{c}}^\gamma}, \quad i = 1, \ldots, n,$$

where $e_{\mathbf{c}}^\gamma$, the powers of t, are given by

$$e_{\mathbf{c}}^\gamma := \langle \mathbf{c}, \gamma\rangle - \langle \mathbf{a}_i, \gamma\rangle + \nu_i(\mathbf{c}) - \nu_i(\mathbf{a}_i), \quad \forall\ \mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\},$$

they are always positive by (8). By (9), γ may be removed from $e_{\mathbf{c}}^\gamma$. We denote the resulting expressions by $e_{\mathbf{c}}$. More explicitly,

$$e_{\mathbf{c}} := (\mathbf{c} - \mathbf{a}_i)A^{-1}\begin{bmatrix} \nu_1(\mathbf{b}_1) - \nu_1(\mathbf{a}_1) \\ \vdots \\ \nu_n(\mathbf{b}_n) - \nu_n(\mathbf{a}_n) \end{bmatrix} + \nu_i(\mathbf{c}) - \nu_i(\mathbf{a}_i). \tag{10}$$

To avoid the powers $e_{\mathbf{c}}$ being too large or too small, it is natural to consider the following minimization problem:

$$\begin{aligned} \text{minimize:}\quad & M \\ \text{subject to:}\quad & 1 \leq e_{\mathbf{c}} \leq M, \\ & \forall\ C = (\{\mathbf{a}_1, \mathbf{b}_1\}, \ldots, \{\mathbf{a}_n, \mathbf{b}_n\}) \in \mathcal{M}_\omega,\ \forall\ \mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\}, \quad i = 1, \ldots, n, \\ & \text{with } \ e_{\mathbf{c}} \text{ defined in (10).} \end{aligned} \tag{11}$$

Apparently, the lifting function $\nu = (\nu_1, \ldots, \nu_n)$ with values obtained by the solution of this minimization problem satisfies (7) and (8) with γ defined in (9). Therefore, the mixed-cell configuration $\mathcal{M}_\nu$ induced by ν coincides with $\mathcal{M}_\omega$. Moreover, the powers of the continuation parameter t in the polyhedral homotopies induced by ν will be much better balanced.

4 Reducing the Size of the Sandwich Model

The minimization problem (11) has $1 + \sum_{i=1}^n \#\mathcal{A}_i$ unknowns, namely, M as well as $\nu_i(\mathbf{c})$ for $\mathbf{c} \in \mathcal{A}_i$, $i = 1, \ldots, n$, and $2(\#\mathcal{M}_\omega)\sum_{i=1}^n(\#\mathcal{A}_i - 2)$ inequalities. For practical considerations, we wish to reduce both the number of unknowns and the number of inequalities in (11). In the following, we will show that for a fixed mixed cell $\bar{C} = (\{\bar{\mathbf{a}}_1, \bar{\mathbf{b}}_1\}, \ldots, \{\bar{\mathbf{a}}_n, \bar{\mathbf{b}}_n\})$, where $\{\bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i\} \subset \mathcal{A}_i$, $i = 1, \ldots, n$, in the mixed-cell configuration $\mathcal{M}_\omega$ with inner normal $(\beta, 1)$, we may set $\nu_i(\bar{\mathbf{a}}_i)$ and $\nu_i(\bar{\mathbf{b}}_i)$ to be zero for $i = 1, \ldots, n$ in (11), so the number of unknowns is reduced by $2n$. But the solution of the new minimization problem defines a lifting function $\nu' = (\nu'_1, \ldots, \nu'_n)$ which induces the same mixed-cell configuration as $\mathcal{M}_\omega$.

