Enumerating Regular Mixed-Cell Configurations*

Tom Michiels[†] Jan Verschelde[‡]

November 24, 1997

Abstract

By means of the Cayley Trick the problem of enumerating all regular fine mixed subdivisions is reduced to enumerating all regular triangulations. The set of all regular triangulations is well-understood thanks to the bijection with the vertices of the secondary polytope. However, since we are only interested in the configurations of mixed cells in a mixed subdivision, we want to avoid dealing with other cells. We propose an operator derived from the bistellar flip for regular triangulations to modify a mixed-cell configuration.

1 Introduction

Mixed subdivisions were introduced by Sturmfels in [10] and used in [11] to extend Viro's theorem [14] for constructing algebraic curves with prescribed topology to complete intersections. As proposed in [5], by enumeration of all alternating mixed cells, the number of real roots of an asymptotic polynomial system can be bounded. Although the conjectured combinatorial upper bound was proved true in [6] for a non-asymptotic challenge problem, a counter example was given in [8]. Still, mixed-cell configurations are of vital importance in the resolution of sparse polynomial systems, see e.g. [12] and [13]. The algorithm we propose allows an explicit classification of all mixed-cell configurations for a given tuple of point configurations. Hereby we have developed a tool to investigate properties such as the maximal (minimal) number of mixed cells, the number of mixed-cell configurations, and symmetry relations.

The enumeration of all regular triangulations is well-understood thanks to the bijection with the vertices of the secondary polytope (see [4, Chapter 7] and [15, Lecture 9]). The edges of the secondary polytope correspond with small local changes of the triangulations called bistellar flips. As described in [2] and [9], all regular triangulations can be enumerated by application of bistellar flips. Regular mixed subdivisions can be constructed from regular triangulations by means of the Cayley Trick [4, Proposition 1.7, page 274]. Since we are only interested in the mixed cells of regular mixed subdivisions, we only want to perform bistellar

*This research was conducted as part of project G.0261.96 (Counting and computing all isolated solutions of systems of nonlinear equations) funded by the Fund for Scientific Research Flanders (F.W.O.-Vlaanderen) Belgium. The work was initiated while the second author was post-doctor at K.U.Leuven.

[†]Katholieke Universiteit Leuven, Department of Computer Science, Celestijnenlaan 200 A, B-3001 Heverlee, Belgium. E-mail: Tom.Michiels@cs.kuleuven.ac.be. Corresponding author.

[‡]Michigan State University, Department of Mathematics, East Lansing, MI 48824-1027, U.S.A. E-mail: jan@math.msu.edu or jan.verschelde@na-net.ornl.gov.

flips which actually change the mixed cells within a mixed subdivision. Our method refines the bistellar flip so that we can restrict ourselves to regular mixed-cell configurations.

The paper is structured as follows. In sections 2 and 3 we recapitulate the definitions of regular triangulations, fine mixed subdivisions, mixed-cell configurations, and we present the basic ideas of the enumeration algorithms. We describe in section 4 how bistellar flips operate on mixed cells. How these bistellar flips modify mixed-cell configurations is discussed in section 5. This leads to a method for enumerating all regular mixed-cell configurations without traversing all regular mixed subdivisions, as discussed in section 6.

2 Regular Triangulations and Bistellar Flips

A *point configuration* $A = \{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}\}$ is a finite subset of $\mathbb{R}^d$. A *subdivision* $\mathcal{S}$ of A is a collection of *cells* $I \subset A$, $\dim(\text{aff}(I)) = d$, such that the intersection of the convex hull of two cells is either empty or a proper common face and $\cup_{I \in S} \text{conv}(I) = \text{conv}(A)$. If all cells in a subdivision have cardinality $d+1$, then we have a *triangulation* $\mathcal{T}$. A subdivision $\mathcal{S}$ of A is *regular* if a *lifting function* $\omega : A \to \mathbb{R}$ exists such that the facets of the lower convex hull of $\widehat{A} = \{(\mathbf{a}, \omega(\mathbf{a})) \mid \mathbf{a} \in A\}$ are spanned by lifted cells $\widehat{I}$, $\forall I \in \mathcal{S}$. The values $\omega(\mathbf{a})$ are called the *lifting values* of the subdivision. If ω is generic, then the induced subdivision is a *regular triangulation.* All ω that induce the same $\mathcal{T}$ satisfy a system of linear inequalities. In [7] it is shown that only for $n \leq d+3$ we have that every subdivision of A is regular.

A *circuit* Z is an affinely dependent finite subset of $\mathbb{R}^d$ with all proper subsets of Z affinely independent. There is, up to a nonzero real multiple, a unique affine relation between the elements of a circuit Z: $\sum_{\mathbf{z} \in Z} \gamma_{\mathbf{z}} \mathbf{z} = 0$, $\sum_{\mathbf{z} \in Z} \gamma_{\mathbf{z}} = 0$, with all $\gamma_{\mathbf{z}}$ different from zero. Every circuit $Z \subset \mathbb{R}^d$ has exactly two triangulations (see [4, Proposition 1.2, page 217]): $\mathcal{T}_+ = \{Z \setminus \{\mathbf{z}\} \mid \gamma_{\mathbf{z}} > 0\}$ and $\mathcal{T}_- = \{Z \setminus \{\mathbf{z}\} \mid \gamma_{\mathbf{z}} < 0\}$. Moreover, they are regular.

