Polyhedral Methods for Space Curves Exploiting Symmetry Applied to the Cyclic *n*-roots Problem

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Outline

1. Problem Statement
   - exploiting symmetry when solving polynomial systems
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   - computing pretropisms
   - Puiseux series for algebraic sets
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   - exploiting symmetry
   - a tropical version of Backelin's Lemma

exploiting symmetry

The solution sets of many polynomial systems arising in practical applications are invariant under permutations of the variables.

Solutions belong to orbits, so just compute one generator per orbit.

Our problems with exploiting symmetry started about 20 years ago...
joint with K. Gatermann: **Symmetric Newton polytopes for solving sparse polynomial systems.** *Adv. Appl. Math.*, 16(1):95–127, 1995.

Observe that, even if the coefficients of a system could be generic, often the Newton polytopes have a symmetric structure.

Exploiting symmetry with polyhedral methods 20 years ago
was restricted to isolated solutions.

Today: exploiting symmetry in positive dimensional solution sets.

polynomial systems

Consider $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, a system of equations defined by

- N polynomials $\mathbf{f} = (f_0, f_1, \ldots, f_{N-1})$,
- in n variables $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$.

A polynomial in n variables consists of a vector of nonzero complex coefficients with corresponding exponents in A:

$$f_k(\mathbf{x}) = \sum_{\mathbf{a} \in A_k} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C} \setminus \{0\}, \quad \mathbf{x}^{\mathbf{a}} = x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}.$$

Input data:

1. $A = (A_0, A_1, \ldots, A_{N-1})$ are sets of exponents, the *supports*. For $\mathbf{a} \in \mathbb{Z}^n$, we consider *Laurent* polynomials, $f_k \in \mathbb{C}[\mathbf{x}^{\pm 1}]$ $\Rightarrow$ only solutions with coordinates in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ matter.
2. $\mathbf{c}_A = (\mathbf{c}_{A_0}, \mathbf{c}_{A_1}, \ldots, \mathbf{c}_{A_{N-1}})$ are vectors of complex coefficients. Although A is exact, the coefficients may be approximate.

the cyclic 4-roots system

$$\mathbf{f}(\mathbf{x}) = \left\{ \begin{array}{c} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0 x_1 + x_1 x_2 + x_2 x_3 + x_3 x_0 = 0 \\ x_0 x_1 x_2 + x_1 x_2 x_3 + x_2 x_3 x_0 + x_3 x_0 x_1 = 0 \\ x_0 x_1 x_2 x_3 - 1 = 0 \end{array} \right.$$

Cyclic 4-roots $\mathbf{x} = (x_0, x_1, x_2, x_3)$ correspond to complex circulant Hadamard matrices:

$$H = \left[\begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \\ x_3 & x_0 & x_1 & x_2 \\ x_2 & x_3 & x_0 & x_1 \\ x_1 & x_2 & x_3 & x_0 \end{array} \right], \qquad \begin{array}{l} |x_k| = 1, k = 1,2,3,4 \\ \\ H^* H = 4 I_4. \end{array}$$

- Haagerup: for prime p, there are $\left(\begin{array}{c} 2p-2 \\ p-1 \end{array} \right)$ isolated roots.
- Backelin: for $n = \ell m^2$, there are infinitely many cyclic n-roots.

solving polynomial systems

Systems like cyclic n-roots are

- Sparse: relative to the degrees of the polynomials,

  few monomials appear with nonzero coefficients
  $\Rightarrow$ fewer roots than the Bézout bounds.

- Symmetric: solutions are invariant under permutations, $n = 4$:
  $(x_0, x_1, x_2, x_3) \rightarrow (x_1, x_2, x_3, x_0)$ and $(x_0, x_1, x_2, x_3) \rightarrow (x_3, x_2, x_1, x_0)$
  generate the permutation group.
  In addition: $(x_0, x_1, x_2, x_3) \rightarrow (x_0^{-1}, x_1^{-1}, x_2^{-1}, x_3^{-1})$.

- Not pure dimensional, for prime n, all solutions are isolated,
  but for $n = \ell m^2$, we have positive dimensional solution sets.

Our solution is to apply a hybrid symbolic-numeric approach.