For the fixed mixed cell $\bar{C} = (\{\bar{\mathbf{a}}_1, \bar{\mathbf{b}}_1\}, \ldots, \{\bar{\mathbf{a}}_n, \bar{\mathbf{b}}_n\}) \in \mathcal{M}_\omega$ with inner normal $(\beta, 1)$, we define a lifting function $\omega' = (\omega'_1, \ldots, \omega'_n)$, where $\omega'_i : \mathcal{A}_i \to \mathbb{R}$, $i = 1, \ldots, n$, as follows: for $i = 1, \ldots, n$ and $\mathbf{c} \in \mathcal{A}_i$,

$$\omega'_i(\mathbf{c}) := \langle \mathbf{c}, \beta \rangle - \langle \bar{\mathbf{a}}_i, \beta \rangle + \omega_i(\mathbf{c}) - \omega_i(\bar{\mathbf{a}}_i). \tag{12}$$

Then ω'_i vanishes at both $\bar{\mathbf{a}}_i$ and $\bar{\mathbf{b}}_i$, $i = 1, \ldots, n$. Let $\mathcal{M}_{\omega'}$ be the mixed-cell configuration in the subdivision $S_{\omega'}$ of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ induced by ω'.

Lemma 1 *$\mathcal{M}_{\omega'} = \mathcal{M}_\omega$. More precisely, $C = (C_1, \ldots, C_n) \in \mathcal{M}_\omega$ with inner normal $(\alpha, 1)$ with respect to ω if and only if $C = (C_1, \ldots, C_n) \in \mathcal{M}_{\omega'}$ with inner normal $(\alpha - \beta, 1)$ with respect to ω'.*

PROOF: Let $C_i = \{\mathbf{a}_i, \mathbf{b}_i\} \subset \mathcal{A}_i$ for $i = 1, \ldots, n$. Then, $C \in \mathcal{M}_\omega$ with inner normal $(\alpha, 1)$ $\Longleftrightarrow$

$$\left\{ \begin{array}{l} \langle(\mathbf{b}_i, \omega_i(\mathbf{b}_i)), (\alpha, 1)\rangle = \langle(\mathbf{a}_i, \omega_i(\mathbf{a}_i)), (\alpha, 1)\rangle, \\ \langle(\mathbf{c}, \omega_i(\mathbf{c})), (\alpha, 1)\rangle > \langle(\mathbf{a}_i, \omega_i(\mathbf{a}_i)), (\alpha, 1)\rangle, \quad \forall\ \mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\}, \end{array} \right. \qquad i = 1, \ldots, n.$$

Or,

$$\left\{ \begin{array}{l} \langle \mathbf{a}_i - \mathbf{b}_i, \alpha\rangle = \omega_i(\mathbf{b}_i) - \omega_i(\mathbf{a}_i), \\ \langle \mathbf{c} - \mathbf{a}_i, \alpha\rangle > \omega_i(\mathbf{a}_i) - \omega_i(\mathbf{c}), \quad \forall\ \mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\}, \end{array} \right. \qquad i = 1, \ldots, n.$$

On the other hand, $C \in \mathcal{M}_{\omega'}$ with inner normal $(\alpha - \beta, 1)$ $\Longleftrightarrow$

$$\left\{ \begin{array}{l} \langle(\mathbf{b}_i, \omega_i'(\mathbf{b}_i)), (\alpha - \beta, 1)\rangle = \langle(\mathbf{a}_i, \omega_i'(\mathbf{a}_i)), (\alpha - \beta, 1)\rangle \\ \langle(\mathbf{c}, \omega_i'(\mathbf{c})), (\alpha - \beta, 1)\rangle > \langle(\mathbf{a}_i, \omega_i'(\mathbf{a}_i)), (\alpha - \beta, 1)\rangle, \quad \forall\ \mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\}, \end{array} \right. \qquad i = 1, \ldots, n.$$

Or, by (12),

$$\begin{array}{rcl} \langle \mathbf{a}_i - \mathbf{b}_i, \alpha - \beta\rangle & = & \omega_i'(\mathbf{b}_i) - \omega_i'(\mathbf{a}_i) \\ & = & \langle \mathbf{b}_i, \beta\rangle - \langle \bar{\mathbf{a}}_i, \beta\rangle + \omega_i(\mathbf{b}_i) - \omega_i(\bar{\mathbf{a}}_i) \\ & & \qquad -(\langle \mathbf{a}_i, \beta\rangle - \langle \bar{\mathbf{a}}_i, \beta\rangle + \omega_i(\mathbf{a}_i) - \omega_i(\bar{\mathbf{a}}_i)) \\ & = & \langle \mathbf{a}_i - \mathbf{b}_i, -\beta\rangle + \omega_i(\mathbf{b}_i) - \omega_i(\mathbf{a}_i), \end{array}$$