Definition 2.1 Let $\mathcal{T}$ be a triangulation of a point configuration $A \subset \mathbb{R}^d$ and $Z \subset A$ a circuit. The circuit Z is *supported on* $\mathcal{T}$ if there exist subsets $F_1, F_2, \ldots, F_s$ of A, $(s \geq 1)$ and a triangulation $\mathcal{T}_\pm$ (which is either $\mathcal{T}_+$ or $\mathcal{T}_-$) of Z such that:

(a) $\forall I \in \mathcal{T}_\pm, \forall J \in \mathcal{T} : (I \subset J) \Rightarrow (\exists i \in \{1, 2, \ldots, s\} : J = F_i \cup I)$;

(b) $\forall i \in \{1, 2, \ldots, s\}, \forall I \in \mathcal{T}_\pm : F_i \cup I \in \mathcal{T}$.

Note that Definition 2.1 includes the case where $\dim(\text{aff}(A)) = \dim(\text{aff}(Z))$ for which $\mathcal{T}_\pm \subset \mathcal{T}$.

For simplicity we assume that in the sequel $\mathcal{T}_\pm = \mathcal{T}_+$.

Definition 2.2 Let $\mathcal{T}$ be a triangulation of a point configuration $A \subset \mathbb{R}^d$, $Z \subset A$ a circuit supported on $\mathcal{T}$ and $F_1, F_2, \ldots, F_s$ as in Definition 2.1. The *bistellar flip* over Z of $\mathcal{T}$ gives

$$\text{flip}_Z(\mathcal{T}) = (\mathcal{T} \setminus \{F_i \cup I \mid 1 \leq i \leq s, I \in \mathcal{T}_+\}) \cup \{F_i \cup I \mid 1 \leq i \leq s, I \in \mathcal{T}_-\}. \qquad (1)$$

Let $\mathcal{Z}$ be a supported circuit on $\mathcal{T}$. Then $\text{flip}_Z(\mathcal{T})$ is a triangulation, but not necessary a regular triangulation. We say that Z *involves* I, a cell of $\mathcal{T}$ if $I \notin \text{flip}_Z(\mathcal{T})$. Then flip_Z *annihilates* I and *yields* those cells in $\text{flip}_Z(\mathcal{T})$ that are not in $\mathcal{T}$.

The vertices of the secondary polytope $\Sigma(A)$ (see [4, Chapter 7]) are in a one-to-one relation with the regular triangulations of A. The edges of $\Sigma(A)$ correspond with bistellar flips. This justifies the use of bistellar flips to explore this graph. Based on this exploration, an algorithm for the enumeration of all regular triangulations was implemented in [2]. A reverse-search version was proposed in [9]. Fiber polytopes generalize secondary polytopes, see [1] and [15, Lecture 9]. A polytope whose vertices correspond with all (also non-regular) triangulations is defined in [3].

3 The Cayley Trick and Regular Mixed-Cell Configurations

Let $\mathcal{A} = (A_1, A_2, \ldots, A_d)$ with $A_i \subset \mathbb{R}^d$ be a d-tuple of point configurations. $A = \sum A_i = \{\mathbf{a}^{(1)} + \mathbf{a}^{(2)} + \cdots + \mathbf{a}^{(d)} \mid \mathbf{a}^{(i)} \in A_i\}$ is the *Minkowski sum* of $\mathcal{A}$. A *subdivision* $\mathcal{S}$ of $\mathcal{A}$ is a collection of *cells* $\mathcal{C} = (C_1, C_2, \ldots, C_d)$, with $C_i \subset A_i$ such that the collection of $\sum C_i$, $\forall \mathcal{C} \in \mathcal{S}$, is a subdivision of A. If $\forall \mathcal{C} \in \mathcal{S} : \sum \dim(\mathrm{aff}(C_i)) = d$ and moreover[1] $\forall \mathcal{C}^{(1)}, \mathcal{C}^{(2)} \in \mathcal{S}, \mathrm{conv}(\sum(C_i^{(1)} \cap C_i^{(2)})) = \mathrm{conv}(\sum C_i^{(1)}) \cap \mathrm{conv}(\sum C_i^{(2)})$, then $\mathcal{S}$ is *mixed*. $\mathcal{S}$ is *regular* if lifting functions $\omega_1, \omega_2, \ldots, \omega_d$ exist for the sets in $\mathcal{A}$ such that the cells of $\mathcal{S}$ are induced by the facets of the lower convex hull of $\sum \widehat{A}_i$, with $\widehat{A}_i = \{(\mathbf{a}, \omega_i(\mathbf{a})) \mid \mathbf{a} \in A_i\}$. A cell $\mathcal{C}$ is *fine* if $\sum(\#C_i - 1) = d$. A mixed subdivision is *fine* when all its cells are fine. Generic lifting functions induce a fine mixed subdivision. From here on we consider all mixed subdivisions to be regular and fine.

Next is a geometric description of the Cayley Trick, as in [10, Lemma 5.2]. It is formulated algebraically in [4, Proposition 1.7, page 274] and algorithmically in [13, Proposition 3.9]. The Cayley Trick relates a mixed subdivision to a triangulation by embedding all points of the sets A_i in a $(2d-1)$-dimensional space.

Definition 3.1 *The* Cayley embedding κ *is defined as*

$$\kappa : \left(\mathcal{P}(\mathbb{R}^d)\right)^d \to \mathbb{R}^{(2d-1)} : (X_1, X_2, \ldots, X_d) \mapsto \bigcup_{i=1}^{d} X_i \times \{\mathbf{e}^{(i-1)}\} \tag{2}$$

with $\mathbf{e}^{(0)} = \mathbf{0} \in \mathbb{R}^{d-1}$ *and* $\mathbf{e}^{(i)}$ *the standard i-th unit vector of* $\mathbb{R}^{d-1}$.