Puiseux series

The Newton polygon of $f(x_0, x_1)$ is the convex hull, spanned by the exponents (a_0, a_1) of monomials $x_0^{a_0} x_1^{a_1}$ that occur in f with $c_{(a_0,a_1)} \neq 0$.

Theorem (the theorem of Puiseux)

Let $f(x_0, x_1) \in \mathbb{C}(x_0)[x_1]$: f is a polynomial in the variable x_1 and its coefficients are fractional power series in x_0.

The polynomial f has as many series solutions as the degree of f.

Every series solution has the following form:

$$\left\{ \begin{array}{l} x_0 = t^u \\ x_1 = ct^v(1 + O(t)), \quad c \in \mathbb{C}^* \end{array} \right.$$

where (u, v) is an inner normal to an edge of the lower hull of the Newton polygon of f.

The series are computed with the polyhedral Newton-Puiseux method.

limits of space curves

Assume $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ has a solution curve C,
which intersects $x_0 = 0$ at a regular point.

For $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{Z}^n$, consider $\mathbf{x} = \mathbf{z} t^{\mathbf{v}} (1 + O(t))$:

- $x_0 = z_0 t^{v_0}$, for t close to zero, $z_0 \neq 0$ and
- for $k = 1, 2, \ldots, n-1$: $x_k = z_k t^{v_k}(1 + O(t)), z_k \neq 0$.

Substitute $x_0 = z_0 t^{v_0}, x_k = z_k t^{v_k}(1 + O(t))$ in $f_\ell(\mathbf{x}) = \displaystyle\sum_{\mathbf{a} \in A_\ell} c_\ell \mathbf{x}^{\mathbf{a}}$:

$$
\begin{aligned}
f_\ell(\mathbf{x} = \mathbf{z} t^{\mathbf{v}}(1 + O(t)) &= \sum_{\mathbf{a} \in A_\ell} c_{\mathbf{a}} z_0^{a_0} t^{a_0 v_0} \prod_{k=1}^{n-1} z_k t^{a_k v_k} (1 + O(t)) \\
&= \sum_{\mathbf{a} \in A_\ell} c_{\mathbf{a}} \mathbf{z}^{\mathbf{a}} t^{a_0 v_0 + a_1 v_1 + \cdots + a_{n-1} v_{n-1}} (1 + O(t)).
\end{aligned}
$$

Because $\mathbf{z} \in (\mathbb{C}^*)^n$, there must be at least two terms in f_ℓ left as $t \to 0$.

initial forms and tropisms

Denote the inner product of vectors **u** and **v** as $\langle \mathbf{u}, \mathbf{v} \rangle$.

Definition

Let $\mathbf{v} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ be a direction vector. Consider $f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_\mathbf{a} \mathbf{x}^\mathbf{a}$.

The ***initial form of f in the direction* v** is

$$\mathrm{in}_\mathbf{v}(f) = \sum_{\substack{\mathbf{a} \in A \\ \langle \mathbf{a}, \mathbf{v} \rangle = m}} c_\mathbf{a} \mathbf{x}^\mathbf{a}, \quad \text{where } m = \min\{ \langle \mathbf{a}, \mathbf{v} \rangle \mid \mathbf{a} \in A \}.$$

Definition

Let the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ define a curve. A ***tropism*** consists of the leading powers $(v_0, v_1, \ldots, v_{n-1})$ of a Puiseux series of the curve.

The leading coefficients of the Puiseux series satisfy $\mathrm{in}_\mathbf{v}(\mathbf{f})(\mathbf{x}) = \mathbf{0}$.

curves of cyclic 4-roots

$$\mathbf{f}(\mathbf{x}) = \left\{ \begin{array}{c} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0 \end{array} \right.$$

One tropism $\mathbf{v} = (+1, -1, +1, -1)$ with $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{z}) = \mathbf{0}$:

$$\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x}) = \left\{ \begin{array}{c} x_1 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_1x_2x_3 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0. \end{array} \right.$$

We look for solutions of the form

$$(x_0 = t^{+1}, x_1 = z_1t^{-1}, x_2 = z_2t^{+1}, x_3 = z_3t^{-1}).$$

solving the initial form system

Substitute $(x_0 = t^{+1}, x_1 = z_1 t^{-1}, x_2 = z_2 t^{+1}, x_3 = z_3 t^{-1})$:

