Solving Polynomial Systems by Diagonal Homotopies

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Plan of the Talk

1. numerical irreducible decomposition: compute a sequence of "witness sets" to represent the solution set of a polynomial systems, arranged along dimensions.

2. intersect solution sets: given two witness sets, compute all components of their intersection using **diagonal homotopies**.

3. intrinsic coordinates for efficiency: represent the linear spaces to cut out the witness sets by generators instead of equations.

4. equation-by-equation solving: applying diagonal homotopies repeatedly, adding one equation after the other.

5. Examples illustrate effectiveness of the new solver.

Joint work with Andrew Sommese (University of Notre Dame) and Charles Wampler (General Motors Research Laboratories).

This talk is about three papers

A.J. Sommese, J. Verschelde, and C.W. Wampler: **Homotopies for intersecting solution components of polynomial systems.** To appear in ***SIAM J. Numer. Anal.***

A.J. Sommese, J. Verschelde, and C.W. Wampler: **An intrinsic homotopy for intersecting algebraic varieties.** To appear in ***J. Complexity***.

A.J. Sommese, J. Verschelde, and C.W. Wampler: **Solving polynomial systems equation by equation**. In preparation.

Towards a numerical implementation of Kronecker's ideas

M. Giusti and J. Heintz: **La détermination de la dimension et des points isolées d'une variété algébrique peuvent s'effectuer en temps polynomial**. In *Computational Algebraic Geometry and Commutative Algebra, Cortona 1991*, edited by D. Eisenbud and L. Robbiano, Symposia Mathematica XXXIV, pages 216–256, Cambridge UP, 1993.

M. Giusti and J. Heintz: **Kronecker's smart, little black boxes**, In *Foundations of Computational Mathematics*, edited by R.A. DeVore, A. Iserles and E. Süli, pages 69–104, Cambridge UP, 2001.

M. Giusti, G. Lecerf, and B. Salvy: **A Gröbner free alternative for polynomial system solving**, *J. Complexity* 17(1): 154–211, 2001.

G. Lecerf: **Computing the equidimensional decomposition of an algebraic closed set by means of lifting fibers**, *J. Complexity* 19(4): 564–596, 2003.

1. Numerical Irreducible Decomposition

$$f = \begin{bmatrix} (y-x^2)(x^2+y^2+z^2-1)(x-0.5) \\ (z-x^3)(x^2+y^2+z^2-1)(y-0.5) \\ (y-x^2)(z-x^3)(x^2+y^2+z^2-1)(z-0.5) \end{bmatrix} = \mathbf{0}.$$

The **irreducible decomposition** of $Z = f^{-1}(\mathbf{0})$ is

$$Z = Z_2 \cup Z_1 \cup Z_0 = \{Z_{21}\} \cup \{Z_{11} \cup Z_{12} \cup Z_{13} \cup Z_{14}\} \cup \{Z_{01}\}$$

where Z_{21} is the sphere $x^2+y^2+z^2-1=0$;

Z_{11} is the line $(x=0.5, z=0.5^3)$;

Z_{12} is the line $(x=\sqrt{0.5}, y=0.5)$;

Z_{13} is the line $(x=-\sqrt{0.5}, y=0.5)$;

Z_{14} is the twisted cubic $(y-x^2=0, z-x^3=0)$;

Z_{01} is the point $(x=0.5, y=0.5, z=0.5)$.

Representing Positive Dimensional Solutions

Let $f(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} \in \mathbb{C}^3$, define a surface S, a curve C, and a point P.

