Numerical Irreducible Decomposition

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Plan of the Lecture

1. We factor in three stages:
   (a) monodromy grouping of witness points;
   (b) certification of grouping with linear traces;
   (c) interpolation to get polynomials for the factors.
2. Special case: one single multivariate polynomial.

   We remove multiplicies by differentation and use
   a theorem of Marden and Walsh for bound on precision.
3. Applications:
   (a) irreducible components of Griffis-Duffy platforms;
   (b) study singularities of Stewart-Gough platforms.

Recommended Background Literature

S.S. Abhyankar: **Algebraic Geometry for Scientists and Engineers.** AMS, 1990.

E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris: **Geometry of Algebraic Curves**, Volume I. Springer, 1985.

J.E. Marsden: **Basic Complex Analysis.** W.H. Freeman and Company, 1973.

M. Mignotte and D. Ştefănescu: **Polynomials. An Algorithmic Approach**. Springer, 1999.

Factoring Solution Components

Input: $f(\mathbf{x}) = \mathbf{0}$ polynomial system with a positive dimensional solution component, represented by witness set.

coefficients of f known approximately, work with limited precision

Wanted: decompose the component into irreducible factors, for each factor, give its degree and multiplicity.

Symbolic-Numeric issue: essential numerical information (such as degree and multiplicity of each factor), is obtained much faster than the full symbolic representation.

Joint Work with A.J. Sommese and C.W. Wampler

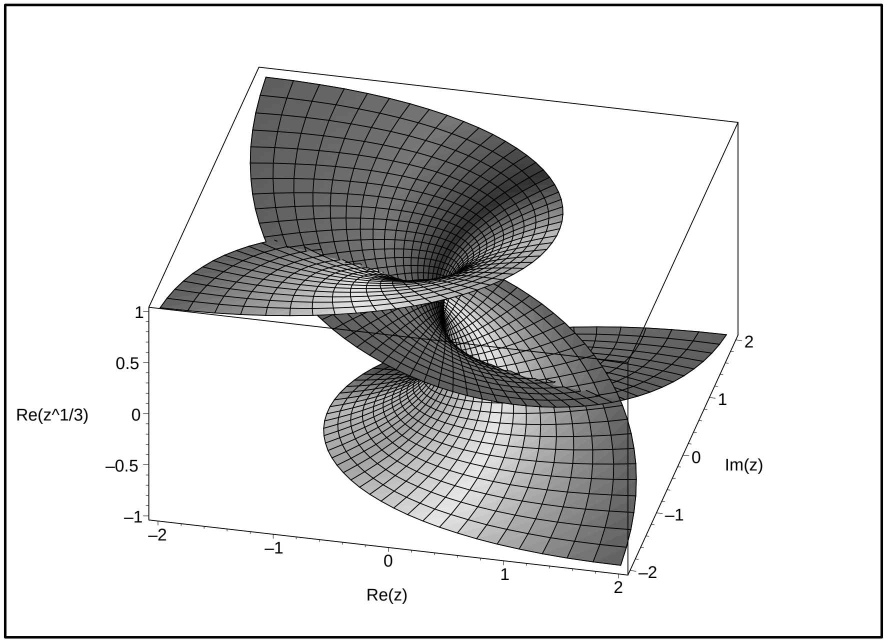
A.J. Sommese, JV and C.W. Wampler: **Using monodromy to decompose solution sets of polynomial systems into irreducible components.** In *Application of Algebraic Geometry to Coding Theory, Physics and Computation*, ed. by C. Ciliberto et al., Proceedings of a NATO Conference, February 25 - March 1, 2001, Eilat, Israel. Pages 297–315, Kluwer AP.

A.J. Sommese, JV and C.W. Wampler: **Symmetric functions applied to decomposing solution sets of polynomial systems.** *SIAM J. Numer. Anal.* 40(6):2026–2046, 2002.

A.J. Sommese, JV and C.W. Wampler: **Numerical Factorization of Multivariate Complex Polynomials.** Manuscript, 2002.

A.J. Sommese, JV and C.W. Wampler: **Numerical irreducible decomposition using PHCpack.** In *Algebra, Geometry, and Software Systems*, edited by M. Joswig and N. Takayama, pages 109–130, Springer-Verlag, 2003.