$\mathrm{in}_{\mathbf{v}}(\mathbf{f})(x_0 = t^{+1}, x_1 = z_1 t^{-1}, x_2 = z_2 t^{+1}, x_3 = z_3 t^{-1})$

$$= \left\{ \begin{array}{c} (1 + z_2)t^{+1} = 0 \\ z_1 + z_1 z_2 + z_2 z_3 + z_3 = 0 \\ (z_1 z_2 + z_3 z_1)t^{+1} = 0 \\ z_1 z_2 z_3 - 1 = 0. \end{array} \right.$$

We find two solutions: $(z_1 = -1, z_2 = -1, z_3 = +1)$ and $(z_1 = +1, z_2 = -1, z_3 = -1)$.

Two space curves $\left(t, -t^{-1}, -t, t^{-1}\right)$ and $\left(t, t^{-1}, -t, -t^{-1}\right)$ satisfy the entire cyclic 4-roots system.

some references

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overview of our polyhedral methods

- finding pretropisms and solving initial forms

  Initial forms with at least two monomials in every equation define the intersection points of the solution set with the coordinate hyperplanes.

- unimodular coordinate transformations

  Via the Smith normal form we obtain nice representations for solutions at infinity.
  Solutions of binomial systems are monomial maps.

- computing terms of Puiseux series

  Although solutions to any initial forms may be monomial maps, in general we need a second term in the Puiseux series expansion to distinguish between

  - a positive dimensional solution set, and
  - an isolated solution at infinity.

the Cayley embedding – an example

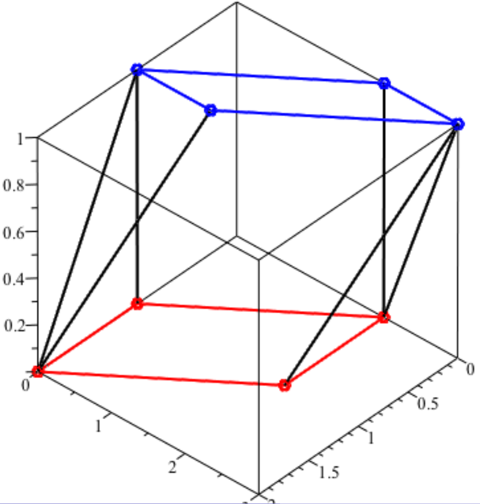
$$\left\{ \begin{array}{l} p = (x_0 - x_1^2)(x_0 + 1) = x_0^2 + x_0 - x_1^2 x_0 - x_1^2 = 0 \\ q = (x_0 - x_1^2)(x_1 + 1) = x_0 x_1 + x_0 - x_1^3 - x_1^2 = 0 \end{array} \right.$$

The Cayley polytope is the convex hull of

$\{(2,0,0),(1,0,0),(1,2,0),(0,2,0)\}$

$\cup$

$\{(1,1,1),(1,0,1),(0,3,1),(0,2,1)\}$.



facet normals and initial forms

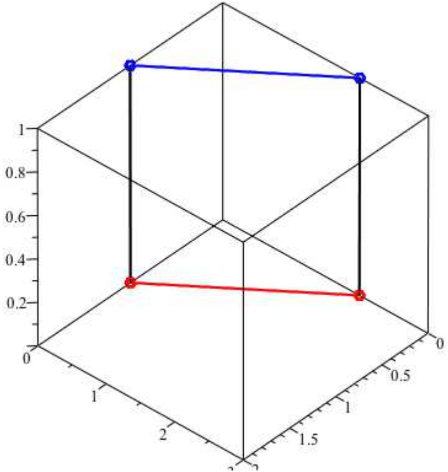
The Cayley polytope has facets spanned by

one edge of the Newton polygon of p

and

one edge of the Newton polygon of q.