Choosing **random** hyperplanes, we cut the solution set of $f(\mathbf{x}) = \mathbf{0}$:

$$
\begin{array}{ccccccc}
\text{cut with} & \text{line} & \subset & \text{plane} & \subset & \text{space} \\
 & \left\{ \begin{array}{l} \mathbf{a}_1\mathbf{x} = b_1 \\ \mathbf{a}_2\mathbf{x} = b_2 \end{array} \right. & \subset & \left\{ \ \mathbf{a}_1\mathbf{x} = b_1 \right. & \subset & \left\{ \ 0 = 0 \right. \\
 & \downarrow & & \downarrow & & \downarrow \\
\text{to find} & \text{surface } S & & \text{curve } C & & \text{point } P
\end{array}
$$

$\mathbf{a}_1, \mathbf{a}_2, b_1, b_2$ **at random** $\Longrightarrow$ **generic points**

avoiding nasty, singular points on solution sets

Witness Sets

A **witness point** is a solution of a polynomial system which lies on a set of generic hyperplanes.

- The **number of generic hyperplanes** used to isolate a point from a solution component equals the **dimension** of the solution component.
- The **number of witness points** on one component cut out by the same set of generic hyperplanes equals the **degree** of the solution component.

A **witness set** for a k-dimensional solution component consists of k random hyperplanes and the set of isolated solutions comprising the intersection of the component with those hyperplanes.

A Cascade of Polynomial Systems

$$
\left\{
\begin{array}{rcl}
f_1(\mathbf{x}) + c_{11}z_1 + c_{12}z_2 & = & 0 \\
f_2(\mathbf{x}) + c_{21}z_1 + c_{22}z_2 & = & 0 \\
f_3(\mathbf{x}) + c_{31}z_1 + c_{32}z_2 & = & 0 \\
L_1(\mathbf{x}) + z_1 \phantom{+ c_{12}z_2} & = & 0 \\
L_2(\mathbf{x}) \phantom{+ c_{11}z_1} + z_2 & = & 0
\end{array}
\right.
\quad \rightarrow \quad
\left\{
\begin{array}{rcl}
f_1(\mathbf{x}) + c_{11}z_1 + c_{12}z_2 & = & 0 \\
f_2(\mathbf{x}) + c_{21}z_1 + c_{22}z_2 & = & 0 \\
f_3(\mathbf{x}) + c_{31}z_1 + c_{32}z_2 & = & 0 \\
L_1(\mathbf{x}) + z_1 \phantom{+ c_{12}z_2} & = & 0 \\
z_2 & = & 0
\end{array}
\right.
$$

$$\downarrow$$

$$L_1 : \mathbf{a_1 x} - b_1$$

$$L_2 : \mathbf{a_2 x} - b_2$$

$$z_1, z_2 : \textit{slack variables}$$

$$c_{ij} \in \mathbb{C}, \textit{random numbers}$$

$$
\left\{
\begin{array}{rcl}
f_1(\mathbf{x}) + c_{11}z_1 & = & 0 \\
f_2(\mathbf{x}) + c_{21}z_1 & = & 0 \\
f_3(\mathbf{x}) + c_{31}z_1 & = & 0 \\
z_1 & = & 0
\end{array}
\right.
$$

A Cascade of Homotopies

Denote $\mathcal{E}_i$ as an embedding of $f(\mathbf{x}) = \mathbf{0}$ with i random hyperplanes and i slack variables $\mathbf{z} = (z_1, z_2, \ldots, z_i)$.

