Enumerating Regular Mixed-Cell Configurations*

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Abstract

By means of the Cayley Trick the problem of enumerating all regular fine mixed subdivisions is reduced to enumerating all regular triangulations. The set of all regular triangulations is well-understood thanks to the bijection with the vertices of the secondary polytope. However, because we are only interested in the configurations of mixed cells in a mixed subdivision, we want to avoid dealing with other cells. We propose an operator derived from the bistellar flip for regular triangulations to modify a mixed-cell configuration.

1 Introduction

Mixed subdivisions were introduced in [10] to extend Viro's theorem [13] for constructing algebraic curves with prescribed topology to complete intersections. They can be used to investigate how many real roots a sparse polynomial system admits asymptotically by counting the contributing mixed cells, as proposed in [5]. Investigating all asymptotic polynomial systems requires the enumeration of all mixed-cell configurations. The algorithm we propose is useful to count the number of real roots of asymptotic polynomial systems ([5]) and to explore the geometric structure of mixed subdivisions (e.g., the connectivity [6], [12]).

In a naive approach one could enumerate all possible combinations of mixed cells and check whether these cells constitute the mixed cells of a regular mixed subdivision. In practice this method is too expensive, even for small examples. A better method is based on a modification operator which guarantees to reach the whole set of regular mixed subdivisions starting at one. Many mixed subdivisions share the same set of mixed cells. We are only interested in the mixed cells of regular mixed subdivisions. Our algorithm for enumerating all regular mixed-cell configurations is based on an operator modifying mixed-cell configurations.

The Cayley Trick [4, Proposition 1.7, page 274] establishes a bijective relation between regular mixed subdivisions and regular triangulations. This reduces the problem of enumerating regular mixed subdivisions to the enumeration of regular triangulations. The regular triangulations are in bijective relation with the vertices of the secondary polytope (see [4, Chapter 7] and [14, Lecture 9]). The edges of the secondary polytope correspond with the local changes of the triangulation. These local changes are called bistellar flips. The enumeration of the

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regular triangulations is done in [2] and [8] by exploring the secondary polytope along the edges. Nonregular triangulations are studied in [3] by means of the universal polytope.

The paper is structured as follows. In sections 2, 3 and 4 we define respectively regular triangulations, fine mixed subdivisions and mixed-cell configurations and present the basic ideas of the enumeration algorithms. We describe in section 5 how bistellar flips operate on mixed cells. How these bistellar flips modify mixed-cell configurations is discussed in section 6. This leads to a method for enumerating all regular mixed-cell configurations without traversing all regular mixed subdivisions, as discussed in section 7.

2 Regular Triangulations and Bistellar Flips

A *point configuration* $A = \{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}\}$ is a finite subset of $\mathbb{R}^d$. A *subdivision* $\mathcal{S}$ of A is a collection of *cells* $I \subset A$, $\dim(\mathrm{aff}(I)) = d$, such that the intersection of the convex hull of two cells is either empty or a proper common face and $\cup_{I \in \mathcal{S}} \mathrm{conv}(I) = \mathrm{conv}(A)$. If all cells in a subdivision have cardinality $d+1$, then we have a *triangulation* $\mathcal{T}$. A subdivision $\mathcal{S}$ of A is *regular* if a *lifting function* $\omega : A \rightarrow \mathbb{R}$ exists such that the facets of the lower hull of $\widehat{A} = \{(\mathbf{a}, \omega(\mathbf{a})) \mid \mathbf{a} \in A\}$ are spanned by lifted cells $\widehat{I}$, $\forall I \in \mathcal{S}$. The values $\omega(\mathbf{a})$ are called the *lifting values* of the subdivision. If ω is generic, then the induced subdivision is a *regular triangulation*.

Circuits (see [14, Lecture 6]) are the appropriate combinatorial tools to exploit relations between triangulations. A *circuit* Z is an affinely dependent finite subset of $\mathbb{R}^d$ with all proper subsets of Z affinely independent. There is, up to a nonzero real multiple, a unique affine relation between the elements of a circuit Z: $\sum_{\mathbf{z} \in Z} \gamma_{\mathbf{z}} \mathbf{z} = 0$, $\sum_{\mathbf{z} \in Z} \gamma_{\mathbf{z}} = 0$, with all $\gamma_{\mathbf{z}}$ different from zero. For our use of circuits we need the following property:

Proposition 2.1 [4, Proposition 1.2, page 217] *Every circuit $Z \subset \mathbb{R}^d$ has exactly two triangulations $\mathcal{T}_+ = \{Z \setminus \{\mathbf{z}\} \mid \gamma_{\mathbf{z}} > 0\}$ and $\mathcal{T}_- = \{Z \setminus \{\mathbf{z}\} \mid \gamma_{\mathbf{z}} < 0\}$. Moreover, they are regular.*

Proof. A simplex of $\mathcal{T}_+$ ($\mathcal{T}_-$) is on the lower hull of $\widehat{A}$ if $\sum_{\mathbf{z} \in Z} \gamma_{\mathbf{z}} \omega(\mathbf{z}) > 0$ (< 0). $\mathcal{T}_+$ and $\mathcal{T}_-$ are the only two triangulations because they contain all d-dimensional simplices and $\forall I_1 \in \mathcal{T}_+, \forall I_2 \in \mathcal{T}_-$: $\mathrm{conv}(I_1)$ and $\mathrm{conv}(I_2)$ overlap. □

Example 2.2 In (1) we represent a circuit Z as a matrix, with the points in its columns.

$$Z = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & -1 \end{bmatrix} \quad \gamma = [-1, +4, -5, +2]^T \quad \text{satisfies} \quad \begin{cases} Z\gamma = \mathbf{0} \\ \sum_{\mathbf{z} \in Z} \gamma_{\mathbf{z}} = 0 \end{cases} \tag{1}$$

In Figure 1, the two triangulations of the circuit are depicted. ■

Proposition 2.1 solves the problem of enumerating regular triangulations for circuits. For larger point configurations, the modification from $\mathcal{T}_+$ into $\mathcal{T}_-$ (and vice versa) is determined by supported circuits.

$$\mathcal{T}_+ = \left\{ Z \setminus \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\},\ Z \setminus \left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\} \right\}$$

$$\mathcal{T}_- = \left\{ Z \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},\ Z \setminus \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \right\}$$

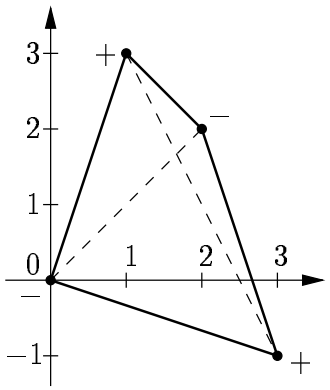

Figure 1: The triangulations of a circuit. The simplices of $\mathcal{T}_+$ (or of $\mathcal{T}_-$) all contain all elements of Z except one of the vertices marked by $+$ (or respectively by $-$).

Definition 2.3 Let $\mathcal{T}$ be a triangulation of a point configuration $A \subset \mathbb{R}^d$ and $Z \subset A$ a circuit. The circuit Z is *supported on* $\mathcal{T}$ if there exist subsets $F_1, F_2, \ldots, F_s$ of A, $(s \geq 1)$ and a triangulation $\mathcal{T}_\pm$ (which is either $\mathcal{T}_+$ or $\mathcal{T}_-$) of Z such that:

(a) $\forall I \in \mathcal{T}_\pm, \forall J \in \mathcal{T} : (I \subset J) \Rightarrow (\exists i \in \{1, 2, \ldots, s\} : J = F_i \cup I)$;

(b) $\forall i \in \{1, 2, \ldots, s\}, \forall I \in \mathcal{T}_\pm : F_i \cup I \in \mathcal{T}$.

Note that Definition 2.3 includes the case where $\dim(\mathrm{aff}(A)) = \dim(\mathrm{aff}(Z))$ for which $\mathcal{T}_\pm \subset \mathcal{T}$.