i.e.,

$$\langle \mathbf{a}_i - \mathbf{b}_i, \alpha\rangle = \omega_i(\mathbf{b}_i) - \omega_i(\mathbf{a}_i), \quad i = 1, \ldots, n$$

and, for $\mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\}$,

$$\begin{array}{rcl} \langle \mathbf{c} - \mathbf{a}_i, \alpha - \beta\rangle & > & \omega_i'(\mathbf{a}_i) - \omega_i'(\mathbf{c}) \\ & = & \langle \mathbf{a}_i, \beta\rangle - \langle \bar{\mathbf{a}}_i, \beta\rangle + \omega_i(\mathbf{a}_i) - \omega_i(\bar{\mathbf{a}}_i) \\ & & \qquad -(\langle \mathbf{c}, \beta\rangle - \langle \bar{\mathbf{a}}_i, \beta\rangle + \omega_i(\mathbf{c}) - \omega_i(\bar{\mathbf{a}}_i)) \\ & = & \langle \mathbf{c} - \mathbf{a}_i, -\beta\rangle + \omega_i(\mathbf{a}_i) - \omega_i(\mathbf{c}), \end{array}$$

i.e.,

$$\langle \mathbf{c} - \mathbf{a}_i, \alpha\rangle > \omega_i(\mathbf{a}_i) - \omega_i(\mathbf{c}), \quad i = 1, \ldots, n.$$

So, the proof of the assertion is achieved. $\square$

Most importantly, a straightforward calculation shows that the polyhedral homotopy, as in (6), induced by the cell $C = (C_1, \ldots, C_n)$ in $\mathcal{M}_\omega$ with inner normal $(\alpha, 1)$ is exactly the same as the one induced by cell $C = (C_1, \ldots, C_n)$ in $\mathcal{M}_{\omega'}$ with inner normal $(\alpha - \beta, 1)$. So, we may solve $G(\mathbf{x}) = \mathbf{0}$ by using the polyhedral homotopies produced by mixed cells in $\mathcal{M}_{\omega'}$ together with corresponding inner normals.

Now, with the lifting ω', we consider the minimization problem

$$\begin{array}{rl} \text{minimize:} & M \\ \text{subject to:} & 1 \leq \langle \mathbf{c} - \mathbf{a}_i, \gamma \rangle + \nu_i(\mathbf{c}) - \nu_i(\mathbf{a}) \leq M, \\ & \forall\ C = (\{\mathbf{a}_1, \mathbf{b}_1\}, \ldots, \{\mathbf{a}_n, \mathbf{b}_n\}) \in \mathcal{M}_{\omega'},\ \forall\ \mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\}, \\ & \nu_i(\bar{\mathbf{a}}_i) = \nu_i(\bar{\mathbf{b}}_i) = 0, \quad i = 1, \ldots, n. \end{array} \tag{13}$$

Here, γ can be expressed, as in (9), as a linear combination of the values of ν_i's:

$$\gamma^T = \begin{bmatrix} \mathbf{a}_1 - \mathbf{b}_1 \\ \vdots \\ \mathbf{a}_n - \mathbf{b}_n \end{bmatrix}^{-1} \begin{bmatrix} \nu_1(\mathbf{b}_1) - \nu_1(\mathbf{a}_1) \\ \vdots \\ \nu_n(\mathbf{b}_n) - \nu_n(\mathbf{a}_n) \end{bmatrix}.$$

This problem has $2n$ fewer unknowns than the original problem in (11), and its solution yields a lifting function $\nu' = (\nu'_1, \ldots, \nu'_n)$. As before, the mixed-cell configuration $\mathcal{M}_{\nu'}$ induced by the lifting ν' is the same as $\mathcal{M}_{\omega'}$.