We denote the inverse of the Cayley embedding by κ^{-1}. There is a one-to-one correspondence between the regular mixed subdivisions of $\mathcal{A}$ and the regular triangulations of $\kappa(\mathcal{A})$. This is expressed in the following proposition:

Proposition 3.2 (The Cayley Trick) *Consider a tuple $\mathcal{A} = (A_1, A_2, \ldots, A_d)$ of point configurations and ω a lifting function for $\kappa(\mathcal{A})$. A cell $\mathcal{C} = (C_1, C_2, \ldots, C_d)$ belongs to the regular mixed subdivision of $\mathcal{A}$ induced by the lifting function $\omega \circ \kappa$ if and only if the simplex $\kappa(\mathcal{C})$ belongs to the regular triangulation of $\kappa(\mathcal{A})$ induced by ω.*

The triangulation of $\kappa(\mathcal{A})$ corresponding with a mixed subdivision $\mathcal{S}$ is denoted by $\kappa(\mathcal{S})$. In Figure 1 we show an example. As suggested in [5], all regular fine mixed subdivisions of $\mathcal{A}$ can be enumerated by applying the algorithms developed in [2] to $\kappa(\mathcal{A})$.

Since the Cayley Trick establishes a bijection, the notion of a bistellar flip can be formulated for mixed subdivisions. A d-tuple $\mathcal{Z}$ is a *mixed circuit* if $\kappa(\mathcal{Z})$ is a circuit. $\mathcal{Z}$ is

[1] Tien-Yien Li pointed out that this property was missing in current definitions of mixed subdivisions. Consider for example for the unit square $A_1 = \{\mathbf{0}, \mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(1)} + \mathbf{e}^{(2)}\}$ and the edge $A_2 = \{\mathbf{0}, \mathbf{e}^{(1)}\}$ the subdivision $\{(A_1, \{\mathbf{0}\}), (A_1, \mathbf{e}^{(1)})\}$.

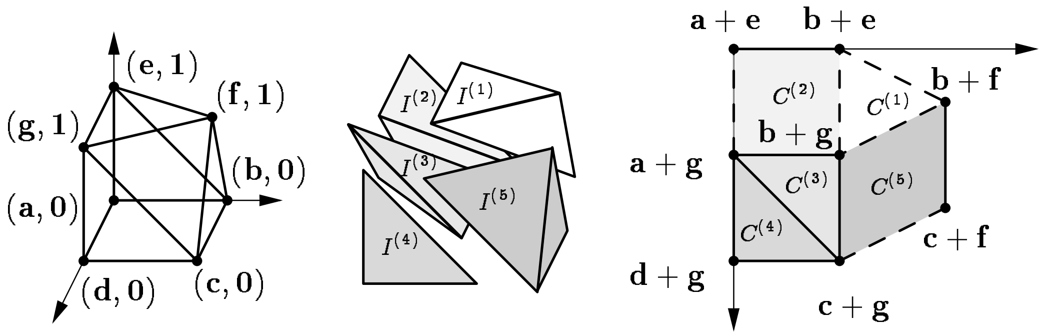

Figure 1: From left to right: the Cayley-embedded point configuration, the triangulation $\kappa(\mathcal{S})$, and the mixed subdivision $\mathcal{S}$.

supported on $\mathcal{S}$ if $\kappa(\mathcal{Z})$ is supported on $\kappa(\mathcal{S})$. $\text{flip}_{\mathcal{Z}}(\mathcal{S}^{(1)}) = \mathcal{S}^{(2)}$ if $\text{flip}_{\kappa(\mathcal{Z})}(\kappa(\mathcal{S}^{(1)})) = \kappa(\mathcal{S}^{(2)})$. Bistellar flips on mixed subdivisions can be used to explore the connected graph of all mixed subdivisions of a tuple of point configurations. This is demonstrated in Figure 2.

A cell of a fine mixed subdivision is called *mixed* if it can be written as the Minkowski sum of edges. To us, two mixed subdivisions are equivalent if they share the same mixed cells. The equivalence classes are called *mixed-cell configurations*. We denote a mixed-cell configuration as a set of mixed cells $\mathcal{M} = \{\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \ldots, \mathcal{C}^{(m)}\}$. Bistellar flips applied to mixed-cell configurations are well-defined, because of the following theorem.

Theorem 3.3 (Main Theorem) *If $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are mixed subdivisions of $\mathcal{A}$, sharing the same mixed cells and $\mathcal{Z}$ is a mixed circuit supported on both, then* $\text{flip}_{\mathcal{Z}}(\mathcal{S}^{(1)})$ *and* $\text{flip}_{\mathcal{Z}}(\mathcal{S}^{(2)})$ *have the same mixed cells.*

We postpone the proof until Section 5.

A mixed circuit $\mathcal{Z}$ is *supported* on a mixed-cell configuration if there is a corresponding mixed subdivision on which $\mathcal{Z}$ is supported. Now we define $\text{flip}_{\mathcal{Z}}(\mathcal{M}^{(1)}) = \mathcal{M}^{(2)}$ if there exist mixed subdivisions $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ where $\mathcal{M}^{(1)} \subset \mathcal{S}^{(1)}$, $\mathcal{M}^{(2)} \subset \mathcal{S}^{(2)}$ and $\text{flip}_{\mathcal{Z}}(\mathcal{S}^{(1)}) = \mathcal{S}^{(2)}$. All mixed-cell configurations can be enumerated by exploring their connected graph by means of bistellar flips. As with triangulations, bistellar flips need to be followed by a check on the regularity.

4 Bistellar Flips and Mixed Cells

In this section we investigate which bistellar flips are of interest. In fact we will characterize the mixed circuits on which these bistellar flips are based.