Because the affine coefficients of the circuit are determined up to a nonzero real multiple, the difference between $\mathcal{T}_+$ and $\mathcal{T}_-$ is rather arbitrary. Therefore, we assume in the sequel that $\mathcal{T}_\pm = \mathcal{T}_+$.

Definition 2.4 Let $\mathcal{T}$ be a triangulation of a point configuration $A \subset \mathbb{R}^d$, $Z \subset A$ a circuit supported on $\mathcal{T}$ and $F_1, F_2, \ldots, F_s$ as in Definition 2.3. The *bistellar flip* over Z of $\mathcal{T}$ gives

$$\mathrm{flip}_Z(\mathcal{T}) = (\mathcal{T} \setminus \{F_i \cup I \mid 1 \leq i \leq s, I \in \mathcal{T}_+\}) \cup \{F_i \cup I \mid 1 \leq i \leq s, I \in \mathcal{T}_-\}\,. \qquad (2)$$

Let $\mathcal{Z}$ be a supported circuit on $\mathcal{T}$. Consider a cell $I \in \mathcal{T}$. We say that Z *involves* I if $I \notin \mathrm{flip}_Z(\mathcal{T})$. Then flip_Z *annihilates* I and *yields* those cells in $\mathrm{flip}_Z(\mathcal{T})$ that are not in $\mathcal{T}$.

Proposition 2.5 *If $\mathcal{T}$ a triangulation of a point configuration $A \subset \mathbb{R}^d$ and $Z \subset A$ a circuit supported on $\mathcal{T}$, then $\mathrm{flip}_Z(\mathcal{T})$ is a triangulation.*

Proof. $\forall \mathbf{x} \in \mathrm{conv}(A)$, $\exists I \in \mathcal{T}$: $\mathbf{x} \in \mathrm{conv}(I)$. Either $I \in \mathrm{flip}_Z(\mathcal{T})$ or $\mathbf{x} \in \mathrm{conv}(F_i \cup I_+)$, with $I_+ \in \mathcal{T}_+$. In the latter case $\mathbf{x}$ is a convex combination of $\mathbf{f} \in \mathrm{conv}(F_i)$ and $\mathbf{z} \in \mathrm{conv}(I_+)$. Because $\mathcal{T}_-$ is a triangulation of Z, $\forall \mathbf{z} \in \mathrm{conv}(Z)$, $\exists I_- \in \mathcal{T}_-$: $\mathbf{z} \in \mathrm{conv}(I_-)$. Whence $\mathbf{x}$ is a convex combination of $\mathbf{f} \in \mathrm{conv}(F_i)$ and $\mathbf{z} \in \mathrm{conv}(I_-)$. In both cases, $\cup_{I \in \mathrm{flip}_Z(T)} \mathrm{conv}(I) = \mathrm{conv}(A)$. Because simplices in $\mathcal{T}_-$ do not overlap, $\mathrm{flip}_Z(\mathcal{T})$ is a triangulation. □

See Figure 2 for some examples.

The constraints on the lifting values that induce a particular triangulation can be formulated as linear inequalities in the lifting values. The solutions of these inequalities determine a polyhedral cone. Since all generic sets of lifting values induce a regular triangulation, these cones form a fan. This fan is polyhedral: it is the normal fan of the *secondary polytope* $\Sigma(A)$.

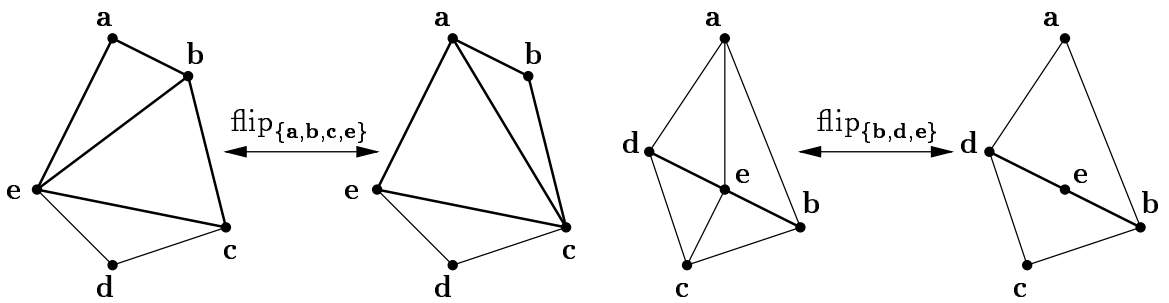

Figure 2: Bistellar flips of triangulations. The affine hull of the circuit that leads to the left bistellar flip is two-dimensional, whereas it is one-dimensional on the right.

Theorem 2.6 *For any point configuration A, there exists a polytope $\Sigma(A)$ such that:*
1. *Vertices of $\Sigma(A)$ are in one-to-one correspondence with the regular triangulations of A.*
2. *The normal cone of lifting values that induce a regular triangulation coincides with the normal cone of the corresponding vertex of $\Sigma(A)$.*
3. *Two vertices t_1 and t_2 with corresponding triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ are joined by an edge if and only if there exists a circuit $Z \in A$ supported on $\mathcal{T}_1$ such that $\mathcal{T}_2 = \text{flip}_Z(\mathcal{T}_1)$.*

The characterization of $\Sigma(A)$ can be found in [4, Chapter 7, Definition 1.6, page 220]. Proofs of the first two items of Theorem 2.6 are contained in [4, Theorem 1.7, page 221], while the third one is proven in [4, Theorem 2.11, page 233]. We refer to [14, Lecture 9] for another introduction to secondary polytopes.

Not all triangulations generated by bistellar flips are regular. Therefore, a regularity check must follow the application of a bistellar flip. According to Theorem 2.6, the graph formed by the regular triangulations as nodes and bistellar flips as links, is connected. This justifies the use of bistellar flips to explore this graph. Based on this exploration, an algorithm for the enumeration of all regular triangulations was implemented in [2]. A reverse-search version was proposed in [8].

3 Regular Mixed Subdivisions and the Cayley Trick

Let $\mathcal{A} = (A_1, A_2, \ldots, A_d)$ be a d-tuple of point configurations. $A = \sum A_i = \{\mathbf{a}^{(1)} + \mathbf{a}^{(2)} + \cdots + \mathbf{a}^{(d)} \mid \mathbf{a}^{(i)} \in A_i\}$ is the *Minkowski sum* of $\mathcal{A}$. A *subdivision* $\mathcal{S}$ of $\mathcal{A}$ is a collection of *cells* $\mathcal{C} = (C_1, C_2, \ldots, C_d)$, with $C_i \subset A_i$ such that the collection of $\sum C_i$, $\forall \mathcal{C} \in \mathcal{S}$, is a subdivision of A. If $\forall \mathcal{C} \in \mathcal{S} : \sum \dim(\text{aff}(C_i)) = d$ and moreover[1]: $\forall \mathcal{C}^{(1)}, \mathcal{C}^{(2)} \in \mathcal{S}, \text{conv}(\sum (C_i^{(1)} \cap C_i^{(2)})) = \text{conv}(\sum C_i^{(1)}) \cap \text{conv}(\sum C_i^{(2)})$, then $\mathcal{S}$ is *mixed*. $\mathcal{S}$ is *regular* if lifting functions $\omega_1, \omega_2, \ldots, \omega_d$ exist for the sets in $\mathcal{A}$ such that the cells of $\mathcal{S}$ are induced by the facets of the lower hull of $\sum \widehat{A}_i$, with $\widehat{A}_i = \{(\mathbf{a}, \omega_i(\mathbf{a})) \mid \mathbf{a} \in A_i\}$. A cell $\mathcal{C}$ is *fine* if $\sum (\#C_i - 1) = d$. A mixed subdivision is *fine* when all its cells are fine. Generic lifting functions induce a fine mixed subdivision. From here on we consider all mixed subdivisions to be regular and fine.