The Riemann Surface of $z^3 - w = 0$:



R.M. Corless and D.J. Jeffrey: **Graphing elementary Riemann surfaces.** *SIGSAM Bulletin* 32(1):11–17, 1998.

Monodromy to Decompose Solution Components

Given: a system $f(\mathbf{x}) = \mathbf{0}$; and $W = (Z, L)$:

$$\text{for all } \mathbf{w} \in Z : f(\mathbf{w}) = \mathbf{0} \text{ and } L(\mathbf{w}) = \mathbf{0}.$$

Wanted: partition of Z so that all points in a subset of Z lie on the same irreducible factor.

Example: does $f(x, y) = xy - 1 = 0$ factor?

Consider $H(x, y, \theta) = \begin{cases} xy - 1 = 0 \\ x + y = 4e^{i\theta} \end{cases}$ for $\theta \in [0, 2\pi]$.

For $\theta = 0$, we start with two real solutions. When $\theta > 0$, the solutions turn complex, real again at $\theta = \pi$, then complex until at $\theta = 2\pi$. Back at $\theta = 2\pi$, we have again two real solutions, but their order is permuted $\Rightarrow$ irreducible.

Connecting Witness Points

1. For two sets of hyperplanes K and L, and a random $\gamma \in \mathbb{C}$

$$H(\mathbf{x}, t, K, L, \gamma) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ \gamma K(\mathbf{x})(1-t) + L(\mathbf{x})t = \mathbf{0} \end{cases}$$

   We start paths at $t = 0$ and end at $t = 1$.

2. For $\alpha \in \mathbb{C}$, trace the paths defined by $H(\mathbf{x}, t, K, L, \alpha) = \mathbf{0}$.

   For $\beta \in \mathbb{C}$, trace the paths defined by $H(\mathbf{x}, t, L, K, \beta) = \mathbf{0}$.

   Compare start points of first path tracking with end points of second path tracking. Points which are permuted belong to the same irreducible factor.

3. Repeat the loop with other hyperplanes.

Linear Traces – an example

$$\text{Consider} \quad f(x, y(x)) = (y - y_1(x))(y - y_2(x))(y - y_3(x))$$
$$= y^3 - t_1(x)y^2 + t_2(x)y - t_3(x)$$

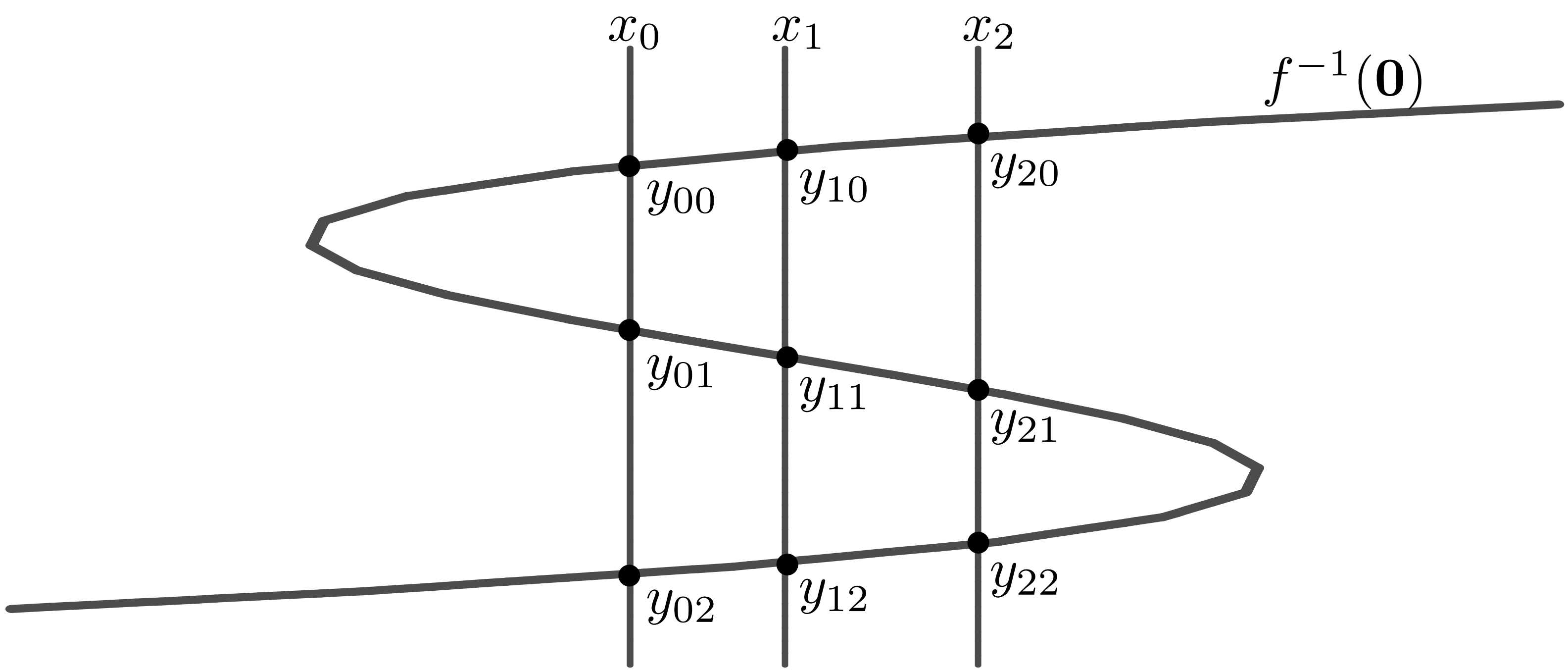
We are interested in the linear trace: $t_1(x) = c_1x + c_0$.