Theorem 4.1 *Consider a mixed circuit $\mathcal{Z}$ involving a mixed cell $\mathcal{C}$. Then there exists a $k \in \{1, 2, \ldots, d\}$, a set $D \subset \{1, 2, \ldots, d\} \setminus \{k\}$, and a point $\mathbf{c} \in A_k$ such that:*

$$Z_i = C_i \quad \textit{if} \quad i \in D \tag{3}$$

$$Z_i = \emptyset \quad \textit{if} \quad i \in \{1, 2, \ldots, d\} \setminus (D \cup \{k\}) \tag{4}$$

and

$$Z_k = C_k \cup \{\mathbf{c}\} \qquad \textit{or} \qquad Z_k = \{\mathbf{a}, \mathbf{c}\} \textit{ where } \mathbf{a} \in C_k. \tag{5}$$

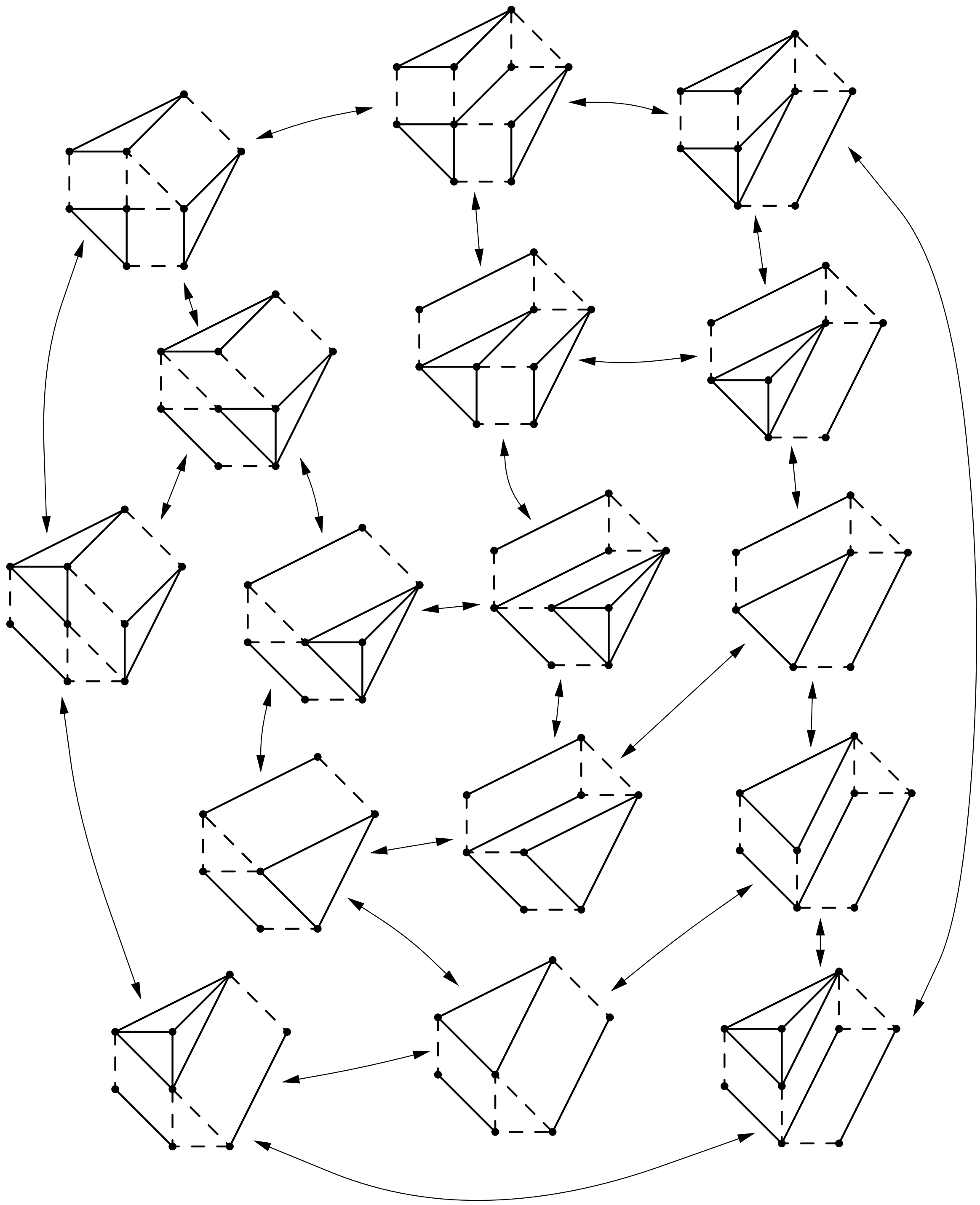

Figure 2: The enumeration of all mixed subdivisions of a triangle (in dashed lines) and a configuration of four points (in solid lines). The arrows represent the bistellar flip operations.