Next we formulate a geometric description (see [12, Proposition 3.9]) of the Cayley Trick as proven in [4, Proposition 1.7, page 274].

[1] Tien-Yien Li pointed out that this property was missing in current definitions of mixed subdivisions.

Definition 3.1 *The* Cayley embedding κ *is defined as*

$$\kappa\left((X_1, X_2, \ldots, X_d)\right) = \bigcup_{i=1}^{d} X_i \times \{\mathbf{e}^{(i-1)}\} \tag{3}$$

with $\mathbf{e}^{(0)} = \mathbf{0} \in \mathbb{R}^{d-1}$ *and* $\mathbf{e}^{(i)}$ *the standard* i*-th unit vector of* $\mathbb{R}^{d-1}$.

The Cayley embedding is a bijection. The inverse of the Cayley embedding is denoted by κ^{-1}. There is a one-to-one correspondence between the regular mixed subdivisions of $\mathcal{A}$ and the regular triangulations of $\kappa(\mathcal{A})$. This is expressed in the following proposition:

Proposition 3.2 (The Cayley Trick) *Consider a tuple* $\mathcal{A} = (A_1, A_2, \ldots, A_d)$ *of point configurations and* ω *a lifting function for* $\kappa(\mathcal{A})$. *A cell* $\mathcal{C} = (C_1, C_2, \ldots, C_d)$ *belongs to the regular mixed subdivision of* $\mathcal{A}$ *induced by the lifting function* $\omega \circ \kappa$ *if and only if the simplex* $\kappa(\mathcal{C})$ *belongs to the regular triangulation of* $\kappa(\mathcal{A})$ *induced by* ω.

The triangulation of $\kappa(\mathcal{A})$ corresponding with a mixed subdivision $\mathcal{S}$ is denoted by $\kappa(\mathcal{S})$.

Example 3.3 We consider a square spanned by $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ and a triangle spanned by $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. In Figure 3 we show a triangulation $\kappa(\mathcal{S})$ of $\kappa(\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}, \{\mathbf{e}, \mathbf{f}, \mathbf{g}\})$ and the corresponding mixed subdivision $\mathcal{S}$. To illustrate Proposition (3.2), formula (4) makes the relation κ explicit. ■

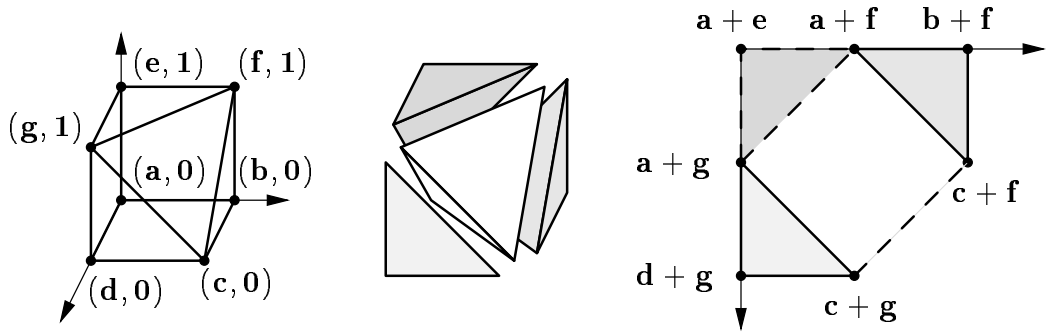

Figure 3: From left to right: the Cayley-embedded point configuration, the triangulation $\kappa(\mathcal{S})$, and the mixed subdivision $\mathcal{S}$.

$$\begin{array}{rcl}
\kappa(\{\mathbf{a}\}, \{\mathbf{e}, \mathbf{f}, \mathbf{g}\}) & = & \{(\mathbf{a}, \mathbf{0}), (\mathbf{e}, \mathbf{1}), (\mathbf{f}, \mathbf{1}), (\mathbf{g}, \mathbf{1})\} \\
\kappa(\{\mathbf{a}, \mathbf{c}, \mathbf{d}\}, \{\mathbf{g}\}) & = & \{(\mathbf{a}, \mathbf{0}), (\mathbf{c}, \mathbf{0}), (\mathbf{d}, \mathbf{0}), (\mathbf{g}, \mathbf{1})\} \\
\kappa(\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}, \{\mathbf{f}\}) & = & \{(\mathbf{a}, \mathbf{0}), (\mathbf{b}, \mathbf{0}), (\mathbf{c}, \mathbf{0}), (\mathbf{f}, \mathbf{1})\} \\
\kappa(\{\mathbf{a}, \mathbf{c}\}, \{\mathbf{f}, \mathbf{g}\}) & = & \{(\mathbf{a}, \mathbf{0}), (\mathbf{c}, \mathbf{0}), (\mathbf{f}, \mathbf{1}), (\mathbf{g}, \mathbf{1})\}
\end{array} \tag{4}$$

As suggested in [5], all regular fine mixed subdivisions can be enumerated by applying the algorithms developed in [2] to $\kappa(\mathcal{A})$.

Because the Cayley Trick establishes a bijection, the notion of a bistellar flip can be formulated for mixed subdivisions. A d-tuple $\mathcal{Z}$ is a *circuit* if $\kappa(\mathcal{Z})$ is a circuit. $\mathcal{Z}$ is *supported* on $\mathcal{S}$ if $\kappa(\mathcal{Z})$ is supported on $\kappa(\mathcal{S})$. $\text{flip}_{\mathcal{Z}}(\mathcal{S}^{(1)}) = \mathcal{S}^{(2)}$ if $\text{flip}_{\kappa(\mathcal{Z})}(\kappa(\mathcal{S}^{(1)})) = \kappa(\mathcal{S}^{(2)})$.

Bistellar flips on mixed subdivisions can be used to explore the connected graph of all mixed subdivisions of a tuple of point configurations. This is demonstrated in Figure 4.