Theorem (Sommese - Verschelde): *J. Complexity* 16(3):572–602, 2000

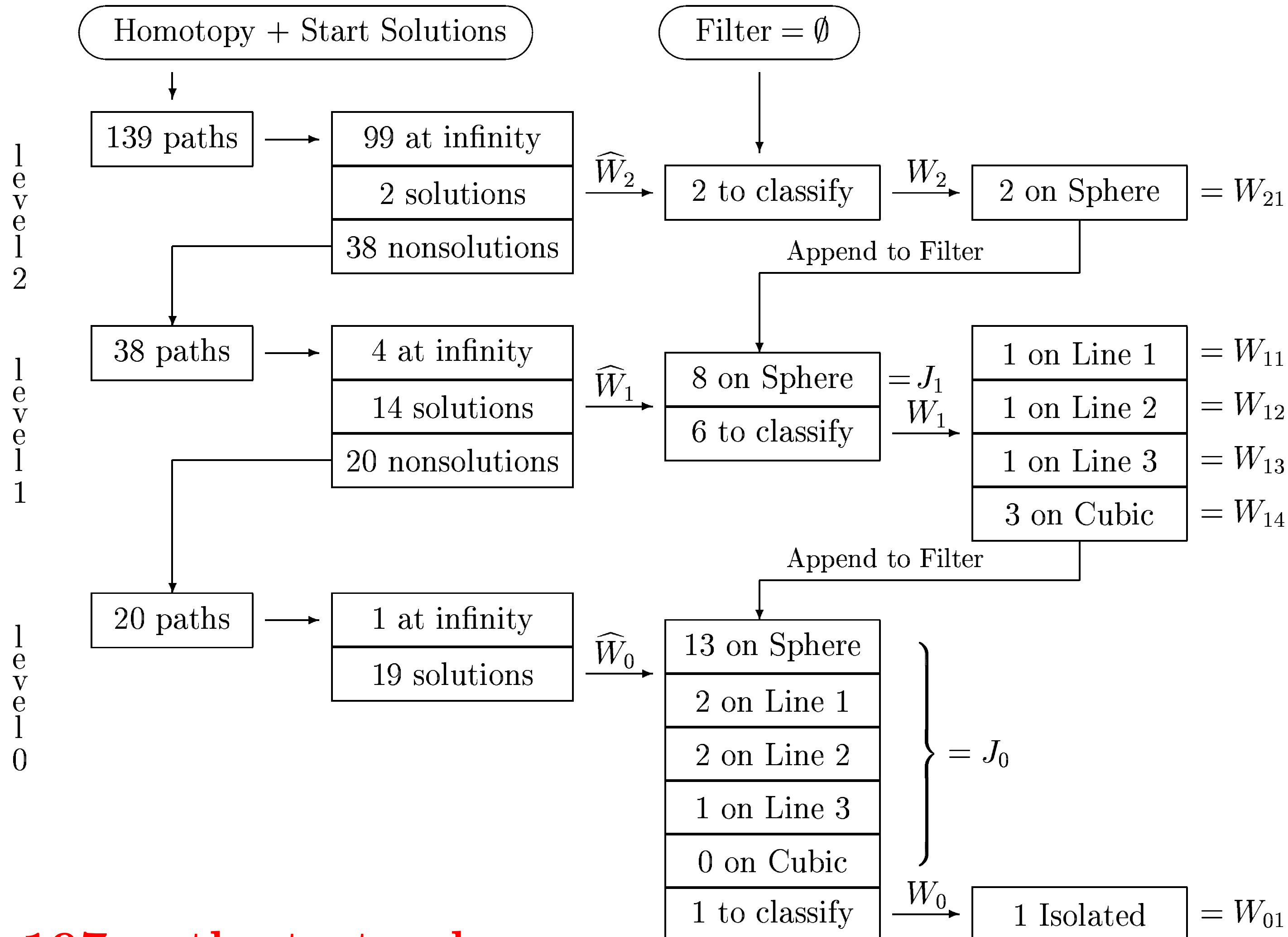
1. Solutions with $(z_1, z_2, \ldots, z_i) = \mathbf{0}$ contain $\deg W$ generic points on every i-dimensional component W of $f(\mathbf{x}) = \mathbf{0}$.
2. Solutions with $(z_1, z_2, \ldots, z_i) \neq \mathbf{0}$ are regular; and

solution paths defined by

$$H_i(\mathbf{x}, \mathbf{z}, t) = t\mathcal{E}_i(\mathbf{x}, \mathbf{z}) + (1-t)\left(\begin{array}{c} \mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) \\ z_i \end{array}\right) = \mathbf{0}$$

starting at $t = 1$ with all solutions with $z_i \neq 0$ reach at $t = 0$ all isolated solutions of $\mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) = \mathbf{0}$.