Proof. By definition of a bistellar flip, $\exists k \in \{1,2,\ldots,d\}$, $\mathbf{c} \in A_k$ such that $\mathcal{Z}$ can be written as:

$$\mathcal{Z} = (B_1, B_2, \ldots, B_{k-1}, B_k \cup \{\mathbf{c}\}, B_{k+1}, \ldots, B_d), \quad B_i \subset C_i. \tag{6}$$

Since $\mathcal{Z}$ is a mixed circuit, there exists a unique (up to a nonzero real multiple) affine relation γ between the points of $\kappa(\mathcal{Z}) = \bigcup Z_i \times \{\mathbf{e}^{(i-1)}\}$:

$$\sum_{i=1}^{d} \sum_{\mathbf{z} \in Z_i \times \{\mathbf{e}^{(i-1)}\}} \gamma_{\mathbf{z}} \mathbf{z} = \mathbf{0}, \quad \sum_{i=1}^{d} \sum_{\mathbf{z} \in Z_i \times \{\mathbf{e}^{(i-1)}\}} \gamma_{\mathbf{z}} = 0, \quad \text{and} \quad \gamma_{\mathbf{z}} \neq 0,\ \forall \mathbf{z} \in Z_i, \forall i. \tag{7}$$

The unit vectors $\mathbf{e}^{(i-1)}$ are affinely independent. Thus coefficients of points contained in one set $A_i \times \{\mathbf{e}^{(i-1)}\}$ add up to zero, whence

$$\sum_{\mathbf{z} \in Z_i \times \{\mathbf{e}^{(i-1)}\}} \gamma_{\mathbf{z}} = 0, \quad \forall i = 1,2,\ldots,d. \tag{8}$$

This implies $\#Z_i \neq 1$. For $i \in \{1,2,\ldots,d\} \setminus \{k\}$, $\#Z_i \in \{0,2\}$ which implies (3) and (4). For $i = k$, $\#Z_i \in \{2,3\}$, whence (5). □

If $\#Z_k = 2$, then the mixed circuit is called *even*. If $\#Z_k = 3$, then the mixed circuit is called *odd*.

Example 4.2 Consider $\mathcal{A} = (\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\},\{\mathbf{e},\mathbf{f},\mathbf{g}\})$ and the mixed subdivision $\mathcal{S}$ as in Figure 1. We examine the following mixed circuits supported on $\mathcal{S}$:

$$\mathcal{Z}^{(1)} = (\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\},\{\}), \quad \mathcal{Z}^{(2)} = (\{\mathbf{a},\mathbf{b}\},\{\mathbf{e},\mathbf{f},\mathbf{g}\}), \quad \text{and} \quad \mathcal{Z}^{(3)} = (\{\mathbf{b},\mathbf{c}\},\{\mathbf{e},\mathbf{g}\}). \tag{9}$$

Only $\mathcal{Z}^{(2)}$ and $\mathcal{Z}^{(3)}$ involve a mixed cell and thus they satisfy Theorem 4.1. ■

The affine relationship can be formulated as a linear relation between the vectors spanned by pairs of points of each set A_i. For the odd mixed circuits this gives:

$$\sum_{i \in D \cup \{k\}} \gamma_{(\mathbf{b}^{(i)},\mathbf{e}^{(i-1)})}(\mathbf{b}^{(i)} - \mathbf{a}^{(i)}) + \gamma_{(\mathbf{c}^{(k)},\mathbf{e}^{(k-1)})}(\mathbf{c}^{(k)} - \mathbf{a}^{(k)}) = \mathbf{0} \tag{10}$$

where $\forall i \in D : Z_i = \{\mathbf{a}^{(i)}, \mathbf{b}^{(i)}\}$ and $Z_k = \{\mathbf{a}^{(k)}, \mathbf{b}^{(k)}, \mathbf{c}^{(k)}\}$. For the even mixed circuits the linear relation can be written as:

$$\sum_{i \in D} \gamma_{(\mathbf{b}^{(i)},\mathbf{e}^{(i-1)})}(\mathbf{b}^{(i)} - \mathbf{a}^{(i)}) + \gamma_{(\mathbf{c}^{(k)},\mathbf{e}^{(k-1)})}(\mathbf{c}^{(k)} - \mathbf{a}^{(k)}) = \mathbf{0} \tag{11}$$

where $Z_k = \{\mathbf{b}^{(k)}, \mathbf{c}^{(k)}\}$.

Since the edges of a mixed cell are linearly independent vectors, they can serve as a basis to denote the other points. All candidate circuits can be computed by writing every point in the bases spanned by the edges of every cell. Not all circuits generated in this way are supported. Nor will all the supported mixed circuits lead to a regular mixed cell configuration.

The cost of enumerating all candidate mixed circuits is given by the next proposition.

Proposition 4.3 *The computation of all even and odd mixed circuits of $\mathcal{A} = (A_1, A_2, \ldots, A_d)$ w.r.t. the mixed-cell configuration $\mathcal{M}$ requires*

$$\mathcal{O}\left(\#\mathcal{M} \times \left(d^3 + d^2 \sum_{i=1}^{d} (\#A_i - 2)\right)\right) \tag{12}$$

arithmetical operations.

Proof. For every mixed cell, an LU-decomposition is computed, which requires $\mathcal{O}(d^3)$ arithmetical operations. This decomposition can be used to compute the coefficients of the mixed circuit, which takes $\mathcal{O}(d^2)$ operations per mixed circuit. For the i-th component we can have at most $\#A_i - 2$ mixed circuits. At most $\sum_{i=1}^{d} \#A_i - 2$ mixed circuits are computed for every mixed cell. □

5 Bistellar Flips operating on Mixed-Cell Configurations

In this section we describe how bistellar flips operate on mixed-cell configurations. In particular, we will prove Theorem 3.3, which establishes the extension of the definition of bistellar flips to mixed-cell configurations.

Recall the following notations. Given a mixed circuit $\mathcal{Z}$ supported on a mixed subdivision $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)} = \text{flip}_{\mathcal{Z}}(\mathcal{S}^{(1)})$. Let $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ respectively be the mixed cells of $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$. We denote $\mathcal{T}^{(1)} = \kappa(\mathcal{S}^{(1)})$, $\mathcal{T}^{(2)} = \kappa(\mathcal{S}^{(2)})$ and $Z = \kappa(\mathcal{Z})$ with $\mathcal{T}_+$ and $\mathcal{T}_-$ the two regular triangulations of Z. By definition we know that Z is supported on $\mathcal{T}^{(1)}$, with $F_1, F_2, \ldots, F_s$ as in Definition 2.1.