Consider $\mathbf{v} = (2, 1, 0)$.



$$\left\{ \begin{array}{l} \mathrm{in}_{(2,1)}(p) = \mathrm{in}_{(2,1)}\left(x_0^2 + x_0 - x_1^2 x_0 - x_1^2\right) = x_0 - x_1^2 \\ \mathrm{in}_{(2,1)}(q) = \mathrm{in}_{(2,1)}\left(x_0 x_1 + x_0 - x_1^3 - x_1^2\right) = x_0 - x_1^2 \end{array} \right.$$

computing all pretropisms

Definition

A nonzero vector $\mathbf{v}$ is a ***pretropism*** for the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ if $\#\mathrm{in}_{\mathbf{v}}(f_k) \geq 2$ for all $k = 0, 1, \ldots, N-1$.

Application of the Cayley embedding to $(A_0, A_1, \ldots, A_{N-1})$:

$$E = \{ (\mathbf{a}, \mathbf{0}) \mid \mathbf{a} \in A_0 \} \cup \bigcup_{k=1}^{N-1} \{ (\mathbf{a}, \mathbf{e}_k) \mid \mathbf{a} \in A_k \} \subset \mathbb{Z}^{n+N-1},$$

where $\mathbf{0}, \mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0)$, …, $\mathbf{e}_{N-1} = (0, 0, \ldots, 1)$ span the standard unit simplex in $\mathbb{R}^{N-1}$.

Definition

Given a tuple of Newton polytopes $\mathbf{P}$ of a system $\mathbf{f}(\mathbf{x}) \subset \mathbf{0}$, ***the tropical prevariety*** of $\mathbf{f}$ is the common refinement of the normal cones to the edges of the Newton polytopes in $\mathbf{P}$.

the Cayley embedding and the tropical prevariety

Proposition

Let $E_{\mathbf{A}}$ be the Cayley embedding of the supports $\mathbf{A}$ of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The normals of those facets of $E_{\mathbf{A}}$ that are spanned by at least two points of each support in $\mathbf{A}$ form the tropical prevariety of $\mathbf{f}$.

Proof. Let $\Sigma_{\mathbf{A}} = A_1 + A_2 + \cdots + A_N$ denote the Minkowski sum of the supports in $\mathbf{A}$. Facets of $\Sigma_{\mathbf{A}}$ spanned by at least two points of each support define the generators of the cones of the tropical prevariety.

Cells in a polyhedral subdivision of $E_{\mathbf{A}}$ are in one-to-one correspondence with cells in a polyhedral subdivision of $\Sigma_{\mathbf{A}}$.

This correspondence implies that facet normals of $\Sigma_{\mathbf{A}}$ occur as facet normals of $E_{\mathbf{A}}$.

Thus the set of all facets of $E_{\mathbf{A}}$ gives the tropical prevariety of $\mathbf{f}$. $\square$

cones of pretropisms

Definition

A ***cone of pretropism*** is a polyhedral cone spanned by pretropisms.

If we are looking for an algebraic set of dimension d and

- if there are no cones of vectors perpendicular to edges of the Newton polytopes of $f(\mathbf{x}) = \mathbf{0}$ of dimension d, then the system $f(\mathbf{x}) = \mathbf{0}$ has no solution set of dimension d that intersects the first d coordinate planes properly; otherwise
- if a d-dimensional cone of vectors perpendicular to edges of the Newton polytopes exists, then that cone defines a part of the tropical prevariety.

For the cyclic 9-roots system,
we found a two dimensional cone of pretropisms.

the tropical prevariety of cyclic n-roots

All facets normals of the Cayley polytope computed with `cddlib` on a 3.07GHz Linux computer with 4Gb RAM:

n	#normals	#pretropisms	#generators	user cpu time
8	831	94	11	$<$ 1 sec
9	4,840	276	17	1 sec
12	907,923	5,582	290	148 hours 27 min

Tropical intersections with `Gfan` on a 2.26GHz MacBook:

n	#rays	f-vector	user cpu time
8	94	1 94 108 48	15 sec
9	276	1 276 222 54	1 min 11 sec
12	5,582	1 5582 37786 66382 42540 8712	21 hours 1 min

Note that `Gfan` can exploit permutation symmetry.

increasing cost with increasing dimensions

For the computation of the tropical prevariety,

- the Sage 5.7/Gfan function `tropical_intersection()` ran (with default settings without exploitation of symmetry)
- on an AMD Phenom II X4 820 processor with 6 GB of RAM, running GNU/Linux.