Cascade on the Illustrative Example



197 paths to track

Joint Work with A.J. Sommese and C.W. Wampler

Numerical decomposition of the solution sets of polynomial systems into irreducible components. *SIAM J. Numer. Anal.* 38(6):2022–2046, 2001.

Using monodromy to decompose solution sets of polynomial systems into irreducible components. In *Application of Algebraic Geometry to Coding Theory, Physics and Computation*, ed. by C. Ciliberto et al., Proceedings of a NATO Conference, February 25 - March 1, 2001, Eilat, Israel. Pages 297–315, Kluwer AP.

Symmetric functions applied to decomposing solution sets of polynomial systems. *SIAM J. Numer. Anal.* 40(6):2026–2046, 2002.

Numerical irreducible decomposition using PHCpack. In *Algebra, Geometry, and Software Systems*, edited by M. Joswig and N. Takayama, pages 109–130, Springer-Verlag, 2003.

Bottleneck: compute numerical representations of solution sets *efficiently*, without any assumption on the top dimension.

Membership Test

Does the point $\mathbf{p}$ *belong to a component?*

Given: a point in space $\mathbf{p} \in \mathbb{C}^N$; a system $f(\mathbf{x}) = \mathbf{0}$;

and a witness set W, $W = (Z, L)$:

$$\text{for all } \mathbf{w} \in Z : f(\mathbf{w}) = \mathbf{0} \text{ and } L(\mathbf{w}) = \mathbf{0}.$$

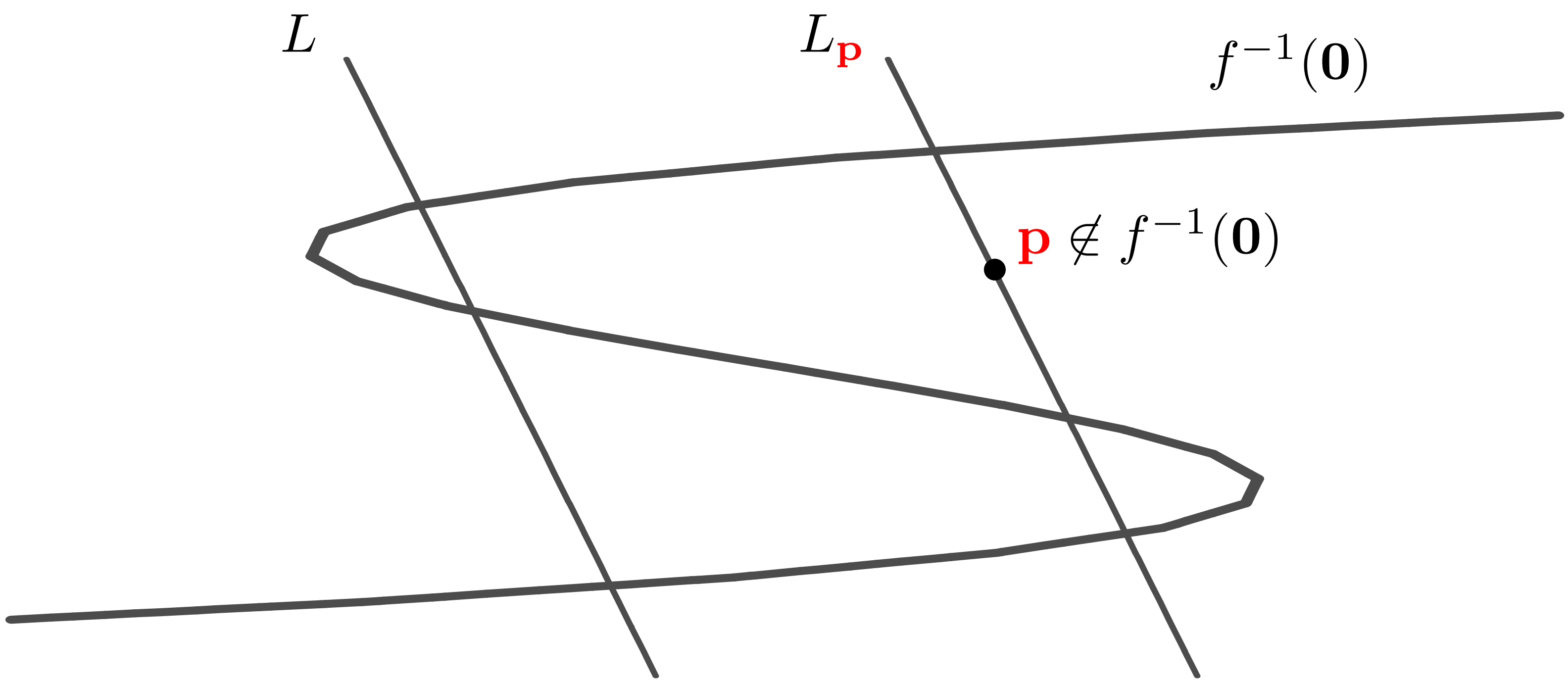
1. Let $L_{\mathbf{p}}$ be a set of hyperplanes through $\mathbf{p}$, and define