Lemma 5.1 *Let Z be a circuit. Consider $I \in \kappa(\mathcal{M}^{(1)})$. Either $I \in \kappa(\mathcal{M}^{(2)})$ or $I = I_+ \cup F_i$ with $I_+ \in \mathcal{T}_+$. In the latter case, $\exists I_- \in \mathcal{T}_- : I_- \cup F_i \in \kappa(\mathcal{M}^{(2)})$.*

Proof. If $I \in \kappa(\mathcal{M}^{(2)})$, then Z does not involve I. Otherwise, by definition of a bistellar flip I can be written as

$$I = I_+ \cup F_i \quad \text{with} \quad I_+ = Z \setminus \{(\mathbf{a}, \mathbf{e}^{(k)})\} \in T_+ \quad \text{and} \quad \mathbf{a} \in Z_k. \tag{13}$$

Since Z is a circuit, there is a affine relation γ between its points, where $\gamma_{\mathbf{a}}$ is chosen to be positive. Because the unit vectors are affinely independent we have:

$$\sum_{\mathbf{x} \in Z_k \times \{\mathbf{e}^{(k)}\}} \gamma_{\mathbf{x}} = 0. \tag{14}$$

Since $\gamma_{\mathbf{a}}$ is chosen to be positive, there exists at least one $\mathbf{b} \in Z_k$ with $\gamma_{\mathbf{b}} < 0$. For this $\mathbf{b}$ we have $I_- = Z \setminus \{(\mathbf{b}, \mathbf{e}^{(k)})\} \in T_-$ and $I_- \cup F_i \in \kappa(\mathcal{M}^{(2)})$. □

In case of an odd mixed circuit, $Z \setminus \{\mathbf{c}\}$ belongs to $\mathcal{T}_+$ or $\mathcal{T}_-$, depending on sign of the affine coefficient $\gamma_{\mathbf{c}}$ of the third point $\mathbf{c}$ of Z_k. In the first case, the bistellar flip annihilates two mixed cells and yields one mixed cell, whereas in the second case, it annihilates one mixed cell and yields two mixed cells, as illustrated in Figure 3.

In the case of an even mixed circuit, for every mixed cell that is annihilated by the flip, exactly one mixed cell is yielded, as illustrated in Figure 4.

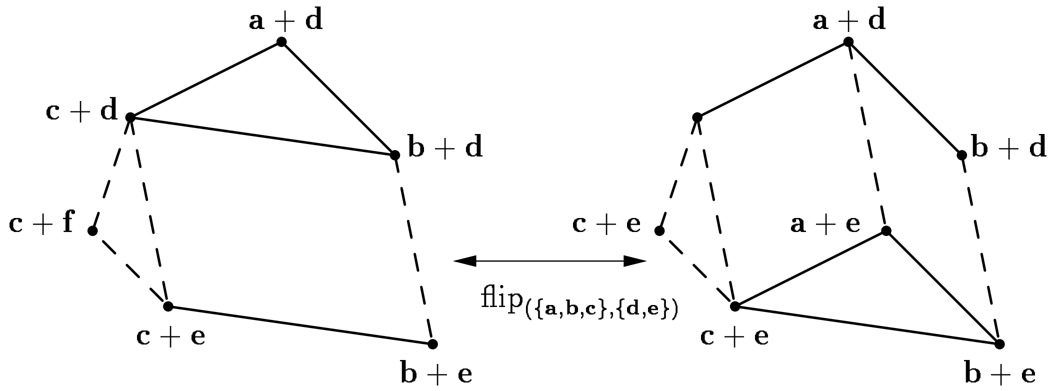

Figure 3: A bistellar flip defined by an odd mixed circuit.

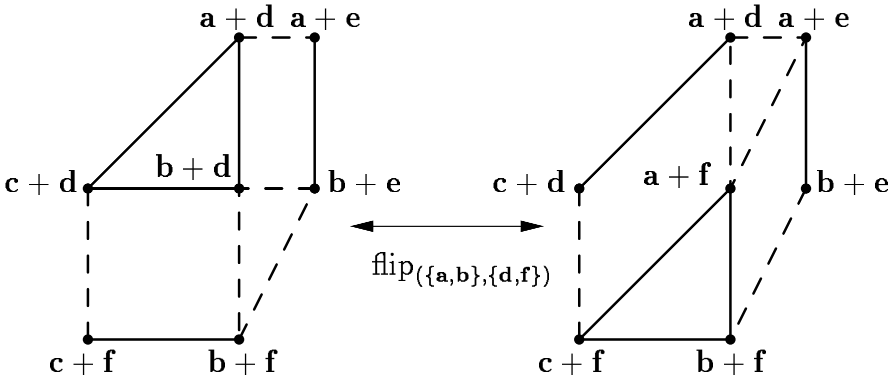

Figure 4: A bistellar flip defined by an even mixed circuit.

Now we can prove the main theorem.

