Developing Solution Sets with Polyhedral Methods (preliminary report)

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Outline

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Problem Statement

Polyhedral homotopies solve polynomial systems via degenerations to initial form systems, systems supported on faces of Newton polytopes:

1. no diverging paths for generic coefficients,
2. the sparser the system, the faster we can solve,
3. as blackbox used for numerical algebraic geometry.

Two questions:

- symbolic-numeric (exact+approximate) data structures ?
- exploitation of (permutation) symmetry ?

A.N. Jensen, H. Markwig, and T. Markwig: *An algorithm for lifting points in a tropical variety*. Collectanea Math. 59(2): 129–165, 2008.

Limits of Algebraic Sets

Let the system $f(\mathbf{x}) = \mathbf{0}$ define a curve and consider

$$\left\{ \begin{array}{c} f(\mathbf{x}) = \mathbf{0} \\ (\ \ell_1(\mathbf{x}) = 0\ )\, t + (\ x_1 - z_1 = 0\ )\,(1-t) \end{array} \right.$$

moving from a general hyperplane $\ell_1(\mathbf{x}) = \mathbf{0}$ to $x_1 - z_1 = 0$,
where z_1 is the first coordinate of $\mathbf{z} \in f^{-1}(\mathbf{0})$.

For t going from 1 to 0 in the homotopy

$$\left\{ \begin{array}{c} f(\mathbf{x}) = \mathbf{0} \\ x_1 - z_1 t = 0 \end{array} \right.$$

we push x_1 outside $\mathbb{C}^*$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

As $t \to 0$, in a polyhedral end game, applying Bernshteĭn's theorem B, f must have ≥ 2 monomials in every equation for a solution $\in (\mathbb{C}^*)^n$.

Initial Forms

System $f(\mathbf{x}) = \mathbf{0}$ has an algebraic set
$\Rightarrow f$ has initial form with ≥ 2 terms/equation.

Let $\mathbf{v} \neq \mathbf{0}$ and denote $\langle \mathbf{a}, \mathbf{v} \rangle = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$.

Then $\mathrm{in}_{\mathbf{v}}(f)$ is the initial form of f in the direction of $\mathbf{v}$:

$$\mathrm{in}_{\mathbf{v}}(f) = \sum_{\substack{\mathbf{a} \in A \\ \langle \mathbf{a}, \mathbf{v} \rangle = m}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad \mathrm{for} \quad f = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$$

where $m = \min\{ \langle \mathbf{a}, \mathbf{v} \rangle \mid \mathbf{a} \in A \}$. We say: A supports f.

A system $f = (f_1, f_2, \ldots, f_n)$ is supported on $(A_1, A_2, \ldots, A_n)$.

We look for $\mathbf{v}$ so that $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \mathbf{0}$ has solutions in $(\mathbb{C}^*)^n$.

Solving the cyclic 4-roots System

$$f(\mathbf{x}) = \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0 \\ x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 = 0 \\ x_1x_2x_3x_4 - 1 = 0 \end{cases}$$

One tropism $\mathbf{v} = (+1, -1, +1, -1)$ with $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{z}) = \mathbf{0}$:

$$\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \begin{cases} x_2 + x_4 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0 \\ x_2x_3x_4 + x_4x_1x_2 = 0 \\ x_1x_2x_3x_4 - 1 = 0 \end{cases} \qquad \begin{cases} x_1 = y_1^{+1} \\ x_2 = y_1^{-1}y_2 \\ x_3 = y_1^{+1}y_3 \\ x_4 = y_1^{-1}y_4 \end{cases}$$

The system $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{y}) = \mathbf{0}$ has two solutions.
We find two solution curves: $(t, -t^{-1}, -t, t^{-1})$ and $(t, t^{-1}, -t, -t^{-1})$.

Sparse Polynomial Systems have Sparse Solutions

the Cayley embedding

A nonzero vector $\mathbf{v}$ is a *pretropism* for the system $f(\mathbf{x}) = \mathbf{0}$
if $\#\mathrm{in}_{\mathbf{v}}(f_k) \geq 2$ for all k ranging from 1 to n.