$$H(\mathbf{x}, t) = \left\{ \begin{array}{c} f(\mathbf{x}) = \mathbf{0} \\ L_{\mathbf{p}}(\mathbf{x})t + L(\mathbf{x})(1 - t) = \mathbf{0} \end{array} \right.$$

2. Trace all paths starting at $\mathbf{w} \in Z$, for t from 0 to 1.

3. The test $(\mathbf{p}, 1) \in H^{-1}(\mathbf{0})$? answers the question above.

Membership Test – an example



$$H(\mathbf{x}, t) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ L_{\mathbf{p}}(\mathbf{x})t + L(\mathbf{x})(1-t) = \mathbf{0} \end{cases}$$

2. Diagonal Homotopies: Problem Statement

Input: two irreducible components A and B, given by polynomial systems f_A and f_B (possibly identical), random hyperplanes L_A and L_B, and the solutions to

$$\left\{ \begin{array}{l} f_A(\mathbf{x}) = \mathbf{0} \\ L_A(\mathbf{x}) = \mathbf{0} \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{l} f_B(\mathbf{x}) = \mathbf{0} \\ L_B(\mathbf{x}) = \mathbf{0} \end{array} \right.$$

$$\#L_A = \dim(A) = a \qquad \#L_B = \dim(B) = b$$

$$\{\ \alpha_1, \alpha_2, \ldots, \alpha_{\deg A}\ \} \qquad \{\ \beta_1, \beta_2, \ldots, \beta_{\deg B}\ \}$$

$$\underbrace{\deg A \text{ generic points}}_{\text{a } \textit{witness set} \text{ for } A} \qquad \underbrace{\deg B \text{ generic points}}_{\text{a } \textit{witness set} \text{ for } B}$$

Output: witness sets for all pure dimensional components of $A \cap B$.

Why new homotopies are needed

stacking two (possibly identical) systems is not sufficient!

For example: find $A \cap B$,

where A is line $x_2 = 0$, solution of $f(x_1, x_2) = x_1 x_2 = 0$,

and B is line $x_1 - x_2 = 0$, solution of $g(x_1, x_2) = x_1(x_1 - x_2) = 0$.

Problem: $A \cap B = (0, 0)$ does not occur as an irreducible

$$\text{solution component of} \left\{ \begin{array}{l} f(x_1, x_2) = x_1 x_2 = 0 \\ g(x_1, x_2) = x_1(x_1 - x_2) = 0. \end{array} \right.$$

Solving Systems restricted to an Algebraic Set

Consider $f(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ over $X \times Y$, Y = parameter space.