As the dimension n increases so does the running time, but the relative cost factors are bounded by n:

n	seconds	hms format	factor
8	16.37	16 s	1.0
9	79.36	1 m 19 s	4.8
10	503.53	8 m 23 s	6.3
11	3898.49	1 h 4 m 58 s	7.7
12	37490.93	10 h 24 m 50 s	9.6

Observe: for $n = 12$, it takes 9.6 times longer than for $n = 11$.

Puiseux series for algebraic sets

Proposition

If $f(\mathbf{x}) = \mathbf{0}$ is in Noether position and defines a d-dimensional solution set in $\mathbb{C}^n$, intersecting the first d coordinate planes in regular isolated points, then there are d linearly independent tropisms $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_{d-1} \in \mathbb{Q}^n$ so that the initial form system $\mathrm{in}_{\mathbf{v}_0}(\mathrm{in}_{\mathbf{v}_1}(\cdots \mathrm{in}_{\mathbf{v}_{d-1}}(f)\cdots))(\mathbf{x} = \mathbf{y}^M) = \mathbf{0}$ has a solution $\mathbf{c} \in (\mathbb{C} \setminus \{0\})^{n-d}$.

This solution and the tropisms are the leading coefficients and powers of a generalized Puiseux series expansion for the algebraic set:

$$
\begin{array}{rclrcl}
x_0 & = & t_0^{v_{0,0}} & x_d & = & c_0 t_0^{v_{0,d}} t_1^{v_{1,d}} \cdots t_{d-1}^{v_{d-1,d}} + \cdots \\
x_1 & = & t_0^{v_{0,1}} t_1^{v_{1,1}} & x_{d+1} & = & c_1 t_0^{v_{0,d+1}} t_1^{v_{1,d+1}} \cdots t_{d-1}^{v_{d-1,d+1}} + \cdots \\
 & \vdots & & & \vdots & \\
x_{d-1} & = & t_0^{v_{0,d-1}} t_1^{v_{1,d-1}} \cdots t_{d-1}^{v_{d-1,d-1}} & x_n & = & c_{n-d-1} t_0^{v_{0,n-1}} t_1^{v_{1,n-1}} \cdots t_{d-1}^{v_{d-1,n-1}} + \cdots
\end{array}
$$

our polyhedral approach

For every d-dimensional cone C of pretropisms:

1. We select d linearly independent generators to form the d-by-n matrix A and the unimodular transformation $\mathbf{x} = \mathbf{y}^M$.
2. If $\mathrm{in}_{\mathbf{v}_0}(\mathrm{in}_{\mathbf{v}_1}(\cdots \mathrm{in}_{\mathbf{v}_{d-1}}(f)\cdots))(\mathbf{x} = \mathbf{y}^M) = \mathbf{0}$ has no solution in $(\mathbb{C}^*)^{n-d}$, then return to step 1 with the next cone C, else continue.
3. If the leading term of the Puiseux series satisfies the entire system, then we report an explicit solution of the system and return to step 1 to process the next cone C.
   Otherwise, we take the current leading term to the next step.
4. If there is a second term in the Puiseux series,
   then we have computed an initial development for an algebraic set and report this development in the output.

Note: to ensure the solution of the initial form system is not isolated, it suffices to compute a series development *for a curve*.

computing the second term

Proposition

Let $\mathbf{v}$ denote the pretropism and $\mathbf{x} = \mathbf{z}^M$ denote the unimodular coordinate transformation, generated by $\mathbf{v}$.
Let $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x} = \mathbf{z}^M)$ denote the transformed initial form system with regular isolated solutions, forming the isolated solutions at infinity of the transformed polynomial system $\mathbf{f}(\mathbf{x} = \mathbf{z}^M)$.

If the substitution of the regular isolated solutions into the transformed polynomial system $\mathbf{f}(\mathbf{x} = \mathbf{z}^M)$ does not satisfy the system entirely, then the constant terms of $\mathbf{f}(\mathbf{x} = \mathbf{z}^M)$ have disappeared, leaving at least one monomial $c_\ell t^{w_\ell}$ for some f_ℓ in $\mathbf{f}(\mathbf{x} = \mathbf{z}^M)$ with minimal value w_ℓ.