Sample the cubic at $x = x_0$ and $x = x_1$. The samples are $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$.

$$\text{Solve} \quad \begin{cases} y_{00} + y_{01} + y_{02} = c_1x_0 + c_0 \\ y_{10} + y_{11} + y_{12} = c_1x_1 + c_0 \end{cases} \quad \text{to find} \quad c_0, c_1.$$

With t_1 we can predict the sum of the y's for a fixed choice of x. For example, samples at $x = x_2$ are $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$. Then, $t_1(x_2) = c_1x_2 + c_0 = y_{20} + y_{21} + y_{22}$.

Linear Traces – example continued



Use $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$ to find the linear trace $t_1(x) = c_0 + c_1 x$.

At $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$: $c_0 + c_1 x_2 = y_{20} + y_{21} + y_{22}$?

Validation of Breakup with Linear Trace

Do we have enough witness points on a factor?

- We may not have enough monodromy loops to connect all witness points on the same irreducible component.
- For a k-dimensional solution component, it suffices to consider a curve on the component cut out by $k-1$ random hyperplanes. The factorization of the curve tells the decomposition of the solution component.
- We have enough witness points on the curve if the value at the linear trace can predict the sum of one coordinate of all points in the set.

Notice: Instead of monodromy, we may enumerate all possible factors and use linear traces to certify. While the complexity of this enumeration is exponential, it works well for low degrees.

Special case: one single polynomial

- Input: $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.
  coefficients known approximately, work with limited precision
- Wanted: write f as product of irreducible factors, as

$$f(\mathbf{x}) = \prod_{i=1}^{N} q_i(\mathbf{x})^{\mu_i}, \quad \sum_{i=1}^{N} \mu_i \deg(q_i) = \deg(f),$$

  every irreducible factor q_i occurs with multiplicity μ_i.

E. Kaltofen: **Challenges of symbolic computation: my favorite open problems**. *J. Symbolic Computation* 29(6): 891–919, 2000.

Related Work

Y. Huang, W. Wu, H.J. Stetter, and L. Zhi: **Pseudofactors of multivariate polynomials.** In *Proceedings of ISSAC 2000*, ed. by C. Traverso, pages 161–168, ACM 2000.

R.M. Corless, M.W. Giesbrecht, M. van Hoeij, I.S. Kotsireas and S.M. Watt: **Towards factoring bivariate approximate polynomials.** In *Proceedings of ISSAC 2001*, ed. by B. Mourrain, pages 85–92, ACM 2001.

A. Galligo and D. Rupprecht: **Semi-numerical determination of irreducible branches of a reduced space curve.** In *Proceedings of ISSAC 2001*, ed. by B. Mourrain, pages 137–142, ACM 2001.