Wanted: Solutions to $f(\mathbf{x}, \mathbf{y}^*) = \mathbf{0}$, for some $\mathbf{y}^* \in Y$.

1. **Choose** a general $\mathbf{y}' \in Y$ $(\mathbf{y}' \neq \mathbf{y}^*)$.

   $D = \#\{\ \mathbf{x} \mid f(\mathbf{x}, \mathbf{y}') = \mathbf{0}\ \}$ is maximal for all $\mathbf{y}' \in Y$.

2. **Construct** a curve $B \subset Y$ connecting $\mathbf{y}'$ to $\mathbf{y}^*$.

3. **Construct** a map $c : [0,1] \times \Gamma \rightarrow B$, $\Gamma = \{\ \gamma \in \mathbb{C} \mid |\gamma| = 1\ \}$,

   so that $c(0, \Gamma) = \mathbf{y}^*$ and $c(1, \Gamma) = \mathbf{y}'$.

4. **Choose** $\gamma \in \Gamma$ at random and track D solution paths defined by the homotopy $f(\mathbf{x}, c(t, \gamma)) = \mathbf{0}$, starting at $t = 1$ at the solutions of $f(\mathbf{x}, \mathbf{y}') = \mathbf{0}$ and ending at $t = 0$ at the desired solutions of $f(\mathbf{x}, \mathbf{y}^*) = \mathbf{0}$.

Abstract Embeddings of Polynomial Systems

X is reduced pure N-dimensional algebraic set
(abstract means: no equations specified for X)

f is system of restrictions of polynomials to X

$$\mathcal{E}(f, \mathbf{x}, \mathbf{z}, Y) = \left[\begin{array}{c} f(\mathbf{x}) + A_2^T \mathbf{z} \\ \mathbf{z} - A_0 - A_1 \mathbf{x} \end{array} \right] \qquad \begin{array}{r} Y = (A_0, A_1, A_2), A_0 \in \mathbb{C}^{N \times 1}, \\ A_1 \in \mathbb{C}^{N \times m}, A_2 \in \mathbb{C}^{N \times N}. \end{array}$$

Stratification: $Y_0 \subset Y_1 \subset \cdots \subset Y_N$, last $N - i$ rows of Y_i are zero.

Cascade of embeddings: $\mathcal{E}_i(f)$ is restricted to Y_i,

$$\mathcal{E}_N(f) = \mathcal{E}(f) \quad \text{and} \quad \mathcal{E}_0(f) = f.$$

A Generalized Cascade of Homotopies

For random $\gamma_i \in \mathbb{C}$, $|\gamma_i| = 1$, the homotopy $H_i(\mathbf{x}, \mathbf{z}, t, Y, \gamma_i)$

$$= \gamma_i \; t \; \mathcal{E}_i(f)(\mathbf{x}, \mathbf{z}, Y_i) + (1-t) \left(\begin{array}{c} \mathcal{E}_{i-1}(f)(\mathbf{x}, \mathbf{z}, Y_{i-1}) \\ z_i \end{array} \right) = \mathbf{0},$$

defines paths starting at $t = 1$ at the solutions of $\mathcal{E}_i(f)$,
ending at $t = 0$ at the solutions of $\mathcal{E}_{i-1}(f)$.

<u>Theorem:</u>

1. Solutions with $\mathbf{z} = (z_1, z_2, \ldots, z_i) \neq \mathbf{0}$ of $H_i(\mathbf{x}, \mathbf{z}, 1, Y, \gamma_i) = \mathbf{0}$ are regular, and stay regular for all $t > 0$.
2. As $t \rightarrow 0$, the solutions of $H_i(\mathbf{x}, \mathbf{z}, t, Y, \gamma_i) = \mathbf{0}$, contain all witness sets on the $(i-1)$-dimensional components of $f^{-1}(\mathbf{0})$.