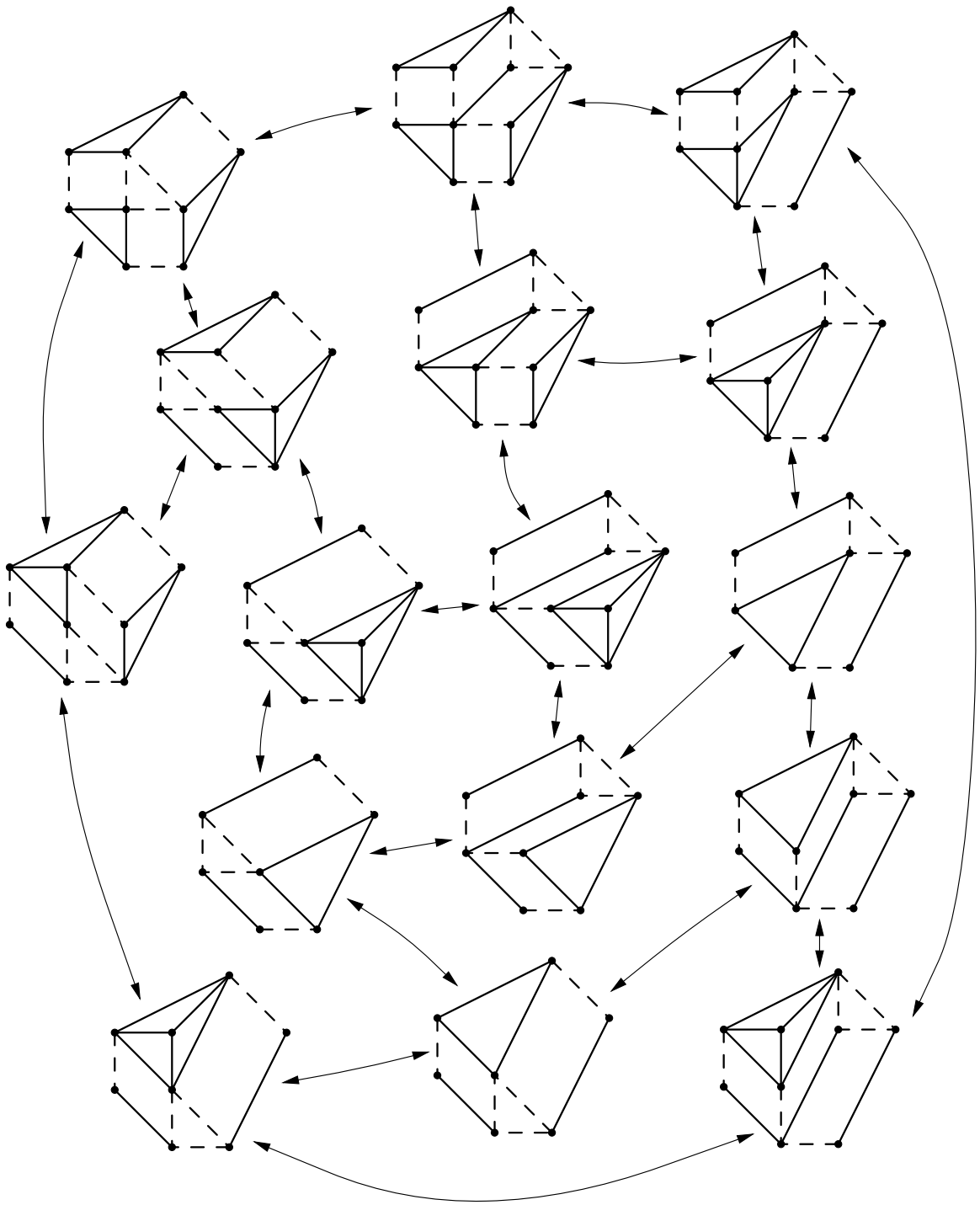

Figure 4: The enumeration of all mixed subdivisions of a triangle (in dashed lines) and a configuration of four points (in solid lines). The arrows represent the bistellar flip operations.

4 Regular Mixed-Cell Configurations

A cell of a fine mixed subdivision is called *mixed* if it can be written as the Minkowski sum of edges. The sum of the volumes of all mixed cells of a mixed subdivision is the *mixed volume* of the tuple $(A_1, A_2, \ldots, A_d)$. Because it is independent of the particular choice of the mixed subdivision, we denote it by $V_d(A_1, A_2, \ldots, A_d)$. The volume of any positive combination of polytopes is a homogeneous polynomial in the factors of this combination. Writing $\mathrm{Vol}_d(A)$ for the volume of the polytope spanned by A, we can state Minkowski's theorem [1] as

$$\mathrm{Vol}_d\left(\sum_{i=1}^{d}\lambda_i A_i\right) = \sum_{i_1=1}^{d}\sum_{i_2=i_1}^{d}\cdots\sum_{i_d=i_{d-1}}^{d} V_d(A_{i_1}, A_{i_2}, \ldots, A_{i_d})\,\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_d}, \quad \lambda_{i_j}\geq 0, \forall j. \quad (5)$$

Mixed subdivisions allow us to calculate *all* the coefficients of the polynomial (5). However, in both real and complex root counting one is only interested in the mixed cells of a mixed subdivision, whose volumes contribute to the mixed volume $V_d(A_1, A_2, \ldots, A_d)$. This is only one coefficient in the polynomial (5). In practice the number of unmixed cells is substantially larger than the number of mixed cells (see [12] for some examples).

To us, two mixed subdivisions are equivalent if they share the same mixed cells. (See [9] for alternative equivalences.) The equivalence classes are called *mixed-cell configurations*. We denote a mixed-cell configuration as a set of mixed cells $\mathcal{M} = \{\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \ldots, \mathcal{C}^{(m)}\}$. Bistellar flips applied to mixed-cell configurations are well-defined, because of the following theorem.

Theorem 4.1 (Main Theorem) *If $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are mixed subdivisions of $\mathcal{A}$, sharing the same mixed cells and $\mathcal{Z}$ is a circuit supported on both, then* $\mathrm{flip}_{\mathcal{Z}}(\mathcal{S}^{(1)})$ *and* $\mathrm{flip}_{\mathcal{Z}}(\mathcal{S}^{(2)})$ *have the same mixed cells.*

We postpone the proof till Section 6.

A circuit $\mathcal{Z}$ is *supported* on a mixed-cell configuration if there is a corresponding mixed subdivision on which $\mathcal{Z}$ is supported. Now we define $\mathrm{flip}_{\mathcal{Z}}(\mathcal{M}^{(1)}) = \mathcal{M}^{(2)}$ if there exist mixed subdivisions $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ where $\mathcal{M}^{(1)} \subset \mathcal{S}^{(1)}$, $\mathcal{M}^{(2)} \subset \mathcal{S}^{(2)}$ and $\mathrm{flip}_{\mathcal{Z}}(\mathcal{S}^{(1)}) = \mathcal{S}^{(2)}$. All mixed-cell configurations can be enumerated by exploring their connected graph by means of bistellar flips. As with triangulations, bistellar flips need to be followed by a check on the regularity. See figure 5 for a nonregular mixed-cell configuration.

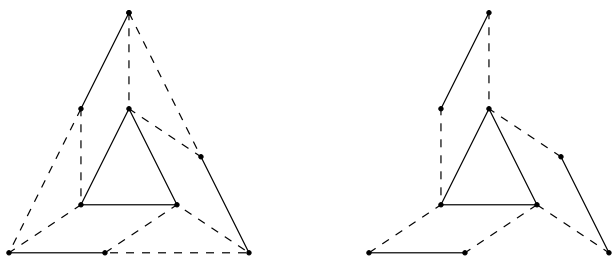

Figure 5: A nonregular fine mixed subdivision with corresponding nonregular mixed-cell configuration. The resemblance with the classical examples of nonregular triangulations in [4, Figure 27, page 219] and [14, page 132] is striking, isn't it?

5 Bistellar Flips and Mixed Cells

In this section we investigate which bistellar flips are of interest. In fact we will characterize the circuits on which these bistellar flips are based.

Theorem 5.1 *Consider a circuit $\mathcal{Z}$ involving a mixed cell $\mathcal{C}$. Then there exists a $k \in \{1,2,\ldots,d\}$, a set $D \subset \{1,2,\ldots,d\} \setminus \{k\}$, and a point $\mathbf{c} \in A_k$ such that:*

$$Z_i = C_i \quad \text{if} \quad i \in D \tag{6}$$

$$Z_i = \emptyset \quad \text{if} \quad i \in \{1,2,\ldots,d\} \setminus (D \cup \{k\}) \tag{7}$$

and

$$Z_k = C_k \cup \{\mathbf{c}\} \quad \text{or} \quad Z_k = \{\mathbf{a},\mathbf{c}\} \text{ where } \mathbf{a} \in C_k. \tag{8}$$

Proof. Because of the definition of a bistellar flip, $\exists k \in \{1,2,\ldots,d\}, \mathbf{c} \in A_k$ such that $\mathcal{Z}$ can be written as:

$$\mathcal{Z} = (B_1, B_2, \ldots, B_{k-1}, B_k \cup \{\mathbf{c}\}, B_{k+1}, \ldots, B_d), \quad B_i \subset C_i. \tag{9}$$

Because $\mathcal{Z}$ is a circuit, there exists a unique (up to a nonzero real multiple) affine relation γ between the points of $\kappa(\mathcal{Z}) = \bigcup Z_i \times \{\mathbf{e}^{(i-1)}\}$:

$$\sum_{i=1}^{d} \sum_{\mathbf{z} \in Z_i \times \{\mathbf{e}^{(i-1)}\}} \gamma_{\mathbf{z}} \mathbf{z} = \mathbf{0}, \quad \sum_{i=1}^{d} \sum_{\mathbf{z} \in Z_i \times \{\mathbf{e}^{(i-1)}\}} \gamma_{\mathbf{z}} = 0, \quad \text{and} \quad \gamma_{\mathbf{z}} \neq 0, \ \forall \mathbf{z} \in Z_i, \forall i. \tag{10}$$

The unit vectors $\mathbf{e}^{(i-1)}$ are affinely independent. Thus coefficients of points contained in one set $A_i \times \{\mathbf{e}^{(i-1)}\}$ add up to zero, whence

$$\sum_{\mathbf{z} \in Z_i \times \{\mathbf{e}^{(i-1)}\}} \gamma_{\mathbf{z}} = 0, \quad \forall i = 1,2,\ldots,d. \tag{11}$$

This implies $\#Z_i \neq 1$. For $i \in \{1,2,\ldots,d\} \setminus \{k\}$, $\#Z_i \in \{0,2\}$ which implies (6) and (7). For $i = k$, $\#Z_i \in \{2,3\}$, whence (8). □

If $\#Z_k = 2$, then the circuit is called *even*. If $\#Z_k = 3$, then the circuit is called *odd*.