The following proposition shows the feasibility of this problem.

Proposition 1 *The linear programming problem (13) has an optimal solution.*

PROOF: Apparently, the values of the lifting function $\omega' = (\omega'_1, \ldots, \omega'_n)$ satisfy

$$0 < \langle \mathbf{c} - \mathbf{a}_i, \alpha \rangle + \omega'_i(\mathbf{c}) - \omega'_i(\mathbf{a}_i), \quad i = 1, \ldots, n, \tag{14}$$

for all $C = (\{\mathbf{a}_1, \mathbf{b}_1\}, \ldots, \{\mathbf{a}_n, \mathbf{b}_n\}) \in \mathcal{M}_{\omega'}$ with inner normal $(\alpha, 1)$ and $\mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\}$, where

$$\alpha^T = \begin{bmatrix} \mathbf{a}_1 - \mathbf{b}_1 \\ \vdots \\ \mathbf{a}_n - \mathbf{b}_n \end{bmatrix}^{-1} \begin{bmatrix} \omega'_1(\mathbf{b}_1) - \omega'_1(\mathbf{a}_1) \\ \vdots \\ \omega'_n(\mathbf{b}_n) - \omega'_n(\mathbf{a}_n) \end{bmatrix}.$$

It follows that the function values of $\nu^{(l)} := l\omega'$ for $l > 0$ also satisfy (14). There exists $l_0 > 0$ such that

$$1 \leq \langle \mathbf{c} - \mathbf{a}_i, l_0\alpha \rangle + \nu_i^{(l_0)}(\mathbf{c}) - \nu_i^{(l_0)}(\mathbf{a}_i), \quad i = 1, \ldots, n, \tag{15}$$

for all $C = (\{\mathbf{a}_1, \mathbf{b}_1\}, \ldots, \{\mathbf{a}_n, \mathbf{b}_n\}) \in \mathcal{M}_{\omega'}$ and $\mathbf{c} \in \mathcal{A}_i \backslash \{\mathbf{a}_i, \mathbf{b}_i\}$. Let M_0 be the maximum of the right-hand side of (15). Then $(\nu^{(l_0)}, M_0)$ is a feasible solution of (13). So the linear programming problem (13) has an optimal solution with the value of M satisfying $1 \le M \le M_0$. $\square$

Remark 1 The pair $(\nu^{(l_0)}, M_0)$ constructed in the proof can serve as a starting point of standard simplex algorithms for solving the linear programming problem in (13).

The number of variables in each inequality in the constraints of (13) is no greater than $2n+2$ which is usually much smaller than the number of variables in (13). This sparsity of the linear programming problem in (13) is exploited in our algorithm by means of efficient storage mechanisms and results in a remarkable speed-up. Some of the inequalities in the constraints of the linear programming problem in (13) are exactly the same, and they can easily be detected by comparisons and deleted when the constraints are being generated.

In the rest of this section, we will show that the constraints in (13) can also be derived from *circuits* [6, 9] of the support $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$. For a mixed cell $C = (\{\mathbf{a}_1, \mathbf{b}_1\}, \ldots, \{\mathbf{a}_n, \mathbf{b}_n\}) \in \mathcal{M}_\omega$ with inner normal $(\alpha, 1)$ and $\mathbf{c} \in \mathcal{A}_j \backslash \{\mathbf{a}_j, \mathbf{b}_j\}$, we have

$$\langle \mathbf{a}_j, \alpha \rangle + \omega_j(\mathbf{a}_j) < \langle \mathbf{c}, \alpha \rangle + \omega_j(\mathbf{c}),$$

or,

$$\langle \mathbf{c} - \mathbf{a}_j, \alpha \rangle + \omega_j(\mathbf{c}) - \omega_j(\mathbf{a}_j) > 0. \tag{16}$$

We will call this inequality the *normal inequality* with respect to C and $\mathbf{c}$. This inequality serves as the building block in our sandwich model in (13).

