Numerical Decomposition of the Intersection of Algebraic Varieties

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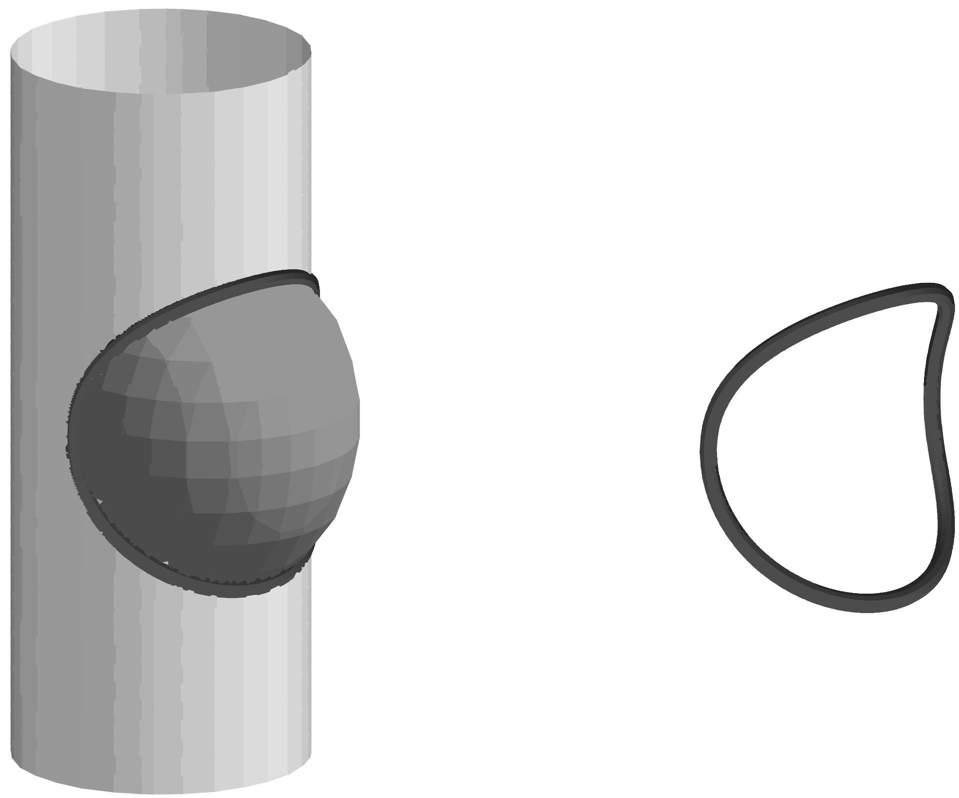
Outline of Talk

1. Motivation and Problem Statement
2. Describing the Intersection of Varieties
3. Decomposing the Diagonal
4. Diagonal Homotopies
5. Computational Experiments

Intersecting a Sphere with a Cylinder



Problem: describe the curve of intersection.

Problems with Previous Homotopies

naive combination of two possibly identical systems is not sufficient!

For example: find $A \cap B$,

where A is line $x_2 = 0$, solution of $f(x_1, x_2) = x_1 x_2 = 0$,

and B is line $x_1 - x_2 = 0$, solution of $g(x_1, x_2) = x_1(x_1 - x_2) = 0$.

Problem: $A \cap B = (0, 0)$ does not occur as an irreducible

solution component of $\left\{ \begin{array}{l} f(x_1, x_2) = x_1 x_2 = 0 \\ g(x_1, x_2) = x_1(x_1 - x_2) = 0. \end{array} \right.$

Representing Positive Dimensional Solutions

Let $f(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} \in \mathbb{C}^3$, define a surface S, a curve C, and a point P.