The minimal exponent w_ℓ is the candidate for the exponent of the second term in the Puiseux series.

series developments for cyclic 8-roots

A pretropism for cyclic 8-roots is $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$.
The corresponding unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$ is

$$\begin{array}{l} x_0 = z_0, \; x_1 = z_1/z_0, \; x_2 = z_2, \; x_3 = z_0 z_3, \\ x_4 = z_4, \; x_5 = z_5, \; x_6 = z_6/z_0, \; x_7 = z_7. \end{array}$$

Solving $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x} = \mathbf{z}^M) = \mathbf{0}$ gives as initial term of the series:

$$\begin{array}{l} z_0 = t, \; z_1 = -I, \; z_2 = \frac{-1}{2} - \frac{I}{2}, \; z_3 = -1, \\ z_4 = 1 + I, \; z_5 = \frac{1}{2} + \frac{I}{2}, \; z_6 = I, \; z_7 = -1 - I, \quad I = \sqrt{-1}. \end{array}$$

The series with its second term is

$$\begin{array}{l} z_0 = t, \; z_1 = -I + (-1 - I)t, \; z_2 = \frac{-1}{2} - \frac{I}{2} + \frac{1}{2}t, \; z_3 = -1, \\ z_4 = 1 + I - t, \; z_5 = \frac{1}{2} + \frac{I}{2} - \frac{1}{2}t, \; z_6 = I + (1 + I)t, \; z_7 = (-1 - I) + t. \end{array}$$

relevant software

- `cddlib` by Komei Fukuda and Alain Prodon implements the double description method to efficiently enumerate all extreme rays of a general polyhedral cone.
- `Gfan` by Anders Jensen to compute Gröbner fans and tropical varieties uses `cddlib`.
- The `Singular` library `tropical.lib` by Anders Jensen, Hannah Markwig and Thomas Markwig for computations in tropical geometry.
- `Macaulay2` interfaces to Gfan.
- `Sage` interfaces to Gfan.
- `PHCpack` (published as Algorithm 795 ACM TOMS) provides our numerical blackbox solver.

computing isolated solutions exploiting symmetry

The first four equations of the cyclic 5-roots system:

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_0 + x_1 + x_2 + x_3 + x_4 = 0 \\ x_0x_1 + x_0x_4 + x_1x_2 + x_2x_3 + x_3x_4 = 0 \\ x_0x_1x_2 + x_0x_1x_4 + x_0x_3x_4 + x_1x_2x_3 + x_2x_3x_4 = 0 \\ x_0x_1x_2x_3 + x_0x_1x_2x_4 + x_0x_1x_3x_4 + x_0x_2x_3x_4 + x_1x_2x_3x_4 = 0. \end{cases}$$

define solution curves. Moreover: $\mathbf{f} = \mathrm{in}_{\mathbf{v}}(\mathbf{f})$, where $\mathbf{v} = (1,1,1,1,1)$.

The first four equations are homogeneous
$\Rightarrow$ we have lines as solution curves

To exploit symmetry, we intersect the generating solution lines of the first four equations with the last equation.

unimodular coordinate transformation

For $\mathbf{v} = (1,1,1,1,1)$ we have the coordinate transformation $\mathbf{x} = \mathbf{z}^M$:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{x} = \mathbf{z}^M : \left\{ \begin{array}{rcl} x_0 & = & z_0 \\ x_1 & = & z_0 z_1 \\ x_2 & = & z_0 z_2 \\ x_3 & = & z_0 z_3 \\ x_4 & = & z_0 z_4. \end{array} \right.$$

Applying $\mathbf{x} = \mathbf{z}^M$ to the first 4 equations of the cyclic 5-roots system:

$$\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x} = \mathbf{z}^M) = \left\{ \begin{array}{l} z_1 + z_2 + z_3 + z_4 + 1 = 0 \\ z_1 z_2 + z_2 z_3 + z_3 z_4 + z_1 + z_4 = 0 \\ z_1 z_2 z_3 + z_2 z_3 z_4 + z_1 z_2 + z_1 z_4 + z_3 z_4 = 0 \\ z_1 z_2 z_3 z_4 + z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 = 0. \end{array} \right.$$