Note that $(\alpha, 1)$ is also the normal of the n-dimensional subspace H of $\mathbb{R}^{n+1}$ spanned by the vectors

$$(\mathbf{b}_1 - \mathbf{a}_1, \omega_1(\mathbf{b}_1) - \omega_1(\mathbf{a}_1)), \ \ldots, \ (\mathbf{b}_n - \mathbf{a}_n, \omega_n(\mathbf{b}_n) - \omega_n(\mathbf{a}_n)).$$

Let $(\mathbf{c} - \mathbf{a}_j, s)$ be the projection of the vector $(\mathbf{c} - \mathbf{a}_j, \omega_j(\mathbf{c}) - \omega_j(\mathbf{a}_j))$ on the subspace H along the direction of its last coordinate. Then $\langle (\alpha, 1), (\mathbf{c} - \mathbf{a}_j, s) \rangle = 0$, or,

$$\langle \mathbf{c} - \mathbf{a}_j, \alpha \rangle + s = 0$$

and (16) becomes

$$\omega_j(\mathbf{c}) - \omega_j(\mathbf{a}) - s > 0.$$

So, the left-hand side of (16) measures the distance between the points $(\mathbf{c} - \mathbf{a}_j, \omega_j(\mathbf{c}) - \omega_j(\mathbf{a}_j))$ and $(\mathbf{c} - \mathbf{a}_j, s)$ as shown in Figure 1.

On the other hand, the Cayley embedding of $\mathcal{A}_i$ into $\mathbb{R}^{2n-1}$ is

$$\tilde{\mathcal{A}}_i = \{(\mathbf{a}, \mathbf{e}_{i-1}) \mid \mathbf{a} \in \mathcal{A}_i\}, \quad i = 1, \ldots, n,$$

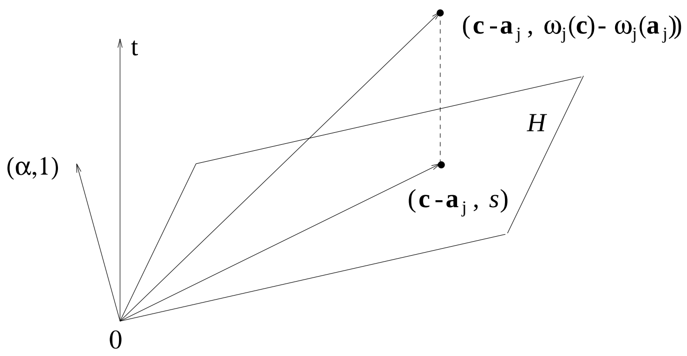

Figure 1

here $\mathbf{e}_0 = \mathbf{0} \in \mathbb{R}^{n-1}$ and $\mathbf{e}_i = (0, \ldots, 1, 0, \ldots, 0)$ is the i-th unit vector in $\mathbb{R}^{n-1}$, $i = 1, \ldots, n-1$. Let $\tilde{\mathcal{A}} = \cup_{i=1}^n \tilde{\mathcal{A}}_i$. Define $\tilde{\omega} : \tilde{\mathcal{A}} \to \mathbb{R}$ by

$$\tilde{\omega}(\mathbf{a}, \mathbf{e}_{i-1}) = \omega_i(\mathbf{a}), \quad \forall\ \mathbf{a} \in \mathcal{A}_i, \quad i = 1, \ldots, n.$$

Then the lifting $\tilde{\omega}$ induces a regular subdivision of $\tilde{\mathcal{A}}$, denoted by $\tilde{S}_{\tilde{\omega}}$.

Proposition 2 (The Cayley Trick [6, 12, 14]) *S_ω is a fine mixed subdivision of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ if and only if $\tilde{S}_{\tilde{\omega}}$ is a triangulation of $\tilde{\mathcal{A}}$. Furthermore, $C = (C_1, \ldots, C_n) \in S_\omega$ if and only if $\bigcup_{i=1}^n \tilde{C}_i \in \tilde{S}_{\tilde{\omega}}$.*