Choosing random hyperplanes, we cut the solution set of $f(\mathbf{x}) = \mathbf{0}$:

$$
\begin{array}{ccccccc}
\text{cut with} & \text{line} & \subset & \text{plane} & \subset & \text{space} \\
 & \left\{ \begin{array}{l} \mathbf{a}_1\mathbf{x} = b_1 \\ \mathbf{a}_2\mathbf{x} = b_2 \end{array} \right. & \subset & \left\{ \ \mathbf{a}_1\mathbf{x} = b_1 \right. & \subset & \left\{ \ 0 = 0 \right. \\
 & \downarrow & & \downarrow & & \downarrow \\
\text{to find} & \text{surface } S & & \text{curve } C & & \text{point } P
\end{array}
$$

A Cascade of Polynomial Systems

$$\left\{\begin{array}{rr} f_1(\mathbf{x}) + c_{11}z_1 + c_{12}z_2 & = 0 \\ f_2(\mathbf{x}) + c_{21}z_1 + c_{22}z_2 & = 0 \\ f_3(\mathbf{x}) + c_{31}z_1 + c_{32}z_2 & = 0 \\ L_1(\mathbf{x}) + z_1 \qquad\quad & = 0 \\ L_2(\mathbf{x}) \qquad\quad + z_2 & = 0 \end{array}\right. \quad \rightarrow \quad \left\{\begin{array}{rr} f_1(\mathbf{x}) + c_{11}z_1 + c_{12}z_2 & = 0 \\ f_2(\mathbf{x}) + c_{21}z_1 + c_{22}z_2 & = 0 \\ f_3(\mathbf{x}) + c_{31}z_1 + c_{32}z_2 & = 0 \\ L_1(\mathbf{x}) + z_1 \qquad\quad & = 0 \\ z_2 & = 0 \end{array}\right.$$

$$\downarrow$$

$$\left\{\begin{array}{rr} f_1(\mathbf{x}) + c_{11}z_1 & = 0 \\ f_2(\mathbf{x}) + c_{21}z_1 & = 0 \\ f_3(\mathbf{x}) + c_{31}z_1 & = 0 \\ z_1 & = 0 \end{array}\right.$$

$$L_1 : \mathbf{a_1 x} - b_1$$

$$L_2 : \mathbf{a_2 x} - b_2$$

$$z_1, z_2 : \textit{slack variables}$$

A Cascade of Homotopies

Denote $\mathcal{E}_i$ as an embedding of $f(\mathbf{x}) = \mathbf{0}$ with i random hyperplanes and i slack variables $\mathbf{z} = (z_1, z_2, \ldots, z_i)$.

Theorem (Sommese - Verschelde): *J. Complexity* 16(3):572–602, 2000

1. Solutions with $(z_1, z_2, \ldots, z_i) = \mathbf{0}$ contain $\deg W$ generic points on every i-dimensional component W of $f(\mathbf{x}) = \mathbf{0}$.
2. Solutions with $(z_1, z_2, \ldots, z_i) \neq \mathbf{0}$ are regular; and

solution paths defined by

$$H_i(\mathbf{x}, \mathbf{z}, t) = t\mathcal{E}_i(\mathbf{x}, \mathbf{z}) + (1-t) \left(\begin{array}{c} \mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) \\ z_i \end{array} \right) = \mathbf{0}$$

starting at $t = 1$ with all solutions with $z_i \neq 0$
reach at $t = 0$ all isolated solutions of $\mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) = \mathbf{0}$.

Witness Sets

A witness point is a solution of a polynomial system which lies on a set of generic hyperplanes.

- The number of generic hyperplanes used to isolate a point from a solution component

  equals the $\boxed{\textbf{dimension}}$ of the solution component.

- The number of witness points on one component cut out by the same set of generic hyperplanes

  equals the $\boxed{\textbf{degree}}$ of the solution component.

A witness set for a k-dimensional solution component consists of k random hyperplanes and the set of isolated solutions comprising the intersection of the component with those hyperplanes.