Example 5.2 Consider $\mathcal{A} = (\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}, \{\mathbf{e},\mathbf{f},\mathbf{g}\})$ and the mixed subdivision $\mathcal{S}$ as in Figure 6. We examine the following circuits supported on $\mathcal{S}$:

$$\mathcal{Z}^{(1)} = (\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}, \{\}), \quad \mathcal{Z}^{(2)} = (\{\mathbf{a},\mathbf{b}\}, \{\mathbf{e},\mathbf{f},\mathbf{g}\}), \quad \text{and} \quad \mathcal{Z}^{(3)} = (\{\mathbf{b},\mathbf{c}\}, \{\mathbf{e},\mathbf{g}\}). \tag{12}$$

Looking at the corresponding triangulation $\kappa(\mathcal{S})$ we see that all these circuits are supported.

1. The circuit $\mathcal{Z}^{(1)}$ only involves the cells $\mathcal{C}^{(3)}$ and $\mathcal{C}^{(4)}$ which are not mixed.
2. $\mathcal{Z}^{(2)}$ is an odd circuit. $\text{flip}_{\mathcal{Z}^{(2)}}$ annihilates the cells $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$. $\mathcal{C}^{(2)} = (\{\mathbf{a},\mathbf{b}\}, \{\mathbf{e},\mathbf{g}\})$ is a mixed cell. $\mathcal{Z}^{(2)}$ can indeed be written as $(\{\mathbf{a},\mathbf{b}\}, \{\mathbf{e},\mathbf{g}\} \cup \{\mathbf{f}\})$.
3. $\mathcal{Z}^{(3)}$ is an even circuit. $\text{flip}_{\mathcal{Z}^{(3)}}$ annihilates the cells $\mathcal{C}^{(1)}$, $\mathcal{C}^{(2)}$, $\mathcal{C}^{(3)}$, and $\mathcal{C}^{(5)}$. $\mathcal{C}^{(2)} = (\{\mathbf{a},\mathbf{b}\}, \{\mathbf{e},\mathbf{g}\})$ and $\mathcal{C}^{(5)} = (\{\mathbf{b},\mathbf{c}\}, \{\mathbf{f},\mathbf{g}\})$ are mixed cells. $\mathcal{Z}^{(3)}$ can indeed be written as $(\{\mathbf{b}\} \cup \{\mathbf{c}\}, \{\mathbf{e},\mathbf{g}\})$ or as $(\{\mathbf{b},\mathbf{c}\}, \{\mathbf{g}\} \cup \{\mathbf{f}\})$. ■

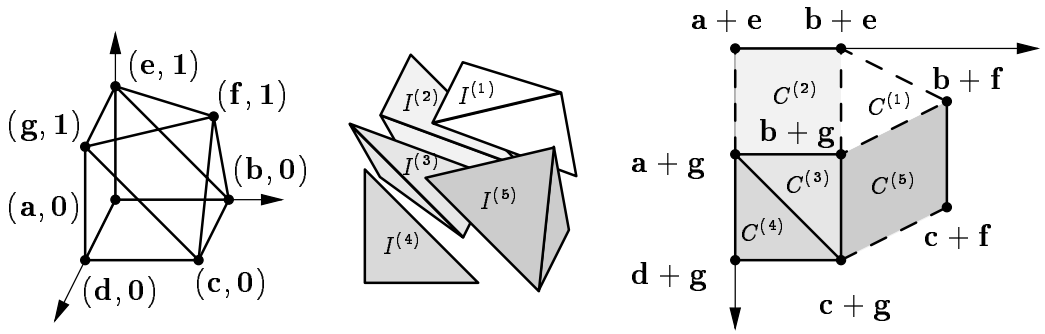

Figure 6: From left to right: the Cayley-embedded point configuration, the triangulation $\kappa(\mathcal{S})$, and the mixed subdivision $\mathcal{S}$.

The affine relationship can be formulated as a linear relation between the vectors spanned by pairs of points of each set A_i. For the odd circuits this gives:

$$\sum_{i \in D \cup \{k\}} \gamma_{(\mathbf{b}^{(i)}, \mathbf{e}^{(i-1)})}(\mathbf{b}^{(i)} - \mathbf{a}^{(i)}) + \gamma_{(\mathbf{c}^{(k)}, \mathbf{e}^{(k-1)})}(\mathbf{c}^{(k)} - \mathbf{a}^{(k)}) = \mathbf{0} \tag{13}$$

where $\forall i \in D : Z_i = \{\mathbf{a}^{(i)}, \mathbf{b}^{(i)}\}$ and $Z_k = \{\mathbf{a}^{(k)}, \mathbf{b}^{(k)}, \mathbf{c}^{(k)}\}$. For the even circuits the linear relation can be written as:

$$\sum_{i \in D} \gamma_{(\mathbf{b}^{(i)}, \mathbf{e}^{(i-1)})}(\mathbf{b}^{(i)} - \mathbf{a}^{(i)}) + \gamma_{(\mathbf{c}^{(k)}, \mathbf{e}^{(k-1)})}(\mathbf{c}^{(k)} - \mathbf{a}^{(k)}) = \mathbf{0} \tag{14}$$

where $Z_k = \{\mathbf{b}^{(k)}, \mathbf{c}^{(k)}\}$.

Example 5.3 The linear relation corresponding to the circuit $\mathcal{Z}^{(2)}$ of Example 5.2 is:

$$(\mathbf{a} - \mathbf{b}) - \frac{1}{2}(\mathbf{e} - \mathbf{g}) + (\mathbf{f} - \mathbf{g}) = \mathbf{0} \quad \text{or} \quad [(\mathbf{a} - \mathbf{b}) \quad (\mathbf{e} - \mathbf{g})] \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} = [\mathbf{f} - \mathbf{g}]\,. \tag{15}$$

The coefficients of $\mathcal{Z}^{(2)}$ are obtained by computing the decomposition of $\mathbf{f} - \mathbf{g} \in A_2$ w.r.t. the vectors spanned by the mixed cell $\mathcal{C}^{(2)} = (\{\mathbf{a}, \mathbf{b}\}, \{\mathbf{e}, \mathbf{g}\})$. This requires the solution of the linear system at the right of (15). The linear relation corresponding to the circuit $\mathcal{Z}^{(3)}$ is

$$(\mathbf{b} - \mathbf{c}) - (\mathbf{e} - \mathbf{g}) = \mathbf{0} \quad \text{or} \quad [(\mathbf{a} - \mathbf{b}) \quad (\mathbf{e} - \mathbf{g})] \begin{bmatrix} 0 \\ -1 \end{bmatrix} = [\mathbf{c} - \mathbf{b}]\,. \tag{16}$$

The matrix notation at the right of (16) expresses the decomposition of $\mathbf{c} - \mathbf{b} \in A_1$ w.r.t. the vectors spanned by the mixed cell $\mathcal{C}^{(2)}$. ■

Because the edges of a mixed cell are linearly independent vectors, they can serve as a basis to denote the other points. In this way we enumerate all possible candidates and compute the affine dependencies. These observations guaranty that Algorithm 5.4, listed below, generates a set of circuits that will at least include those circuits that are supported and involve mixed cells.