By the Cayley embedding, $\mathbf{a}_1, \mathbf{b}_1, \ldots, \mathbf{a}_n, \mathbf{b}_n$ and $\mathbf{c}$ are embedded in $\mathbb{R}^{2n-1}$ as follows:

$$(\mathbf{a}_1, \mathbf{e}_0),\ (\mathbf{b}_1, \mathbf{e}_0),\ \ldots,\ (\mathbf{a}_n, \mathbf{e}_{n-1}),\ (\mathbf{b}_n, \mathbf{e}_{n-1}),\ (\mathbf{c}, \mathbf{e}_{j-1})$$

and these $2n+1$ points induce a unique circuit in $\mathbb{R}^{2n-1}$, i.e.: a unique minimal affinely dependent subset of these $2n+1$ points. So, there exist $\lambda_1^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_n^{(1)}, \lambda_n^{(2)} \in \mathbb{R}$ such that

$$\left\{ \begin{array}{l} \displaystyle\sum_{l=1}^{2} \sum_{i=1}^{n} \lambda_i^{(l)} = 1, \\ \displaystyle(\mathbf{c}, \mathbf{e}_{j-1}) = \sum_{i=1}^{n} (\lambda_i^{(1)}(\mathbf{a}_i, \mathbf{e}_{i-1}) + \lambda_i^{(2)}(\mathbf{b}_i, \mathbf{e}_{i-1})). \end{array} \right. \tag{17}$$

From (17), as in Theorem 4.1 of [9], we have

$$\left\{ \begin{array}{l} \lambda_j^{(1)} + \lambda_j^{(2)} = 1, \\ \lambda_i^{(1)} + \lambda_i^{(2)} = 0 \quad \text{for } i \neq j. \end{array} \right. \tag{18}$$

It follows that

$$\sum_{i=1}^{n}(\lambda_i^{(1)}(\mathbf{a}_i,\mathbf{e}_{i-1})+\lambda_i^{(2)}(\mathbf{b}_i,\mathbf{e}_{i-1}))=\sum_{i=1}^{n}\lambda_i^{(2)}(\mathbf{b}_i-\mathbf{a}_i,\mathbf{0})+(\mathbf{a}_j,\mathbf{e}_{j-1})$$

and so,

$$\mathbf{c}-\mathbf{a}_j=\sum_{i=1}^{n}\lambda_i^{(2)}(\mathbf{b}_i-\mathbf{a}_i). \tag{19}$$

Now, $\bigcup_{i=1}^{n}\tilde{C}_i\in\tilde{S}_{\tilde{\omega}}$ implies

$$\tilde{\omega}(\mathbf{c},\mathbf{e}_{j-1})>\sum_{i=1}^{n}(\lambda_i^{(1)}\tilde{\omega}(\mathbf{a}_i,\mathbf{e}_{i-1})+\lambda_i^{(2)}\tilde{\omega}(\mathbf{b}_i,\mathbf{e}_{i-1})). \tag{20}$$

By (18) and the definition of $\tilde{\omega}$, (20) can be written as

$$\omega_j(\mathbf{c})-\omega_j(\mathbf{a}_j)>\sum_{i=1}^{n}\lambda_i^{(2)}(\omega_i(\mathbf{b}_i)-\omega_i(\mathbf{a}_i)),$$

or,

$$\omega_j(\mathbf{c})-\omega_j(\mathbf{a}_j)-\sum_{i=1}^{n}\lambda_i^{(2)}(\omega_i(\mathbf{b}_i)-\omega_i(\mathbf{a}_i))>0. \tag{21}$$

We will call this inequality the *circuit inequality* with respect to C and $\mathbf{c}$.

Since $(\mathbf{c}-\mathbf{a}_j,s)$ is in the subspace H and thus can be written as a unique linear combination of $(\mathbf{b}_i-\mathbf{a}_i,\omega_i(\mathbf{b}_i)-\omega_i(\mathbf{a}_i))$, $i=1,\dots,n$, (19) leads to

$$s=\sum_{i=1}^{n}\lambda_i^{(2)}(\omega_i(\mathbf{b}_i)-\omega_i(\mathbf{a}_i)).$$

So, the left-hand side of (21) also represents the distance between the point $(\mathbf{c}-\mathbf{a}_j,\omega_j(\mathbf{c})-\omega_j(\mathbf{a}_j))$ and $(\mathbf{c}-\mathbf{a}_j,s)$. Therefore, the normal inequality (16) is the same as the circuit inequality (21).