Every tropism is a pretropism, but not every pretropism is a tropism, as pretropisms depend only on supports $A = (A_1, A_2, \ldots, A_n)$ of f.

Via the Cayley embedding we reduce A to *one* set:

$$E_A = (A_1 \times \{\mathbf{0}\}) \cup (A_2 \times \{\mathbf{e}_1\}) \cup \cdots \cup (A_n \times \{\mathbf{e}_{n-1}\})$$

where $\mathbf{e}_k$ is the k-th $(n-1)$-dimensional unit vector.

Claim: enumerating all facet normals to $\mathrm{conv}(E_A)$ yields all tropisms.

1. Tropisms for curves are normals to facets spanned by at least two points of each support.
2. Tropisms for surfaces are cones spanned by tropisms for curves.

running `cddlib`

We run `cddlib` Version 0.94b of Komei Fukuda to compute H-representation of the points in the Cayley embedding.

The H-representation of a polytope contains all facet inequalities, all half planes that define the polytope.

- On cyclic 8-roots: 831 facet normals, computed in less than one second.
- On cyclic 9-roots: 4,840 facet normals, computed in just one second.
- On cyclic 12-roots: 907,923 facet normals, took about 148.5 hours (one week).

Ran on one core of 3.07Ghz Linux with 4Gb RAM.

Processing Pretropisms

Filtering the normals to the facets of the Cayley polytope:

- Some facets are spanned by only one vertex of a polytope.
- Exploitation of permutation symmetry, for example:
  cyclic n-roots has group of size $2n$, generated by

$$\mathbf{x} \mapsto (x_2, x_3, \ldots, x_n, x_1) \quad \text{and} \quad \mathbf{x} \mapsto (x_n, x_{n-1}, \ldots, x_2, x_1).$$

  It suffices to process one pretropism per orbit.
- We let $x_1 = t^{v_1}$, $t \rightarrow 0$, need *positive* first component: $v_1 > 0$.

Processing pretropisms in two stages:

1. Find the leading coefficient of the Puiseux series.
   A solution to the initial form system may satisfy the entire system!
2. Find the second term of the Puiseux series.
   Then we have a valid starting point to develop the algebraic set with symbolic or numeric methods.

An Illustrative Example

for a numerical irreducible decomposition

$$f(x_1, x_2, x_3) = \left\{ \begin{array}{r} (x_2 - x_1^2)(x_1^2 + x_2^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ (x_2 - x_1^2)(x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{array} \right.$$

$$f^{-1}(\mathbf{0}) = Z = Z_2 \cup Z_1 \cup Z_0 = \{Z_{21}\} \cup \{Z_{11} \cup Z_{12} \cup Z_{13} \cup Z_{14}\} \cup \{Z_{01}\}$$

1. Z_{21} is the sphere $x_1^2 + x_2^2 + x_3^2 - 1 = 0$,
2. Z_{11} is the line $(x_1 = 0.5, x_3 = 0.5^3)$,
3. Z_{12} is the line $(x_1 = \sqrt{0.5}, x_2 = 0.5)$,
4. Z_{13} is the line $(x_1 = -\sqrt{0.5}, x_2 = 0.5)$,
5. Z_{14} is the twisted cubic $(x_2 - x_1^2 = 0, x_3 - x_1^3 = 0)$,
6. Z_{01} is the point $(x_1 = 0.5, x_2 = 0.5, x_3 = 0.5)$.

The Illustrative Example

numerically computing positive dimensional solution sets

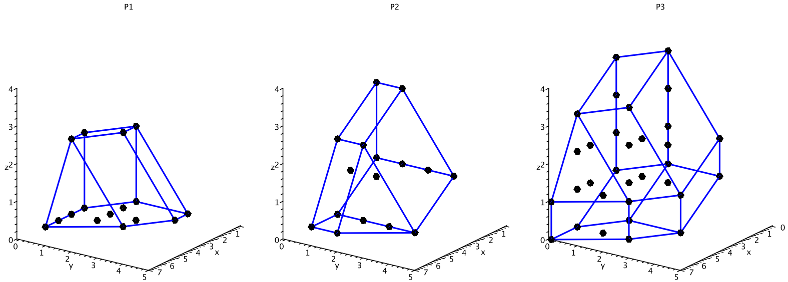
Used in two papers in numerical algebraic geometry:

- first cascade of homotopies: 197 paths
  A.J. Sommese, J. Verschelde, and C.W. Wampler: *Numerical decomposition of the solution sets of polynomial systems into irreducible components*. SIAM J. Numer. Anal. 38(6):2022–2046, 2001.