Algorithm 5.4 *Compute the set of circuits with corresponding affine coefficients.*

Input: $\mathcal{A} = (A_1, A_2, \ldots, A_d)$; *point configurations*
$\mathcal{M}$. *a mixed-cell configuration*
Output: $\mathfrak{Z} = \{ (\mathcal{Z}, \gamma) \mid \mathcal{Z}$ involves a $\mathcal{C} \in \mathcal{M} \}$. *circuits + coefficients*

for all $\mathcal{C} \in \mathcal{M}$, $C_i = \{\mathbf{a}^{(i)}, \mathbf{b}^{(i)}\}$ **loop** *enumerate cells in* $\mathcal{M}$

$$[L, U] := LUdecompose \begin{pmatrix} (\mathbf{b}^{(1)} - \mathbf{a}^{(1)}) \\ (\mathbf{b}^{(2)} - \mathbf{a}^{(2)}) \\ \vdots \\ (\mathbf{b}^{(d)} - \mathbf{a}^{(d)}) \end{pmatrix}^T ;$$

LU-decomposition of basis $B = LU, B, L, U \in \mathbb{R}^{d \times d}$ *with* L = *lower triangular and* U = *upper triangular*

for all $k \in \{1, 2, \ldots, d\}$ **loop** *enumerate all components*
for all $\mathbf{c} \in (A_k \setminus C_k)$ **loop** *enumerate other points*
$\alpha := U^{-1}L^{-1}(\mathbf{c} - \mathbf{a}^{(k)})^T$; *decomposition of* $\mathbf{c} - \mathbf{a}^{(k)}$
for all $i \in \{1, 2, \ldots, d\} \setminus \{k\}$ **loop** *collect nonempty sets*
if $\alpha_i \neq 0$ **then**
$Z_i := C_i$;
$\gamma_{(\mathbf{a}^{(i)}, \mathbf{e}^{(i-1)})} := \alpha_i$; *affine coefficients*
$\gamma_{(\mathbf{b}^{(i)}, \mathbf{e}^{(i-1)})} := -\alpha_i$;
else
$Z_i := \emptyset$;
end
end loop;
if $\alpha_k = 0$ **then** *check* (13) *and* (14)
$Z_k := \{\mathbf{a}^{(k)}, \mathbf{c}\}$; *even circuit*
$\gamma_{(\mathbf{c}, \mathbf{e}^{(k-1)})} := 1$; *affine coefficients*
$\gamma_{(\mathbf{a}^{(k)}, \mathbf{e}^{(k-1)})} := -1$;
else if $\alpha_k = 1$ **then**
$Z_k := \{\mathbf{b}^{(k)}, \mathbf{c}\}$; *even circuit*
$\gamma_{(\mathbf{c}, \mathbf{e}^{(k-1)})} := 1$; *affine coefficients*
$\gamma_{(\mathbf{b}^{(k)}, \mathbf{e}^{(k-1)})} := -1$;
else
$Z_k := \{\mathbf{a}^{(k)}, \mathbf{b}^{(k)}, \mathbf{c}\}$; *odd circuit*
$\gamma_{(\mathbf{c}, \mathbf{e}^{(k-1)})} := 1$; *affine coefficients*
$\gamma_{(\mathbf{a}^{(k)}, \mathbf{e}^{(k-1)})} := \alpha_k - 1$;
$\gamma_{(\mathbf{b}^{(k)}, \mathbf{e}^{(k-1)})} := -\alpha_k$;
end if;
$\mathfrak{Z} = \mathfrak{Z} \cup \{(\mathcal{Z}, \gamma)\}$; *update circuits + coefficients*
end for;
end for;
end for.

The cost of Algorithm 5.4 is given by the next proposition.

Proposition 5.5 *The computation of all even and odd circuits of $\mathcal{A} = (A_1, A_2, \ldots, A_d)$ w.r.t. the mixed-cell configuration $\mathcal{M}$ requires*

$$\mathcal{O}\left(\#\mathcal{M} \times \left(d^3 + d^2 \sum_{i=1}^{d} (\#A_i - 2)\right)\right) \tag{17}$$

arithmetical operations.

Proof. For every mixed cell, an LU-decomposition is computed, which requires $\mathcal{O}(d^3)$ arithmetical operations. This decomposition can be used to compute the coefficients of the circuit, which takes $\mathcal{O}(d^2)$ operations per circuit. For the i-th component we can have at most $\#A_i - 2$ circuits. At most $\sum_{i=1}^{d} \#A_i - 2$ circuits are computed for every mixed cell. □

6 Bistellar Flips operating on Mixed-Cell Configurations

In this section we describe how bistellar flips operate on mixed-cell configurations. In particular, we will prove Theorem 4.1, which gives foundation for the extension of the definition of bistellar flips to mixed-cell configurations.

Recall the following notations. Given a circuit $\mathcal{Z}$ supported on a mixed subdivision $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)} = \text{flip}_{\mathcal{Z}}(\mathcal{S}^{(1)})$. Let $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ respectively be the mixed cells of $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$. We denote $\mathcal{T}^{(1)} = \kappa(\mathcal{S}^{(1)})$, $\mathcal{T}^{(2)} = \kappa(\mathcal{S}^{(2)})$ and $Z = \kappa(\mathcal{Z})$ with $\mathcal{T}_+$ and $\mathcal{T}_-$ the two regular triangulations of Z. By definition we know that Z is supported on $\mathcal{T}^{(1)}$. As in Definition 2.3 we can associate a tuple of sets $F_1, F_2, \ldots, F_s$ to define the bistellar flip flip_Z.

Lemma 6.1 *Let Z be an odd circuit. Consider $I \in \kappa(\mathcal{M}^{(1)})$. Either $I \in \kappa(\mathcal{M}^{(2)})$ or $I = I_+ \cup F_i$ with $I_+ \in \mathcal{T}_+$. In the latter case, $\exists I_- \in \mathcal{T}_- : I_- \cup F_i \in \kappa(\mathcal{M}^{(2)})$.*

Proof. If $I \in \kappa(\mathcal{M}^{(2)})$, then Z does not involve I. Otherwise, by definition of a bistellar flip I can be written as

$$I = I_+ \cup F_i \quad \text{with} \quad I_+ \in T_+. \tag{18}$$

By Theorem 5.1, the odd circuit Z equals

$$Z = Z_k \times \{\mathbf{e}^{(k-1)}\} \cup \bigcup_{i \in D \setminus \{k\}} Z_i \times \{\mathbf{e}^{(i-1)}\}, \tag{19}$$

with $\#Z_k = 3$ and $\#Z_i = 2$ for $i \in D \setminus \{k\}$. Since $\kappa^{-1}(I)$ is a mixed cell, $\#(I \cap \mathbb{R}^d \times \{\mathbf{e}^{(k-1)}\}) = 2$. This implies $I_+ = Z \setminus \{\mathbf{a}\}$ with $\mathbf{a} \in Z_k \times \{\mathbf{e}^{(k-1)}\}$. Because $\sum_{\mathbf{z} \in Z_k} \gamma_{\mathbf{z}} = 0$, there must be at least one point $\mathbf{b} \in Z_k$ with $\gamma_{\mathbf{b}} < 0$. Take $I_- = Z \setminus \{\mathbf{b}\}$, then $I_- \in \mathcal{T}_-$ and $I_- \cup F_i = I \setminus \{\mathbf{a}\} \cup \{\mathbf{b}\} \in \kappa(\mathcal{M}^{(2)})$. □

Depending on sign of the affine coefficient $\gamma_{\mathbf{c}}$ of the third point $\mathbf{c}$ of Z_k, $Z \setminus \{\mathbf{c}\}$ belongs to $\mathcal{T}_+$ or $\mathcal{T}_-$. In the first case, the bistellar flip annihilates two mixed cells and yields one mixed cell, whereas in the second case, it annihilates one mixed cell and yields two mixed cells, as illustrated in Figure 7.