A. Galligo and D. Rupprecht: **Irreducible decomposition of curves.** *J. Symbolic Computation* 33(5):661–677, 2002.

T. Sasaki: **Approximate multivariate polynomial factorization based on zero-sum relations.** In *Proceedings of ISSAC 2001*, ed. by B. Mourrain, pages 284–291, ACM 2001.

R.M. Corless, A. Galligo, I.S. Kotsireas, and S.M. Watt: **A geometric-numeric algorithm for absolute factorization of multivariate polynomials.** In *Proceedings of ISSAC 2002*, ed. by T. Mora, pages 37–45, ACM 2002.

E. Kaltofen and J. May: **On approximate irreducibility of polynomials in several variables.** To appear in *Proceedings of ISSAC 2003*.

Dealing with Multiplicities

On a factor of degree d and multiplicity μ,

we find d clusters, each of μ witness points.

Choose $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and compute

$$g(\mathbf{x}) := \left(v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + \cdots + v_n \frac{\partial}{\partial x_n} \right)^{\mu - 1} f(\mathbf{x}).$$

Then apply the techniques to the multiplicity one roots of $g(\mathbf{x})$ corresponding to the clusters.

Using a theorem of Marden and Walsh

Assume d is the degree of $f(z)$, $f \in \mathbb{C}[z]$;

μ is the multiplicity of a root of f;

z_0 is the center of the cluster around the multiple root;

$\Delta_r(z_0) = \{\ z \in \mathbb{C} \mid |z - z_0| \leq r\ \}$ contains the cluster;

r is the radius of the disk $\Delta_r(z_0)$;

R is largest such that $\{\ z \in \mathbb{C} \mid |z - z_0| \geq R\ \}$ contains all other $d - \mu$ roots of f.

If $\frac{R}{r} \geq \frac{2\binom{d}{\mu}}{d-\mu+1}$, then $f^{(k)}$ has exactly $\mu - k$ roots in $\Delta_r(z_0)$, for $k = 1, 2, \ldots, \mu - 1$.

Applying the bound for R/r

Given a cluster of μ roots (and $d-\mu$ other roots), compute

- z_0 as the average of the roots in the cluster;
- r as the largest distance of the roots in the cluster to z_0;
- R as the smallest distance of the other $d-\mu$ roots to z_0.

$$\frac{R}{r} \geq \frac{2\binom{d}{\mu}}{d-\mu+1} \quad \Rightarrow \quad r \leq R \left(\frac{d-\mu+1}{2\binom{d}{\mu}} \right)$$

We obtain a bound on r, the precision of the roots in the cluster, in order for the successive derivatives of f to be safe.

Numerical Limitations

- Evaluation of high degree polynomials is numerically unstable:

$$f(x) = (x_0 + tv)^d = \sum_{k=0}^{d} \binom{d}{k} x_0^{d-k} v^k t^k = 0,$$

  for example, $d = 30$ and $k = 15$: nine decimal places in $\binom{d}{k}$.

- Working precision determines accuracy of factorization:

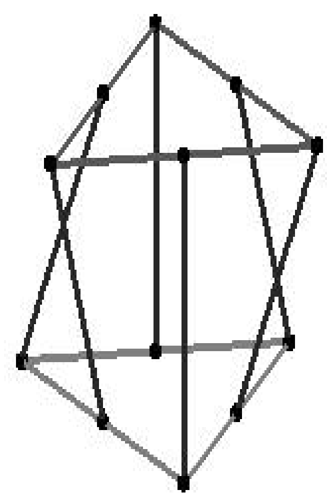
$$f(x, y) = xy + 10^{-16}$$

  – will factor when working with double precision floats;

  – will not factor as soon as precision is high enough.

Application I: Architecturally Singular Platforms

Special Griffis-Duffy type



- Base and endplate are equilateral triangles.
- Legs connect vertices to midpoints.

Results of Husty and Karger

Self-motions of Griffis-Duffy type parallel manipulators. In *Proc. 2000 IEEE Int. Conf. Robotics and Automation* (CDROM), 2000.