- equation-by-equation solver: 13 paths
  A.J. Sommese, J. Verschelde, and C.W. Wampler: *Solving polynomial systems equation by equation*. In Algorithms in Algebraic Geometry, Volume 146 of The IMA Volumes in Mathematics and Its Applications, pages 133–152, Springer-Verlag, 2008.

The mixed volume of the Newton polytopes of this system is 124.
By theorem A of Bernshteĭn, the mixed volume is an upper bound on the number of isolated solutions.

Three Newton Polytopes



$$f(x_1, x_2, x_3) = \begin{cases} (x_2 - x_1^2)(x_1^2 + x_2^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ (x_2 - x_1^2)(x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

Gift Wrapping for Tropisms

Gift wrapping is an algorithm to compute the convex hull, *every* $(d-2)$*-dimensional face of a* d*-dimensional polytope is the intersection of two facets.*

We call a *face(t) pretropism* an inner normal to a face(t) common to all Newton polytopes. Then a pretropism is an *edge pretropism*.

For the illustrative example, the facet pretropisms are

- $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(-1,-1,-1)$,
- the inner normals to the unit simplex,
- the Newton polytope of the common factor.

Looking for edge pretropisms: first look at a pair of polytopes.

Looking for Solution Curves

The twisted cubic is $(x_1 = t, x_2 = t^2, x_3 = t^3)$.

We look for solutions of the form

$$\begin{cases} x_1 = t^{v_1}, & v_1 > 0, \\ x_2 = c_2 t^{v_2}, & c_2 \in \mathbb{C}^*, \\ x_3 = c_3 t^{v_3}, & c_3 \in \mathbb{C}^*. \end{cases}$$

Substitute $x_1 = t, x_2 = c_2 t^2, x_3 = c_3 t^3$ into f

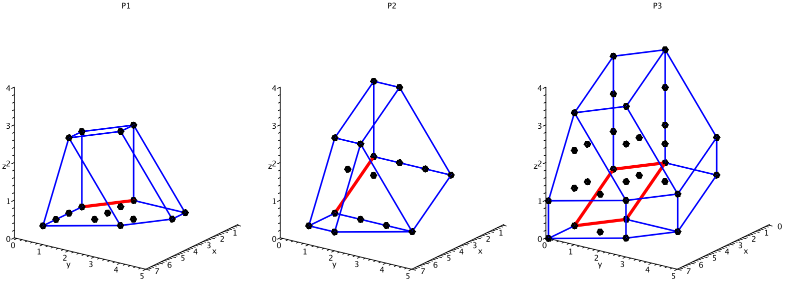
$$f(x_1 = t, x_2 = c_2 t^2, x_3 = c_3 t^3) = \begin{cases} (0.5c_2 - 0.5)t^2 + O(t^3) = 0 \\ (0.5c_3 - 0.5)t^3 + O(t^5) = 0 \\ 0.5(c_2 - 1.0)(c_3 - 1.0)t^5 + O(t^7) \end{cases}$$

$\rightarrow$ conditions on c_2 and c_3.

How to see $(v_1, v_2, v_3) = (1, 2, 3)$?

Faces of Newton Polytopes

Looking at the Newton polytopes in the direction $\mathbf{v} = (1, 2, 3)$:



Selecting those monomials supported on the faces

$$\mathrm{in}_{\mathbf{v}} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} 0.5x_2 - 0.5x_1^2 = 0 \\ 0.5x_3 - 0.5x_1^3 = 0 \\ -0.5x_2x_1^3 - 0.5x_3x_1^2 + 0.5x_3x_2 + 0.5x_1^5 = 0 \end{array} \right.$$

Degenerating the Sphere

$$f(x_1,x_2,x_3) = \begin{cases} (x_2 - x_1^2)(x_1^2 + x_2^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ (x_2 - x_1^2)(x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