Proof of Theorem 3.3.
Lemma 5.1 implies that the F_i's necessary to compute $\mathcal{M}^{(2)}$ can be deduced from $\mathcal{M}^{(1)}$. □

The deduction of the necessary F_i's can be done by iterating over all mixed cells and comparing them one-by-one with the circuit. The bistellar flip can be performed during this iteration. The whole operation can be done using $\mathcal{O}(d \times \#\mathcal{M}^{(1)})$ arithmetical operations. When not all mixed cells are present to perform the operation for an odd mixed circuits for which mixed cells collapse, the algorithm reports failure. Otherwise, there is still no guarantee that the resulting set of mixed cells is mixed-cell configuration or a regular mixed-cell configuration. Since the operation does not change the total mixed volume of the cells, a regularity check determines whether the mixed cells constitute a regular mixed-cell configuration.

6 Enumerating Regular Mixed-Cell Configurations.

Algorithm 6.1 outlines how the graph of regular mixed-cell configurations is explored.

Algorithm 6.1 *Enumerate all regular mixed-cell configurations.*

Input: $\mathcal{A} = (A_1, A_2, \ldots, A_d)$;	*point configurations*
$\quad\mathcal{M}$.	*initial mixed-cell configuration*
Output: $\mathfrak{M}$.	*all regular mixed-cell configurations*
$\mathfrak{M}' := \{\mathcal{M}\}$;	*yet to explore*
$\mathfrak{M} := \{\}$;	*already explored*
while $\exists \mathcal{M} \in \mathfrak{M}'$ **loop**	*unexplored mixed cell configuration*
$\quad \mathfrak{M}' = \mathfrak{M}' \setminus \{\mathcal{M}\}$; $\mathfrak{M} = \mathfrak{M} \cup \{\mathcal{M}\}$;	*update* $\mathfrak{M}$ *and* $\mathfrak{M}'$
$\quad$ **for all** $\mathcal{C} \in \mathcal{M}$ **loop**	*enumerate cells in* $\mathcal{M}$
$\quad\quad$ **for all** $\mathcal{Z}$, $\mathcal{Z}$ *involves* $\mathcal{C}$ **loop**	*enumerate circuits*
$\quad\quad\quad \mathcal{M}' := \mathrm{flip}_{\mathcal{Z}}(\mathcal{M})$;	*perform flip*
$\quad\quad\quad$ **if** $\mathrm{flip}_{\mathcal{Z}}$ *succeeded*	*successful flip*
$\quad\quad\quad\quad$ **if** $\mathcal{M}' \not\in \mathfrak{M} \cup \mathfrak{M}'$ **and** Is_Regular($\mathcal{M}'$)	*regularity check*
$\quad\quad\quad\quad\quad$ **then** $\mathfrak{M}' := \mathfrak{M}' \cup \{\mathcal{M}'\}$;	*update* $\mathfrak{M}$
$\quad\quad\quad\quad$ **end if**;	
$\quad\quad\quad$ **end if**;	
$\quad\quad$ **end loop**;	
$\quad$ **end loop**;	
end while.	

Figure 5 illustrates the enumeration of all regular mixed-cell configurations.

The major cost in Algorithm 6.1 is the regularity check. Regularity is determined by the feasibility of a linear system of inequalities. We denote the cost of solving a system of x unknowns and y inequalities by $\mathrm{lp}(x, y)$.

Proposition 6.2 *The computation of all mixed-cell configurations of* $\mathcal{A} = (A_1, A_2, \ldots, A_d)$ *requires at most*

$$\mathcal{O}\left(\#\mathfrak{M} \times (m \times s \times \mathrm{lp}(s, m \times s))\right) \tag{15}$$

arithmetical operations, where $m = \max\limits_{\mathcal{M}\in\mathfrak{M}} \#\mathcal{M}$ *and* $s = \sum\limits_{i=1}^{d}(\#A_i - 2)$.

Proof. Every cell can be involved in at most s circuits. s inequalities on the lifting values determine whether a mixed cell is on the lower hull of the lifted point configuration. This gives $m \times s$ inequalities used to check the regularity of a mixed cell configuration.

Given a collection of mixed cells $\mathcal{M}$, the test on the regularity requires the feasibility of at most $m \times s$ linear inequalities on the lifting values, at a cost $\mathrm{lp}(s, m \times s)$. For one mixed-cell configuration $\mathcal{M}$, there are at most m mixed cells and at most s supported circuits yielding at most $m \times s$ regularity checks on the mixed-cell configurations $\mathcal{M}'$. This stage is applied at most $\#\mathfrak{M}$ times. □

The factor $m \times s$ in Proposition 6.2 bounds the number of bistellar flips which can be performed starting at one mixed-cell configuration. This bound is a strong overestimation because not all circuits correspond to valid bistellar flips and not all mixed-cell configurations are regular. To deal with the space complexity, reverse search can be applied on Algorithm 6.1, in a similar way as the enumeration of all regular triangulations as described in [9].