A Numerical Embedding

Let X be an N-dimensional solution component of $g(\mathbf{x}) = \mathbf{0}$,
a system of n equations $g = (g_1, g_2, \ldots, g_n)$ in $\mathbf{x} \in \mathbb{C}^m$.

Randomize g to have as many equations as co-dimension of X:

$$G(\mathbf{x}) := \mathcal{R}(g(\mathbf{x}), m - N) = \Lambda g(\mathbf{x}), \quad \Lambda \in \mathbb{C}^{(m-N)\times n},$$

where Λ is a random matrix.

In the cascade of homotopies, replace $\mathcal{E}_i(f)$ by $\begin{bmatrix} G(\mathbf{x}) \\ \mathcal{E}_i(f)(\mathbf{x}, \mathbf{z}) \end{bmatrix}$.

Decomposing the Diagonal

Given two irreducible components A and B in $\mathbb{C}^k$,
consider their product $X := A \times B \subset \mathbb{C}^{k+k}$.

Then $A \cap B \cong X \cap \Delta$ where Δ is the diagonal of $\mathbb{C}^{k+k}$ defined by

$$\delta(\mathbf{u}, \mathbf{v}) := \left[\begin{array}{c} u_1 - v_1 = 0 \\ u_2 - v_2 = 0 \\ \vdots \\ u_k - v_k = 0 \end{array} \right] \quad \text{on } X.$$

Notice: δ plays role of f in the abstract embedding.

Input Data for Diagonal Homotopies

Let $A \in \mathbb{C}^k$ be an irreducible component of $f_A^{-1}(\mathbf{0})$, $\dim A = a$; and
$B \in \mathbb{C}^k$ be an irreducible component of $f_B^{-1}(\mathbf{0})$, $\dim B = b$.

Assuming $a \geq b$ and $B \not\subseteq A$, then $\dim(A \cap B) \leq b - 1$.

Randomize: $F_A(\mathbf{u}) := \mathcal{R}(f_A, k-a)$ and $F_B(\mathbf{v}) := \mathcal{R}(f_B, k-b)$.

$A \times B$ is a solution component of $\mathcal{F}(\mathbf{u}, \mathbf{v}) := \begin{bmatrix} F_A(\mathbf{u}) \\ F_B(\mathbf{v}) \end{bmatrix} = \mathbf{0}$.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_{\deg A}\}$ satisfy $F_A(\mathbf{u}) = \mathbf{0}$ and $L_A(\mathbf{u}) = \mathbf{0}$; and
$\{\beta_1, \beta_2, \ldots, \beta_{\deg B}\}$ satisfy $F_B(\mathbf{v}) = \mathbf{0}$ and $L_B(\mathbf{v}) = \mathbf{0}$,

where $L_A(\mathbf{u}) = \mathbf{0}$ is a system of a general hyperplanes; and
$L_B(\mathbf{v}) = \mathbf{0}$ is a system of b general hyperplanes.

Diagonal Homotopies, when $a + b < k$

Randomize the diagonal $D(\mathbf{u}, \mathbf{v}) := \mathcal{R}(\delta(\mathbf{u}, \mathbf{v}), a + b)$.

At the start of the cascade (denote $\mathbf{z}_{1:b} = (z_1, z_2, \ldots, z_b)^T$):

$$\mathcal{E}_b(\mathbf{u}, \mathbf{v}, \mathbf{z}_{1:b}) = \left[\begin{array}{c} \mathcal{F}(\mathbf{u}, \mathbf{v}) \\ \mathcal{R}(D(\mathbf{u}, \mathbf{v}), z_1, \ldots, z_b; a + b) \\ \mathbf{z}_{1:b} - \mathcal{R}(1, \mathbf{u}, \mathbf{v}; b) \end{array} \right] = \mathbf{0}.$$

The homotopy $\left[t\ \gamma \left[\begin{array}{c} \mathcal{F}(\mathbf{u}, \mathbf{v}) \\ L_A(\mathbf{u}) \\ L_B(\mathbf{v}) \\ \mathbf{z}_{1:b} \end{array} \right] + (1 - t)\ \mathcal{E}_b(\mathbf{u}, \mathbf{v}, \mathbf{z}_{1:b}) \right] = \mathbf{0}$

starts the cascade at $t = 1$, at the $\deg A \times \deg B$ solutions,
at the product $\{(\alpha_1, \beta_1), (\alpha_1, \beta_2), \ldots, (\alpha_{\deg A}, \beta_{\deg B})\} \subset \mathbb{C}^{2k}$.