The special Griffis-Duffy platforms *move*:

- Case 1: Plates not equal, legs not equal.
  - Curve is degree 20 in Euler parameters.
  - Curve is degree 40 in position.
- Case 2: Plates congruent, legs all equal.
  - Factors are degrees $(4 + 4) + 6 + 2 = 16$ in Euler parameters.
  - Factors are degrees $(8 + 8) + 12 + 4 = 32$ in position.

Question: *Can we confirm these results numerically?*

Components of Griffis-Duffy Platforms

Solution components by degree

Husty & Karger		SVW	
Euler	Position	Study	Position
General Case			
20	40	28	40
Legs equal, Plates equal			
		6	8
4	8	6	8
4	8	6	8
6	12	6	12
2	4	4	4
16	32	28	40

Griffis-Duffy Platforms: Factorization

Case A: One irreducible component of degree 28 (general case).

Case B: Five irreducible components of degrees 6, 6, 6, 6, and 4.

user cpu on 800Mhz	Case A	Case B
witness points	1m 12s 480ms	
monodromy breakup	33s 430ms	27s 630ms
Newton interpolation	1h 19m 13s 110ms	2m 34s 50ms

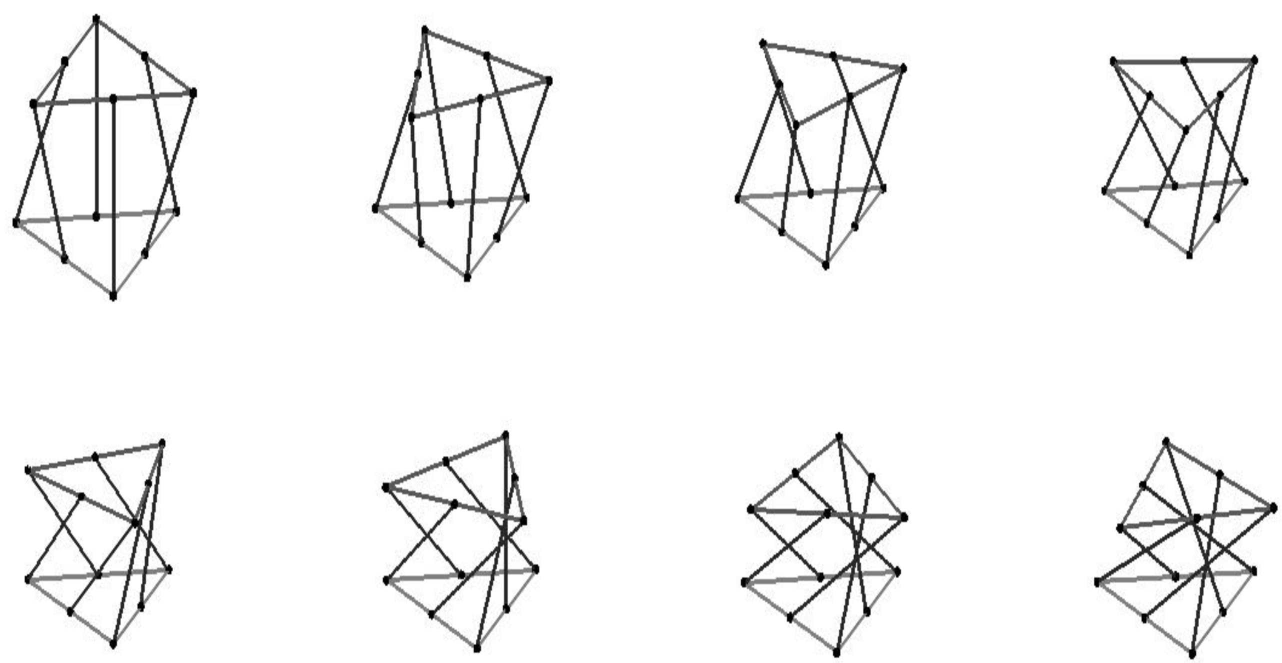
32 decimal places used to interpolate polynomial of degree 28