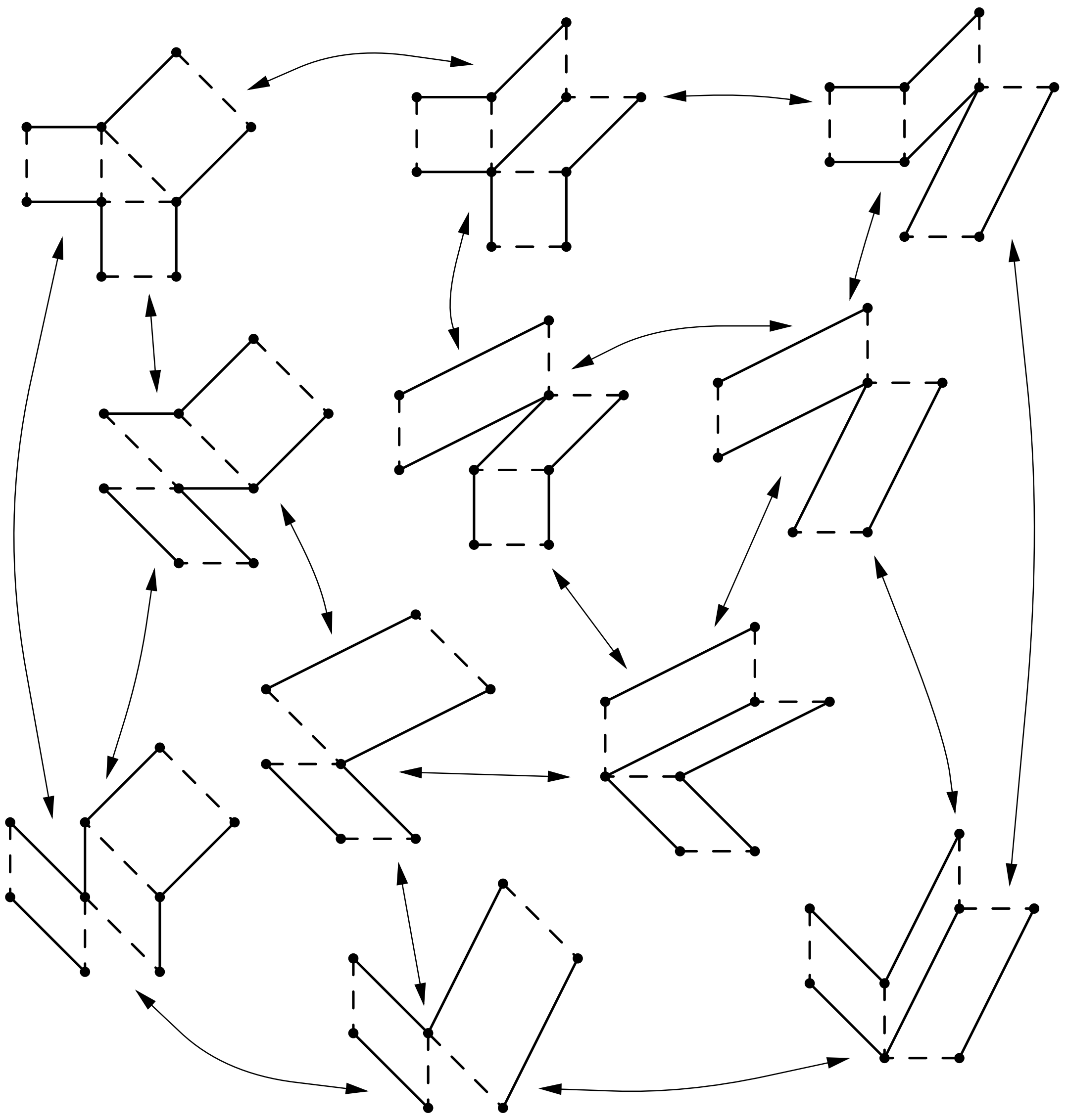

Figure 5: The enumeration of all mixed-cell configurations, corresponding to Figure 2.

References

[1] L.J. Billera and B. Sturmfels. Fiber polytopes. *Ann. of Math.*, 135(3):527–549, 1992.

[2] J.A. de Loera. *Triangulations of Polytopes and Computational Algebra.* PhD thesis, Cornell University, 1995.

[3] J.A. de Loera, S. Hosten, F. Santos, and B. Sturmfels. The polytope of all triangulations of a point configuration. *Documenta Mathematica*, 1:103–119, 1996. Available from `http://www.math.uiuc.edu/documenta/`.

[4] I.M. Gel'fand, M.M. Kapranov, and A.V. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants.* Birkhäuser, Boston, 1994.

[5] I. Itenberg and M.-F. Roy. Multivariate Descartes' rule. *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 37(2):337–346, 1996. Available from `http://cdns.emis.de/journals/BAG/`.

[6] J.C. Lagarias and T.J. Richardson. Multivariate Descartes' rule of signs and Sturmfels's challenge problem. *The Mathematical Intelligencer*, 19(3):9–15, 1997.

[7] C.W. Lee. Regular triangulations of convex polytopes. In P. Gritzmann and B. Sturmfels, editors, *Applied Geometry and Discrete Mathematics - The Victor Klee Festschrift*, volume 4 of *DIMACS Series*, pages 443–456. AMS, Providence, R.I., 1991.

[8] T.Y. Li and X. Wang. On multivariate Descartes' rule – a counterexample. Manuscript, 1997.

[9] T. Masada, H. Imai, and Imai K. Enumeration of regular triangulations. In *Proceedings of the Twelfth Annual Symposium on Computational Geometry*, pages 224–233. ACM, 1996.

[10] B. Sturmfels. On the Newton polytope of the resultant. *Journal of Algebraic Combinatorics*, 3:207–236, 1994.

[11] B. Sturmfels. Viro's theorem for complete intersections. *Annali della Scuola Normale di Pisa*, 21(3):377–386, 1994.

[12] J. Verschelde and K. Gatermann. Symmetric Newton polytopes for solving sparse polynomial systems. *Adv. Appl. Math.*, 16(1):95–127, 1995.

[13] J. Verschelde, K. Gatermann, and R. Cools. Mixed-volume computation by dynamic lifting applied to polynomial system solving. *Discrete Comput. Geom.*, 16(1):69–112, 1996.

[14] O.Y. Viro. Gluing of plane real algebraic curves and constructions of curves of degrees 6 and 7. In L.D. Faddeev and A.A. Mal'cev, editors, *Topology*, volume 1060 of *Lecture Notes in Mathematics*, pages 187–200. Springer–Verlag, 1984.

[15] G.M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics.* Springer–Verlag, New York, 1995.