A similar lemma can be formulated for even circuits.

Lemma 6.2 *Let Z be an even circuit. Consider $I \in \kappa(\mathcal{M}^{(1)})$. Either $I \in \kappa(\mathcal{M}^{(2)})$ or $I = I_+ \cup F_i$ with $I_+ \in \mathcal{T}_+$. In the latter case, $\exists I_- \in \mathcal{T}_- : I_- \cup F_i \in \kappa(\mathcal{M}^{(2)})$.*

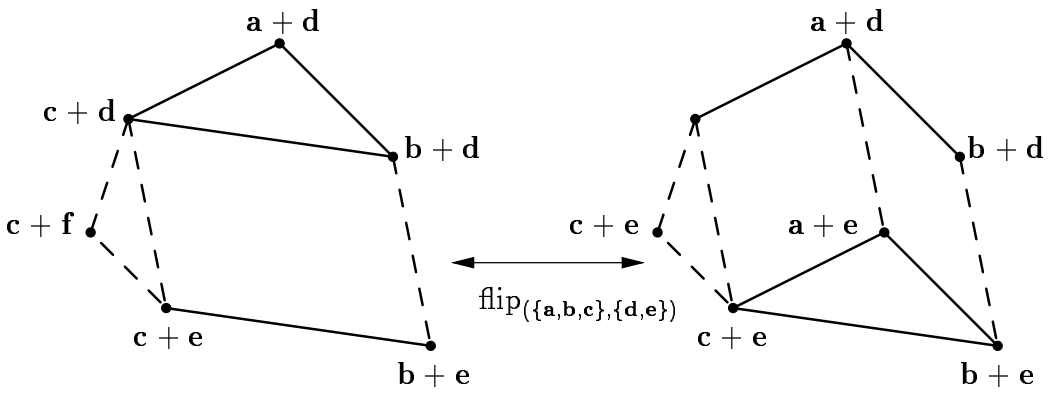

Figure 7: A bistellar flip defined by an odd circuit.

Proof. If $I \in \kappa(\mathcal{M}^{(2)})$, then Z does not involve I. Otherwise, I can be written as

$$I = I_+ \cup F_i \quad \text{with} \quad I_+ \in \mathcal{T}_+. \tag{20}$$

By Theorem 5.1, the even circuit Z equals

$$Z = \bigcup_{i \in D} Z_i \times \{\mathbf{e}^{(i-1)}\}, \tag{21}$$

with $\#Z_i = 2$ for $i \in D$. I can be written as $I = Z \setminus \{\mathbf{a}\}$ with $\gamma_{\mathbf{a}} > 0$ and $\mathbf{a} \in Z_i \times \{\mathbf{e}^{(i-1)}\}$ for some $i \in D$. Let $\mathbf{b}$ be the second point of $Z_i \times \{\mathbf{e}^{(i-1)}\}$. Define $I_- = Z \setminus \{\mathbf{b}\}$. Because $\gamma_{\mathbf{b}} = -\gamma_{\mathbf{a}} < 0$, $I_- \in Z_-$. I_- satisfies $I_- \cup F_i = I_+ \setminus \{\mathbf{a}\} \cup \{\mathbf{b}\} \in \kappa(\mathcal{M}^{(2)})$. □

In Figure 8 we see that for every mixed cell that is annihilated by the flip, exactly one mixed cell is yielded.

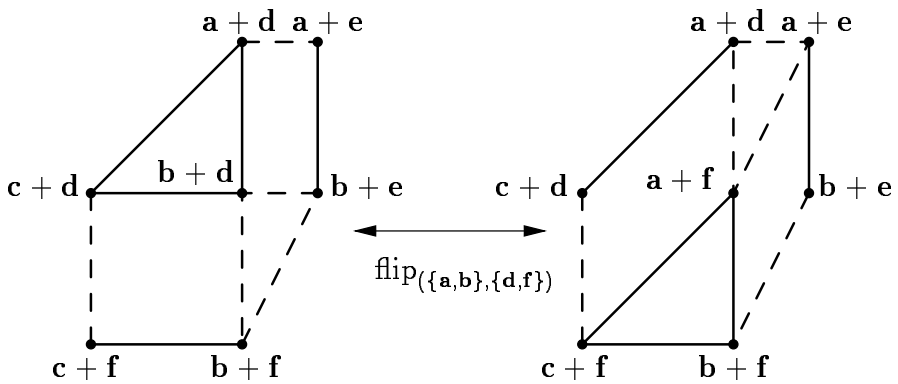

Figure 8: A bistellar flip defined by an even circuit.

Now we can prove the main theorem.

Proof of Theorem 4.1. Given a mixed subdivision $\mathcal{S}^{(1)}$, a circuit $\mathcal{Z}$, $\mathcal{S}^{(2)} = \text{flip}_{\mathcal{Z}}(\mathcal{S}^{(1)})$ and $\mathcal{M}^{(1)} \subset \mathcal{S}^{(1)}, \mathcal{M}^{(2)} \subset \mathcal{S}^{(2)}$ the corresponding mixed-cell configurations. If $\text{flip}_{\mathcal{Z}}$ annihilates

any of the mixed cells, then $\mathcal{Z}$ is an even or an odd circuit. From the lemmas 6.2 and 6.1 we know that the $F_{l^{(1)}}, F_{l^{(2)}}, \ldots, F_{l^{(r)}}$, with $\{l^{(1)}, l^{(2)}, \ldots, l^{(r)}\} \subseteq \{1, 2, \ldots, s\}$ necessary to compute $\mathcal{M}^{(2)}$ can be deduced from $\mathcal{M}^{(1)}$. □

The execution of a bistellar flip on one mixed cell consists of checking the appearance of all but one point of the circuit in the cell. Note that if $\mathcal{Z}$ is not supported, it is impossible to perform the bistellar flip. In particular: if pairs of mixed cells are annihilated, as in Lemma 6.1, then they must be both in the mixed-cell configuration.

Proposition 6.3 $\mathrm{flip}_{\mathcal{Z}}(\mathcal{M}^{(1)})$ *can be performed in* $\mathcal{O}(d \times \#\mathcal{M}^{(1)})$ *arithmetical operations.*

7 Enumerating Regular Mixed-Cell Configurations.

In this section we describe an algorithm for enumerating all mixed-cell configurations. This is done by exploring nodes of the connected graph of mixed-cell configurations by means of bistellar flips. A basic version is sketched in Algorithm 7.1.

Algorithm 7.1 *Enumerate all regular mixed-cell configurations.*

Input: $\mathcal{A} = (A_1, A_2, \ldots, A_d)$; *point configurations*
$\mathcal{M}$. *initial mixed-cell configuration*
Output: $\mathfrak{M}$. *all regular mixed-cell configurations*

$\mathfrak{M}' := \{\mathcal{M}\}$; *yet to explore*
$\mathfrak{M} := \{\}$; *already explored*
while $\exists \mathcal{M} \in \mathfrak{M}'$ **loop** *unexplored mixed cell configuration*
  $\mathfrak{M}' = \mathfrak{M}' \setminus \{\mathcal{M}\}$; $\mathfrak{M} = \mathfrak{M} \cup \{\mathcal{M}\}$; *update $\mathfrak{M}$ and $\mathfrak{M}'$*
  for all $\mathcal{C} \in \mathcal{M}$ **loop** *enumerate cells in $\mathcal{M}$*
    for all $\mathcal{Z}$, $\mathcal{Z}$ *involves* $\mathcal{C}$ **loop** *enumerate circuits*
      $\mathcal{M}' := \mathrm{flip}_{\mathcal{Z}}(\mathcal{M})$; *perform flip*
      if $\mathcal{M}' \notin \mathfrak{M} \cup \mathfrak{M}'$ **and** Is_Regular($\mathcal{M}'$) *regularity check*
       then $\mathfrak{M}' := \mathfrak{M}' \cup \{\mathcal{M}'\}$; *update $\mathfrak{M}$*
      end if;
    end loop;
  end loop;
end while.