As $x_1 = t \to 0$:

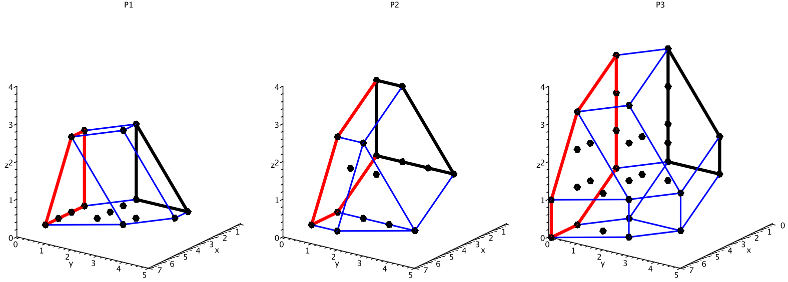
$$\mathrm{in}_{(1,0,0)} f(x_1,x_2,x_3) \begin{cases} x_2(x_2^2 + x_3^2 - 1)(-0.5) = 0 \\ x_3(x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ x_2 x_3(x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

As $x_2 = s \to 0$:

$$\mathrm{in}_{(0,1,0)} f(x_1,x_2,x_3) \begin{cases} -x_1^2(x_1^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(-0.5) = 0 \\ -x_1^2(x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

More Faces of Newton Polytopes

Looking at the Newton polytopes along $\mathbf{v} = (\mathbf{1,0,0})$ and $\mathbf{v} = (0, 1, 0)$:



$$\mathrm{in}_{(1,0,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} x_2(x_2^2 + x_3^2 - 1)(-0.5) \\ x_3(x_2^2 + x_3^2 - 1)(x_2 - 0.5) \\ x_2 x_3(x_2^2 + x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

$$\mathrm{in}_{(0,1,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} -x_1^2(x_1^2 + x_3^2 - 1)(x_1 - 0.5) \\ (x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(-0.5) \\ -x_1^2(x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

Faces of Faces

The sphere degenerates to circles at the coordinate planes.

$$\mathrm{in}_{(1,0,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} x_2(x_2^2 + x_3^2 - 1)(-0.5) \\ x_3(x_2^2 + x_3^2 - 1)(x_2 - 0.5) \\ x_2 x_3(x_2^2 + x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

$$\mathrm{in}_{(0,1,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} -x_1^2(x_1^2 + x_3^2 - 1)(x_1 - 0.5) \\ (x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(-0.5) \\ -x_1^2(x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

Degenerating even more:

$$\mathrm{in}_{(0,1,0)} \mathrm{in}_{(1,0,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} x_2(x_3^2 - 1)(-0.5) \\ x_3(x_3^2 - 1)(-0.5) \\ x_2 x_3(x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

The factor $x_3^2 - 1$ is shared with $\mathrm{in}_{(1,0,0)} \mathrm{in}_{(0,1,0)} f(x_1, x_2, x_3)$.

Representing a Solution Surface

The sphere is two dimensional, x_1 and x_2 are free:

$$\begin{cases} x_1 = t_1 \\ x_2 = t_2 \\ x_3 = 1 + c_1 t_1^2 + c_2 t_2^2. \end{cases}$$

For $t_1 = 0$ and $t_2 = 0$, $x_3 = 1$ is a solution of $x^3 - 1 = 0$.

Substituting $(x_1 = t_1, x_2 = t_2, x_3 = 1 + c_1 t_1^2 + c_2 t_2^2)$ into the original system gives linear conditions on the coefficients of the second term: $c_1 = -0.5$ and $c_2 = -0.5$.

processing pretropisms for cyclic 8-roots

`cddlib` returned 831 normals to facets of the Cayley polytope

- only 101 were pretropisms
- after permutation symmetry: 11
- up to positive sign of first component: 7

$\Rightarrow$ investigate 7 initial forms, to find 16 curves.