linear trace	4s 750ms	4s 320ms

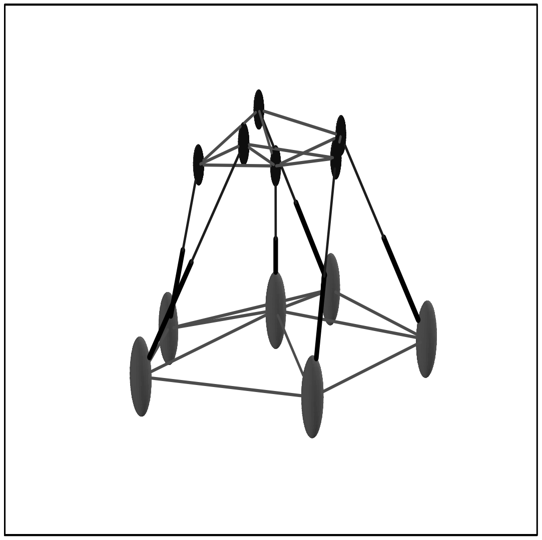
Linear traces replace Newton interpolation:

$\Rightarrow$ **time to factor independent of geometry!**

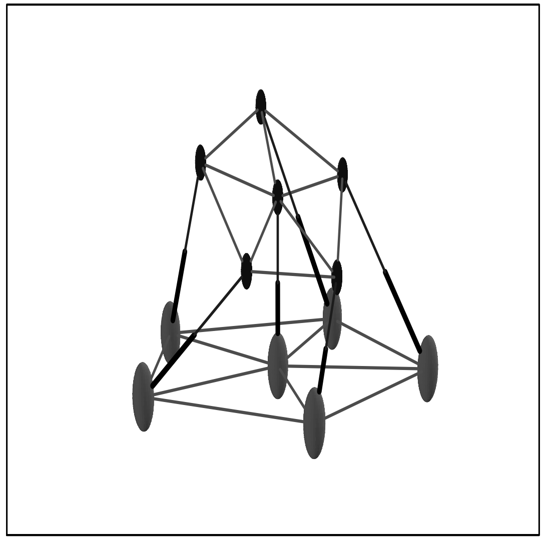
Griffis-Duffy Platforms: an Animation



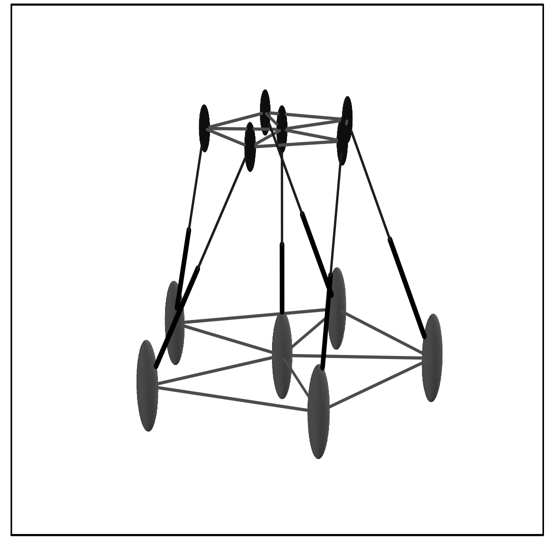
Application II: three Stewart-Gough platforms



General platform,
fixed position



Planar base,
planar platform



Parallel base
and platform

J.P. Merlet: **Parallel Robots**. Kluwer Academic Publishers, 2000.

Singularities of Stewart-Gough Platforms

At singularity, rigidity of device is lost, allowing finite motion which cannot be controlled by leg lengths (*disaster!*).

Denote	$\mathbf{p} \in \mathbb{C}^3$	position of platform;
	$\mathbf{q} \in \mathbb{P}^3$	quaternion defines a rotation;
	$\mathbf{a}_i, \mathbf{b}_i \in \mathbb{C}^3$	ball joints at platform and base, $i = 1, 2, \ldots, 6$;
	$\mathbf{J} \in \mathbb{C}^{6\times 6}$	Jacobian matrix of mapping from platform motion to leg lengths.