Note that Algorithm 5.4 should be applied to compute all involved circuits $\mathcal{Z}$.

The major cost in Algorithm 7.1 is the regularity check. Regularity is determined by the feasibility of a linear system of inequalities. We denote the cost of solving a system of x unknowns and y inequalities by $\mathrm{lp}(x, y)$.

Proposition 7.2 *The computation of all mixed-cell configurations of* $\mathcal{A} = (A_1, A_2, \ldots, A_d)$ *requires at most*

$$\mathcal{O}\left(\#\mathfrak{M} \times (m \times s \times \mathrm{lp}(s, m \times s))\right) \tag{22}$$

arithmetical operations, where $m = \max_{\mathcal{M} \in \mathfrak{M}} \#\mathcal{M}$ *and* $s = \sum_{i=1}^{d} (\#A_i - 2)$.

Proof. Every cell can be involved in at most s circuits. As in Proposition 2.1, with every supported circuit there is a linear inequality on the lifting values that determines whether the induced subdivision is either $\mathcal{T}_+$ or $\mathcal{T}_-$. All inequalities from supported circuits that involve a certain cell constitute a system of at most s linear inequalities in s unknowns.

Given a collection of mixed cells $\mathcal{M}$, the test on the regularity requires the feasibility of at most $m \times s$ linear inequalities on the lifting values, at a cost $\mathrm{lp}(s, m \times s)$. For one mixed-cell configuration $\mathcal{M}$, there are at most m mixed cells and at most s supported circuits yielding at most $m \times s$ regularity checks on the mixed-cell configurations $\mathcal{M}'$. This stage is applied at most $\#\mathfrak{M}$ times. □

The factor $m \times s$ in Proposition 7.2 bounds the number of bistellar flips which can be performed starting at one mixed-cell configuration. This bound is a strong overestimation because not all circuits correspond to valid bistellar flips and because not all mixed-cell configurations are regular.

As pointed out in section 6, bistellar flips can only be performed when all necessary mixed cells are available. Even if all necessary cells are available, the bistellar flip might induce overlapping mixed cells. To avoid the regularity check used to cancel out these spurious configurations, the algorithm could maintain a table of nonoverlapping mixed cells. This table should be consulted before each regularity check.

When a bistellar flip succeeds, the algorithm can lookup the mixed-cell configuration first in the sets of already-explored mixed-cell configurations, before checking the regularity. This can be done explicitly or implicitly by using the polyhedral structure to define a sequential order (reverse search) on the mixed-cell configurations (see [8]). Using this technique for the hypothetical case where no nonregular mixed-cell configurations are reachable, the number of regularity checks can be reduced to $\#\mathfrak{M}$. In this case the regularity check is obsolete.

Figure 9 illustrates the enumeration of all regular mixed-cell configurations using bistellar flips. Compared to the corresponding Figure 4, we see that fewer flipping operations need to be performed.

8 Conclusions

In this paper we have shown how to enumerate the set of all regular mixed-cell configurations of a tuple of point configurations. The proposed algorithm is similar to algorithms to enumerate regular triangulations. It is an exploration of a graph, with the regular mixed-cell configurations as nodes, and modifications, called bistellar flips, as links.

The algorithm can be used to compute upper bounds on the number of real roots of asymptotic polynomial systems. Note however that these bounds are only valid in the asymptotic case, as illustrated by [7]. By application of the algorithm we can obtain an explicit classification of all mixed-cell configurations for a given tuple of point configurations. Hereby we have developed a tool to investigate properties of mixed-cell configurations such as the maximal (minimal) number of mixed cells, and the number of mixed-cell configurations.

Another particular application is the exploitation of symmetry relations in a tuple of point configurations. As an alternative approach to the static methods proposed in [11], the bistellar flip operator developed in this paper could be applied successively on any mixed-cell configuration till a symmetric mixed-cell configuration is obtained.

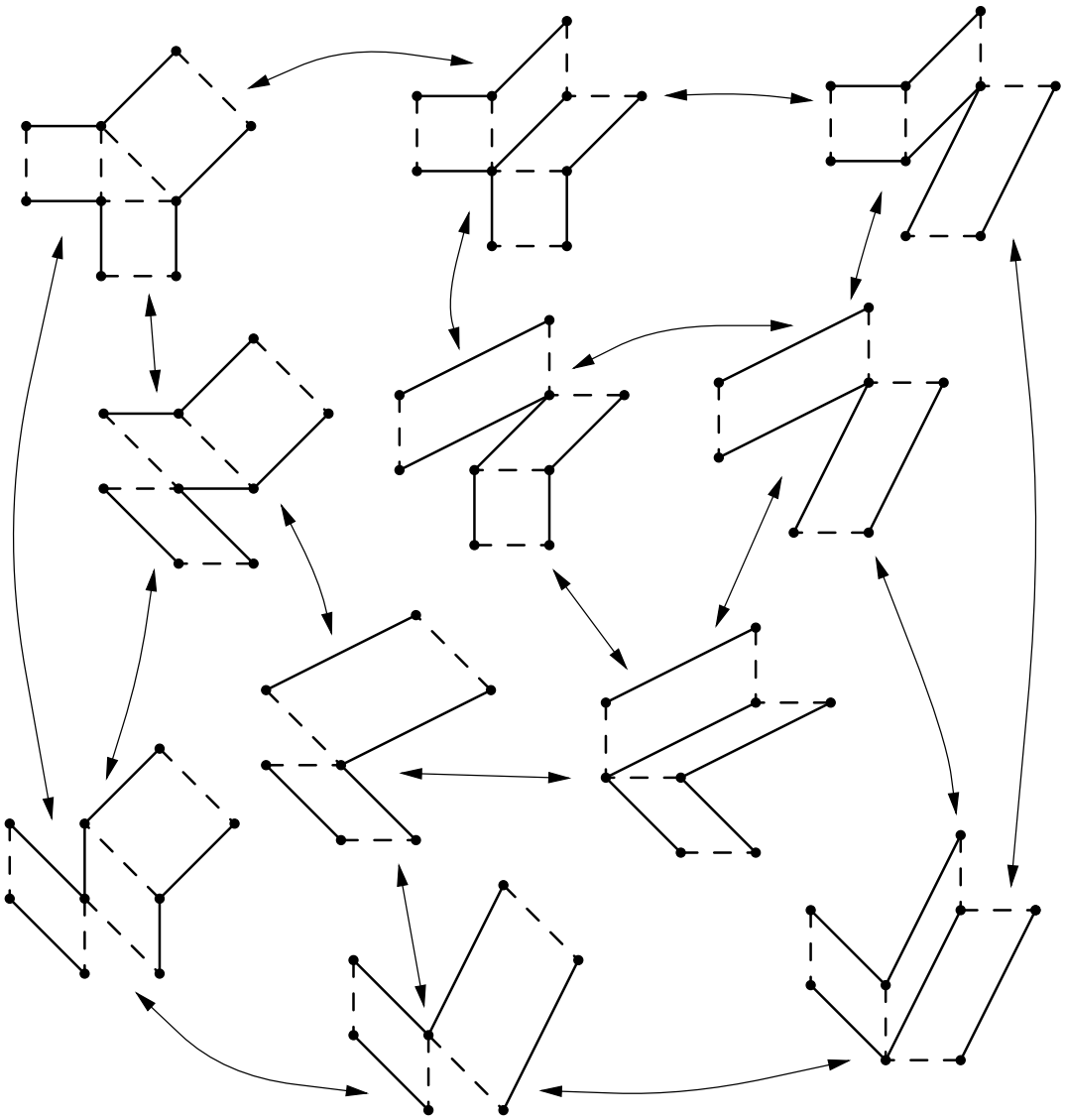

Figure 9: The enumeration of all mixed-cell configurations, corresponding to Figure 4.

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