Problem Statement

Input: two irreducible components A and B, given by polynomial systems f_A and f_B (possibly identical), random hyperplanes L_A and L_B, and the solutions to

$$\underbrace{\begin{array}{c} \left\{ \begin{array}{l} f_A(\mathbf{x}) = \mathbf{0} \\ L_A(\mathbf{x}) = \mathbf{0} \end{array} \right. \\ \#L_A = \dim(A) = a \\ \{\ \alpha_1, \alpha_2, \ldots, \alpha_{\deg A}\ \} \\ \deg A \text{ generic points} \end{array}}_{\text{a } \textit{witness set} \text{ for } A} \quad \text{and} \quad \underbrace{\begin{array}{c} \left\{ \begin{array}{l} f_B(\mathbf{x}) = \mathbf{0} \\ L_B(\mathbf{x}) = \mathbf{0} \end{array} \right. \\ \#L_B = \dim(B) = b \\ \{\ \beta_1, \beta_2, \ldots, \beta_{\deg B}\ \} \\ \deg B \text{ generic points} \end{array}}_{\text{a } \textit{witness set} \text{ for } B}$$

Output: witness sets for all pure dimensional components of $A \cap B$.

Solving Systems restricted to an Algebraic Set

Consider $f(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ over $X \times Y$, Y = parameter space.

Wanted: Solutions to $f(\mathbf{x}, \mathbf{y}^*) = \mathbf{0}$, for some $\mathbf{y}^* \in Y$.

1. Choose a general $\mathbf{y}' \in Y$ $(\mathbf{y}' \neq \mathbf{y}^*)$.

   $D = \#\{ \ \mathbf{x} \mid f(\mathbf{x}, \mathbf{y}') = \mathbf{0} \ \}$ is maximal for all $\mathbf{y}' \in Y$.

2. Construct a curve $B \subset Y$ connecting $\mathbf{y}'$ to $\mathbf{y}^*$.

3. Construct a map $c : [0, 1] \times \Gamma \rightarrow B$, $\Gamma = \{ \ \gamma \in \mathbb{C} \mid |\gamma| = 1 \ \}$,

   so that $c(0, \Gamma) = \mathbf{y}^*$ and $c(1, \Gamma) = \mathbf{y}'$.

4. Choose $\gamma \in \Gamma$ at random and track D solution paths defined by the homotopy $f(\mathbf{x}, c(t, \gamma)) = \mathbf{0}$, starting at $t = 1$ at the solutions of $f(\mathbf{x}, \mathbf{y}') = \mathbf{0}$ and ending at $t = 0$ at the desired solutions of $f(\mathbf{x}, \mathbf{y}^*) = \mathbf{0}$.

Abstract Embeddings of Polynomial Systems

X is reduced pure N-dimensional algebraic set

(abstract means: no equations specified for X)

f is system of restrictions of polynomials to X

$$\mathcal{E}(f, \mathbf{x}, \mathbf{z}, Y) = \begin{bmatrix} f(\mathbf{x}) + A_2^T \mathbf{z} \\ \mathbf{z} - A_0 - A_1 \mathbf{x} \end{bmatrix} \qquad \begin{array}{r} Y = (A_0, A_1, A_2), A_0 \in \mathbb{C}^{N \times 1}, \\ A_1 \in \mathbb{C}^{N \times m}, A_2 \in \mathbb{C}^{N \times N}. \end{array}$$

Stratification: $Y_0 \subset Y_1 \subset \cdots \subset Y_N$, last $N - i$ rows of Y_i are zero.

Cascade of embeddings: $\mathcal{E}_i(f)$ is restricted to Y_i,

$$\mathcal{E}_N(f) = \mathcal{E}(f) \quad \text{and} \quad \mathcal{E}_0(f) = f.$$

A Generalized Cascade of Homotopies

For random $\gamma_i \in \mathbb{C}$, $|\gamma_i| = 1$, the homotopy $H_i(\mathbf{x}, \mathbf{z}, t, Y, \gamma_i)$

$$= \gamma_i \ t \ \mathcal{E}_i(f)(\mathbf{x}, \mathbf{z}, Y_i) + (1-t) \left(\begin{array}{c} \mathcal{E}_{i-1}(f)(\mathbf{x}, \mathbf{z}, Y_{i-1}) \\ z_i \end{array} \right) = \mathbf{0},$$

defines paths starting at $t = 1$ at the solutions of $\mathcal{E}_i(f)$,
ending at $t = 0$ at the solutions of $\mathcal{E}_{i-1}(f)$.