There are 14 solution lines of the form

$$x_0 = t, \; x_1 = t c_1, \; x_2 = t c_2, \; x_3 = t c_3, \; x_4 = t c_4$$

where (c_1, c_2, c_3, c_4) are solutions of $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x} = \mathbf{z}^M) = \mathbf{0}$.

positive dimensional sets of cyclic n-roots

- $n = 8$: Tropisms, their cyclic permutations, and degrees:

$$
\begin{array}{rr}
(1,-1,1,-1,1,-1,1,-1) & 8 \times 2 = 16 \\
(1,-1,0,1,0,0,-1,0) \rightarrow (1,0,0,-1,0,1,-1,0) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,-1,0,0,1,0,-1) \rightarrow (1,0,-1,1,0,-1,0,0) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,-1,1,0,-1,0,0) \rightarrow (1,0,-1,0,0,1,0,-1) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,0,-1,0,1,-1,0) \rightarrow (1,-1,0,1,0,0,-1,0) & 8 \times 2 + 8 \times 2 = 32 \\
 & \text{TOTAL} = 144
\end{array}
$$

- $n = 9$: A 2-dimensional cone of tropisms spanned by $\mathbf{v}_0 = (1,1,-2,1,1,-2,1,1,-2)$ and $\mathbf{v}_1 = (0,1,-1,0,1,-1,0,1,-1)$. Denoting by $u = e^{i2\pi/3}$ the primitive third root of unity, $u^3 - 1 = 0$:

$$
\begin{array}{lll}
x_0 = t_0 & x_3 = u t_0 & x_6 = u^2 t_0 \\
x_1 = t_0 t_1 & x_4 = u t_0 t_1 & x_7 = u^2 t_0 t_1 \\
x_2 = u^2 t_0^{-2} t_1^{-1} & x_5 = t_0^{-2} t_1^{-1} & x_8 = u t_0^{-2} t_1^{-1}.
\end{array}
$$

- $n = 12$: Computed 77 quadratic space curves.

results in the literature

Our results for $n = 9$ and $n = 12$ are in agreement with

- J.C. Faugère. **Finding all the solutions of Cyclic 9 using Gröbner basis techniques.** In *Computer Mathematics - Proceedings of the Fifth Asian Symposium (ASCM 2001)*, pages 1–12. World Scientific, 2001.

- R. Sabeti. **Numerical-symbolic exact irreducible decomposition of cyclic-12.** *London Mathematical Society Journal of Computation and Mathematics*, 14:155–172, 2011.

a tropical version of Backelin's Lemma

Lemma (Tropical Version of Backelin's Lemma)

For $n = m^2\ell$, where $\ell \in \mathbb{N} \setminus \{0\}$ and ℓ is no multiple of k^2, for $k \geq 2$, there is an $(m-1)$-dimensional set of cyclic n-roots, represented exactly as

$$
\begin{array}{rcl}
x_{km+0} & = & u^k t_0 \\
x_{km+1} & = & u^k t_0 t_1 \\
x_{km+2} & = & u^k t_0 t_1 t_2 \\
 & \vdots & \\
x_{km+m-2} & = & u^k t_0 t_1 t_2 \cdots t_{m-2} \\
x_{km+m-1} & = & \gamma u^k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1}
\end{array}
$$

for $k = 0, 1, 2, \ldots, m-1$, free parameters $t_0, t_1, \ldots, t_{m-2}$, constants $u = e^{\frac{i2\pi}{m\ell}}$, $\gamma = e^{\frac{i\pi\beta}{m\ell}}$, with $\beta = (\alpha \mod 2)$, and $\alpha = m(m\ell - 1)$.

summary

Promising results on the cyclic *n*-roots problem give a proof of concept for a new polyhedral method to compute algebraic sets:

- hybrid symbolic-numeric algorithm for Puiseux series;
- for highly structured systems we may find exact monomial maps.

For the computation of pretropisms, we rely on

- `cddlib` on the Cayley embedding of the Newton polytopes, or
- `Gfan` for the tropical intersection.

To process the pretropisms, we

- use `Sage` to extract initial form systems
  and look for the second term in the Puiseux series;
- solve initial form systems with the blackbox solver of `PHCpack`.