Transforming Coordinates

to eliminate one variable

The tropism $\mathbf{v} = (-1, 0, 0, +1, 0, -1, +1, 0)$ defines a change of coordinates:

$$\left\{ \begin{array}{l} z_0 = x_0^{-1} \\ z_1 = x_0^{0} x_1 \\ z_2 = x_0^{0} x_2 \\ z_3 = x_0^{+1} x_3 \\ z_4 = x_0^{0} x_4 \\ z_5 = x_0^{-1} x_5 \\ z_6 = x_0^{+1} x_6 \\ z_7 = x_0^{0} x_7 \end{array} \right. \qquad \mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \left\{ \begin{array}{r} 1 + x_5 = 0 \\ x_1 + x_4 x_5 + x_7 = 0 \\ x_1 x_2 + x_7 x_1 = 0 \\ x_5 x_6 x_7 + x_7 x_1 x_2 = 0 \\ x_4 x_5 x_6 x_7 + x_5 x_6 x_7 x_1 = 0 \\ x_1 x_2 x_3 x_4 x_5 + x_4 x_5 x_6 x_7 x_1 \quad \\ + x_5 x_6 x_7 x_1 x_2 = 0 \\ x_4 x_5 x_6 x_7 x_1 x_2 + x_7 x_1 x_2 x_3 x_4 x_5 = 0 \\ x_1 x_2 x_3 x_4 x_5 x_6 x_7 - 1 = 0 \end{array} \right.$$

After clearing x_0, $\mathrm{in}_{\mathbf{v}}(f)$ consists of 8 equations in 7 unknowns.

The second Term of a Puiseux Expansion

for a component of the cyclic 8-roots system

Because we find a nonzero solution for the y_k coefficients, we use it as the second term of a Puiseux expansion:

$$\left\{ \begin{array}{lll} x_0 = t^1 & & \\ x_1 = (\ 0.5 + 0.5i\ )\ t^0 & + (\ -0.5i\ )\ t & \\ x_2 = (\ 1 + i\ )\ t^0 & + (\ -i\ )\ t & \\ x_3 = (\ -i\ )\ t^0 & + (\ 1 - i\ )\ t & \\ x_4 = (\ -0.5 - 0.5i\ )\ t^0 & + (\ 0.5i\ )\ t & \quad i = \sqrt{-1}. \\ x_5 = (\ -1\ )\ t^0 & + (\ 0\ )\ t & \\ x_6 = (\ i\ )\ t^0 & + (\ -1 + i\ )\ t & \\ x_7 = (\ -1 - i\ )\ t^0 & + (\ i\ )\ t & \end{array} \right.$$

Substitute series in $f(\mathbf{x})$: result is $O(t^2)$.

Note: exploitation of symmetry is immediate.

processing pretropisms for cyclic 12-roots

`cddlib` returned 907,923 normals to facets of the Cayley polytope

- after permutation symmetry: 38,229 remained
- only 290 of those were pretropisms
- up to positive sign of first component: 158 left

Examining $\mathbf{v} = (-1, +1, -1, +1, -1, +1, -1, +1, -1, +1, -1, +1)$:
initial form system has mixed volume 49,816.

(Note: mixed volume of original system is 500,352 and increases to 983,952 after added random hyperplane and slack variable.)

Solving initial form system leads to a solution
that satisfies the entire polynomial system.

An Exact Solution for cyclic 12-roots

For the tropism $\mathbf{v} = (-1, +1, -1, +1, -1, +1, -1, +1, -1, +1, -1, +1)$:

$$
\begin{array}{ll}
z_0 = t^{-1} & z_1 = t\left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right) \\
z_2 = -t^{-1} & z_3 = t\left(-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right) \\
z_4 = t^{-1}\left(-\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) & z_5 = t\left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) \\
z_6 = -t^{-1} & z_7 = t\left(-\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) \\
z_8 = t^{-1} & z_9 = t\left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) \\
z_{10} = t^{-1}\left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right) & z_{11} = t\left(-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right)
\end{array}
$$

makes the system entirely and exactly equal to zero.

Conclusions

An apriori certificate for a solution component consists of

1. a tropism: leading powers of a Puiseux series,
2. a root at infinity: leading coefficients of the Puiseux series,
3. the next term in the Puiseux series.

The certificate is compact and easy to verify with substitution.

Preprocessing for more costly representations:

- either lifting fibers for a geometric resolution,
- or witness sets in a numerical irreducible decomposition.