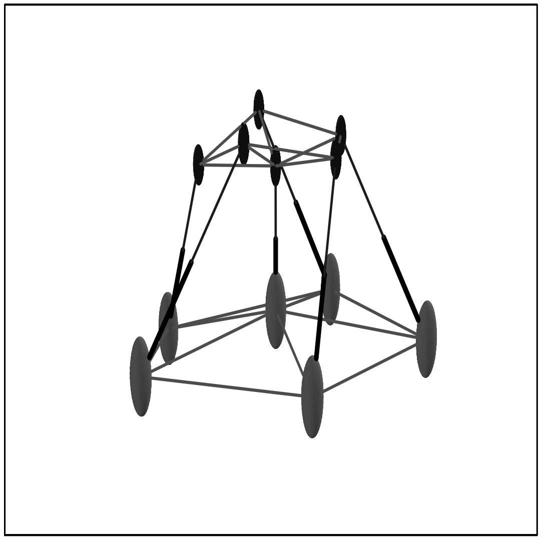
Then the condition on a singular configuration is $\det \mathbf{J} = 0$.

$\det \mathbf{J}$ is a polynomial of degree 1728 in 43 variables: $\mathbf{p}$, $\mathbf{q}$, $\mathbf{a}_i$, $\mathbf{b}_i$.

Merlet. *Int. J. Robotics Research* 8(5):45–56, 1989.

Bayer St-Onge and Gosselin. *Int. J. Robotics Research* 19(3):271–288, 2000.

first general case of a Stewart-Gough platform



General platform,
fixed position

- case of *almost all* manipulators

  $\mathbf{p}, \mathbf{a}_i$, and $\mathbf{b}_i$ are randomly chosen

- $\deg(\det \mathbf{J}) = 12$, homogeneous in $\mathbf{q}$

  the expanded $\det \mathbf{J}$ has 910 terms

- $\det \mathbf{J} = F_1(\mathbf{q})(F_2(\mathbf{q}))^3$

  $\mathbf{q} = (q_0, q_1, q_2, q_3)$ quaternion

  $\deg(F_1) = 6$

  $F_2(\mathbf{q}) = q_0^2 + q_1^2 + q_2^2 + q_3^2$

  F_2 has no physical significance

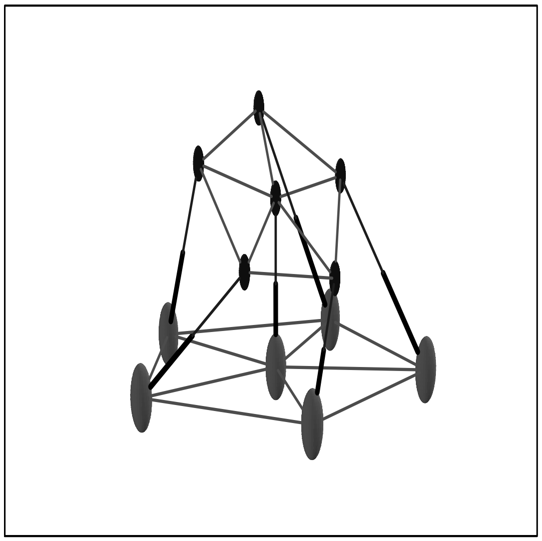
Computational results for first platform

cluster	r	R	R/r
one	1.7E-05	3.4E-01	2.0E+04
two	4.9E-06	1.7E-01	3.6E+04

Lower bound on R/r evaluates to 44.

Elapsed user CPU times on 2.4Ghz WindowsXP							
1.	monodromy grouping	:	0h	6m	40s	469ms
2.	linear traces certification	:	0h	0m	30s	672ms
3.	interpolation at factors	:	1h	41m	53s	78ms
4.	multiplication validation	:	0h	0m	8s	156ms
	total time for all 4 stages	:	1h	49m	12s	391ms

second case: planar base and platform



Planar base
and platform

- ball joints $\mathbf{a}_i$ lie in planar platform
  ball joints $\mathbf{b}_i$ lie in planar base
- $\deg(\det \mathbf{J}) = 12$, homogeneous in $\mathbf{q}$
  the expanded $\det \mathbf{J}$ has 910 terms
- $\det \mathbf{J} = F_1(\mathbf{q})(F_2(\mathbf{q}))^3$
  $\mathbf{q} = (q_0, q_1, q_2, q_3)$ quaternion
  $\deg(F_1) = 6 \quad \deg(F_2) = 2$