<u>Theorem:</u>

1. Solutions with $\mathbf{z} = (z_1, z_2, \ldots, z_i) \neq \mathbf{0}$ of $H_i(\mathbf{x}, \mathbf{z}, 1, Y, \gamma_i) = \mathbf{0}$ are regular, and stay regular for all $t > 0$.
2. As $t \to 0$, the solutions of $H_i(\mathbf{x}, \mathbf{z}, t, Y, \gamma_i) = \mathbf{0}$, contain all witness sets on the $(i-1)$-dimensional components of $f^{-1}(\mathbf{0})$.

A Numerical Embedding

Let X be an N-dimensional solution component of $g(\mathbf{x}) = \mathbf{0}$,
a system of n equations $g = (g_1, g_2, \ldots, g_n)$ in $\mathbf{x} \in \mathbb{C}^m$.

Randomize g to have as many equations as co-dimension of X:

$$G(\mathbf{x}) := \mathcal{R}(g(\mathbf{x}), m - N) = \Lambda g(\mathbf{x}), \quad \Lambda \in \mathbb{C}^{(m-N)\times n},$$

where Λ is a random matrix.

In the cascade of homotopies, replace $\mathcal{E}_i(f)$ by $\begin{bmatrix} G(\mathbf{x}) \\ \mathcal{E}_i(f)(\mathbf{x}, \mathbf{z}) \end{bmatrix}$.

Decomposing the Diagonal

Given two irreducible components A and B in $\mathbb{C}^k$,
consider their product $X := A \times B \subset \mathbb{C}^{k+k}$.

Then $A \cap B \cong X \cap \Delta$ where Δ is the diagonal of $\mathbb{C}^{k+k}$ defined by

$$\delta(\mathbf{u}, \mathbf{v}) := \begin{bmatrix} u_1 - v_1 = 0 \\ u_2 - v_2 = 0 \\ \vdots \\ u_k - v_k = 0 \end{bmatrix} \quad \text{on } X.$$

Notice: δ plays role of f in the abstract embedding.

Input Data for Diagonal Homotopies

Let $A \in \mathbb{C}^k$ be an irreducible component of $f_A^{-1}(\mathbf{0})$, $\dim A = a$; and
$B \in \mathbb{C}^k$ be an irreducible component of $f_B^{-1}(\mathbf{0})$, $\dim B = b$.

Assuming $a \geq b$ and $B \not\subseteq A$, then $\dim(A \cap B) \leq b - 1$.

Randomize: $F_A(\mathbf{u}) := \mathcal{R}(f_A, k - a)$ and $F_B(\mathbf{v}) := \mathcal{R}(f_B, k - b)$.

$A \times B$ is a solution component of $\mathcal{F}(\mathbf{u}, \mathbf{v}) := \begin{bmatrix} F_A(\mathbf{u}) \\ F_B(\mathbf{v}) \end{bmatrix} = \mathbf{0}$.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_{\deg A}\}$ satisfy $F_A(\mathbf{u}) = \mathbf{0}$ and $L_A(\mathbf{u}) = \mathbf{0}$; and
$\{\beta_1, \beta_2, \ldots, \beta_{\deg B}\}$ satisfy $F_B(\mathbf{v}) = \mathbf{0}$ and $L_B(\mathbf{v}) = \mathbf{0}$,

where $L_A(\mathbf{u}) = \mathbf{0}$ is a system of a general hyperplanes; and
$L_B(\mathbf{v}) = \mathbf{0}$ is a system of b general hyperplanes.

Diagonal Homotopies, when $a + b < k$

Randomize the diagonal $D(\mathbf{u}, \mathbf{v}) := \mathcal{R}(\delta(\mathbf{u}, \mathbf{v}), a + b)$.