The above observation stimulates the potential that the number of inequalities in our minimization problem in (13) can be further reduced by verifying the flip operations that involve mixed cells as defined in [9]. In this case we compute the faces of the cone in the secondary fan, see [6, Chapter 7] for precise definitions. Namely, one may check whether the mixed-cell configuration is such that the flips determined by the circuit can be performed. If this is not the case, then the corresponding circuit is not supported and its inequality can be discarded provided that the optimal solution in (13) is not interfered. The problem of implementing this idea efficiently remains unclear at this stage. Another possible improvement is to combine the balancing strategy and the flattening method [14] when the optimal value M obtained by the balancing method is still too large for practical computations.

5 Numerical Experiments

Our algorithm in balancing the powers of t by solving the linear programming problem (13) has been successfully implemented. In our numerical experiments, the non-vertex points in the supports of the polynomial systems are deleted before calculating the mixed-cell configurations, and the mixed-cell configurations $\mathcal{M}_\omega$ are generated by our implementation of the Lift-Prune algorithm [5] with random real liftings. The numerical experiments are done on a PC with a Pentium 166MHz processor, 16Mb RAM, 256Kb cache and Linux operating system. The numerical results of applying our algorithm to several well-known polynomial systems are listed in Tables 1 and 2.

Polynomial System	n	$\#\mathcal{M}_\omega$	Size of the LP in (11)		Size of the LP in (13)	
			Number of Variables	Number of Constraints	Number of Variables	Number of Constraints
Cohn-2 [11]	4	17	31	748	23	690
Cassou-Noguès [8]	4	3	28	114	20	106
Planar 4-bar [10]	4	4	33	192	25	168
Cyclic-6 [5]	6	25	33	1000	21	692
Cyclic-7 [5]	7	126	45	7560	31	4982
Cyclic-8 [5]	8	297	59	24948	43	16118

Table 1: Sizes of the Linear Programming problems. Here, n is the number of variables of the polynomial system and $\#\mathcal{M}_\omega$ is the number of mixed cells in the mixed-cell configuration $\mathcal{M}_\omega$.

The data in Table 1 are generated by the program with one random lifting function ω for each polynomial system. The fourth and fifth columns give the size of the linear programming problem in (11). The last two columns are the size of the linear programming problem in (13) after all repeated constraints are deleted. For cyclic-n polynomial systems, about 1/3 of the constraints are deleted, which results in a considerable speed-up.

For the data in Table 2, we run the algorithm with ten different real random liftings for each polynomial system. We first scale the powers of t in the polyhedral homotopies before balancing so that the lowest power of t in the homotopies is one, and the average of the highest powers

Polynomial System	Average Highest Power of t		Average CPU time	
	Before Balancing	After Balancing	Finding Mixed Cells	Balancing Method
Cohn-2 [11]	1391	85	0.21	0.19
Cassou-Noguès [8]	251	11	0.05	0.03
Planar 4-bar [10]	429	8	0.17	0.08
Cyclic-6 [5]	425	31	0.46	0.17
Cyclic-7 [5]	3152	139	7.1	1.9
Cyclic-8 [5]	10281	398	81	16.6

Table 2: Height of Powers and CPU Time in Seconds. The averages are obtained from ten different random liftings.

of t in the polyhedral homotopies for the ten random liftings are listed in the second column. The third column lists the average of the highest powers of t in the polyhedral homotopies for the ten liftings obtained from the optimal solutions of the corresponding linear programming problems (13). The fourth column gives the average time elapsed for finding all mixed cells. The last column is the average time elapsed for finding the optimal lifting functions ν', including the constructing and solving of the linear programming problems (13). From these results, we see that the highest powers of t in the polyhedral homotopies are tremendously reduced. The overall reduced powers of t in the polyhedral homotopies greatly limit the chance of running into "valleys" which causes failures of curve-tracings.

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