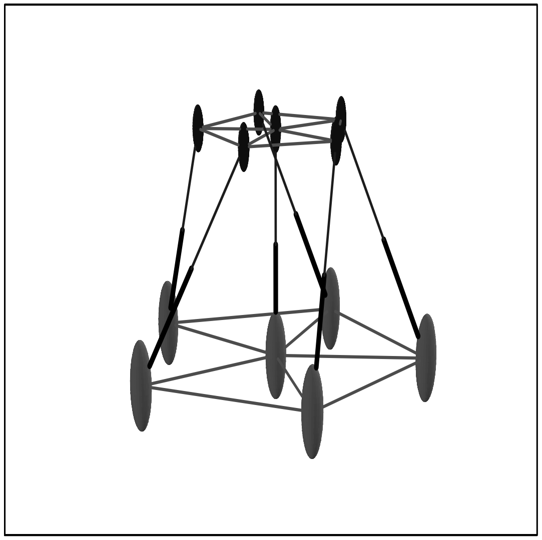
Computational results for second platform

cluster	r	R	R/r
one	6.2E-05	2.4E-01	3.8E+04
two	4.8E-05	6.0E-01	1.2E+04

Lower bound on R/r evaluates to 44.

Elapsed user CPU times on 2.4Ghz WindowsXP			
1.	monodromy grouping	:	0h 17m 34s 735ms
2.	linear traces certification	:	0h 0m 27s 359ms
3.	interpolation at factors	:	1h 24m 45s 766ms
4.	multiplication validation	:	0h 0m 8s 172ms
	total time for all 4 stages	:	1h 42m 56s 32ms

third case: parallel base and platform



Parallel base
and platform

- ball joints $\mathbf{a}_i, \mathbf{b}_i$ in parallel planes, position $\mathbf{p}$ is variable, $q_1 = q_2 = 0$
- $\deg(\det \mathbf{J}) = 15$, in $(\mathbf{p}, \mathbf{q})$
  expanded $\det \mathbf{J}$ has 24 terms,
  much sparser, as $24 << 910$
- $$\det \mathbf{J} = a p_3^3 (q_0 + b q_3)(q_0 + c q_3) \\ (q_0 + i q_3)^5 (q_0 - i q_3)^5$$
  where the constants a, b, c
  depend on the choice of $\mathbf{a}_i, \mathbf{b}_i$

Computational results for third platform

cluster	r	R	R/r
one	5.1E-07	1.0E+00	2.0E+06
two	7.3E-04	3.4E-01	4.7E+02
three	4.0E-03	7.2E-01	1.8E+02

Lower bound on R/r evaluates to 546.

Elapsed user CPU times on 2.4Ghz WindowsXP			
1.	monodromy grouping	:	1m 13s 656ms
2.	linear traces certification	:	0m 3s 891ms
3.	interpolation at factors	:	0m 4s 734ms
4.	multiplication validation	:	0m 1s 657ms
	total time for all 4 stages	:	1m 23s 938ms

Monodromy Compared to the Enumeration Method

Enumeration of all possible factors certified by linear traces outperforms the monodromy algorithm for our application:

User CPU times on 2.4Ghz Windows XP		
case	monodromy	enumeration
1	6m 40s 460ms	40s 750ms
2	17m 34s 735ms	31s 657ms
3	1m 13s 656ms	3s 0ms

Random irreducible polynomials of five monomials:

User CPU times on 2.4Ghz Windows XP		
degree	monodromy	enumeration
10	5s 484ms	312ms
15	8s 187ms	1s 453ms
16	16s 63ms	2s 875ms

Exercises

- Apply `phc -f` to factor

```
x**6 - x**5*y + 2*x**5*z - x**4*y**2 - x**4*y*z+x**3*y**3
 - 4*x**3*y**2*z + 3*x**3*y*z**2 - 2*x**3*z**3 + 3*x**2*y**3*z
 - 6*x**2*y**2*z**2 + 5*x**2*y*z**3 - x**2*z**4 + 3*x*y**3*z**2
 - 4*x*y**2*z**3 + 2*x*y*z**4+y**3*z**3 - y**2*z**4;
```

- Consider the adjacent minors of a general 2×4-matrix:

$$\left[\begin{array}{cccc} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{array} \right] \qquad f(\mathbf{x}) = \left\{ \begin{array}{l} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \\ x_{13}x_{24} - x_{23}x_{14} = 0 \end{array} \right.$$

  Compute the irreducible decomposition of $f^{-1}(\mathbf{0})$.