At the start of the cascade (denote $\mathbf{z}_{1:b} = (z_1, z_2, \ldots, z_b)^T$):

$$\mathcal{E}_b(\mathbf{u}, \mathbf{v}, \mathbf{z}_{1:b}) = \left[\begin{array}{c} \mathcal{F}(\mathbf{u}, \mathbf{v}) \\ \mathcal{R}(D(\mathbf{u}, \mathbf{v}), z_1, \ldots, z_b; a + b) \\ \mathbf{z}_{1:b} - \mathcal{R}(1, \mathbf{u}, \mathbf{v}; b) \end{array} \right] = \mathbf{0}.$$

The homotopy $\left[t\ \gamma \left[\begin{array}{c} \mathcal{F}(\mathbf{u}, \mathbf{v}) \\ L_A(\mathbf{u}) \\ L_B(\mathbf{v}) \\ \mathbf{z}_{1:b} \end{array} \right] + (1 - t)\ \mathcal{E}_b(\mathbf{u}, \mathbf{v}, \mathbf{z}_{1:b}) \right] = \mathbf{0}$

starts the cascade at $t = 1$, at the $\deg A \times \deg B$ solutions,
at the product $\{(\alpha_1, \beta_1), (\alpha_1, \beta_2), \ldots, (\alpha_{\deg A}, \beta_{\deg B})\} \subset \mathbb{C}^{2k}$.

Diagonal Homotopies, when $a + b \geq k$

As $A \cap B \neq \emptyset \Rightarrow \dim(A \cap B) \geq a + b - k$, the cascade starts at

$$\mathcal{E}_b(\mathbf{u}, \mathbf{v}, \mathbf{z}_{(a+b-k+1):b}) = \left[\begin{array}{c} \mathcal{F}(\mathbf{u}, \mathbf{v}) \\ \mathcal{R}(\delta(\mathbf{u}, \mathbf{v}), z_{a+b-k+1}, \ldots, z_b; k) \\ \mathcal{R}(1, \mathbf{u}, \mathbf{v}; a + b - k) \\ \mathbf{z}_{(a+b-k+1):b} - \mathcal{R}(1, \mathbf{u}, \mathbf{v}; k - a) \end{array} \right] = 0,$$

where $\mathbf{z}_{(a+b-k+1):b} = (z_{a+b-k+1}, \ldots, z_b)^T$.

Use $$\left[t\ \gamma \left[\begin{array}{c} \mathcal{F}(\mathbf{u}, \mathbf{v}) \\ L_A(\mathbf{u}) \\ L_B(\mathbf{v}) \\ \mathbf{z}_{(a+b-k+1):b} \end{array} \right] + (1 - t)\mathcal{E}_b(\mathbf{u}, \mathbf{v}, \mathbf{z}_{(a+b-k+1):b}) \right] = \mathbf{0}$$

as before to start the cascade at $t = 1$.

Application: Adding a Leg to a Moving Platform

A special case of a Stewart-Gough platform, proposed in

M. Griffis and J. Duffy: **Method and apparatus for controlling geometrically simple parallel mechanisms with distinctive connections.** US Patent 5,179,525, 1993.

was analyzed in

M.L. Husty and A. Karger: **Self-motions of Griffis-Duffy type parallel manipulators.** In *Proc. 2000 IEEE Int. Conf. Robotics and Automation* (CDROM), 2000.

Formulation of the kinematic equations using Study coordinates has one irreducible curve of degree 28 (plus irrelevant lines).

Intersecting this motion curve with quadratic hypersurface is equivalent to adding seventh leg to the platform, reducing the motion of the platform to a number of fixed postures.

Running the Cascade

- $k = 8$: #variables = #equations of original system
- $a = 7$: dimension of hypersurface, $\deg A = 2$
- $b = 1$: dimension of motion curve, $\deg B = 28$
- $2k + b = 17$: #variables in the cascade
- $\deg A \times \deg B = 56$: #solution paths

20.3 seconds (0.3 minutes) CPU time for 40 intersection points, 16 of the 56 solution paths diverged.

Compared to the direct approach: 108.5 seconds (1.8 minutes) CPU time, for 124 of the 164 solution paths diverged.

done on 2.4 Ghz Linux machine

Conclusions and Future Work

work done:

- generalize cascade to intersect varieties
- implementation of diagonal homotopy runs

in progress:

- avoid doubling of #variables
- equation-by-equation solver