Diagonal Homotopies, when $a + b \geq k$

As $A \cap B \neq \emptyset \Rightarrow \dim(A \cap B) \geq a + b - k$, the cascade starts at

$$\mathcal{E}_b(\mathbf{u}, \mathbf{v}, \mathbf{z}_{(a+b-k+1):b}) = \left[\begin{array}{c} \mathcal{F}(\mathbf{u}, \mathbf{v}) \\ \mathcal{R}(\delta(\mathbf{u}, \mathbf{v}), z_{a+b-k+1}, \ldots, z_b; k) \\ \mathcal{R}(1, \mathbf{u}, \mathbf{v}; a + b - k) \\ \mathbf{z}_{(a+b-k+1):b} - \mathcal{R}(1, \mathbf{u}, \mathbf{v}; k - a) \end{array} \right] = 0,$$

where $\mathbf{z}_{(a+b-k+1):b} = (z_{a+b-k+1}, \ldots, z_b)^T$.

Use $\left[t\, \gamma \left[\begin{array}{c} \mathcal{F}(\mathbf{u}, \mathbf{v}) \\ L_A(\mathbf{u}) \\ L_B(\mathbf{v}) \\ \mathbf{z}_{(a+b-k+1):b} \end{array} \right] + (1 - t)\mathcal{E}_b(\mathbf{u}, \mathbf{v}, \mathbf{z}_{(a+b-k+1):b}) \right] = \mathbf{0}$

as before to start the cascade at $t = 1$.

Application: Adding a Leg to a Moving Platform

A special case of a Stewart-Gough platform, proposed in

M. Griffis and J. Duffy: **Method and apparatus for controlling geometrically simple parallel mechanisms with distinctive connections.** US Patent 5,179,525, 1993.

was analyzed in

M.L. Husty and A. Karger: **Self-motions of Griffis-Duffy type parallel manipulators.** In *Proc. 2000 IEEE Int. Conf. Robotics and Automation* (CDROM), 2000.

Formulation of the kinematic equations using Study coordinates has one irreducible curve of degree 28 (plus irrelevant lines).

Intersecting this motion curve with quadratic hypersurface is equivalent to adding seventh leg to the platform, reducing the motion of the platform to a number of fixed postures.

Running the Cascade

- $k = 8$: #variables = #equations of original system
- $a = 7$: dimension of hypersurface, $\deg A = 2$
- $b = 1$: dimension of motion curve, $\deg B = 28$
- $2k + b = 17$: #variables in the cascade
- $\deg A \times \deg B = 56$: #solution paths

20.3 seconds CPU time to start the cascade tracing 56 paths, followed by the removal of the hyperplane to get to the 40 intersection points (16 of the 56 paths diverged) in 14.4 CPU seconds, so a total 34.7 CPU seconds.

Compared to the direct approach: 108.5 seconds (1.8 minutes) CPU time, for 124 of the 164 solution paths diverged.

done on a 2.4 Ghz Linux machine

3. Intrinsic Coordinates

$f(\mathbf{x}) = 0$ one equation in n unknowns $\mathbf{x} = (x_1, x_2, \ldots, x_n)$

defines a hypersurface in $\mathbb{C}^n$

$$\begin{cases} f(\mathbf{x}) = 0 & n \text{ equations in } n \text{ unknowns } \mathbf{x} \\ L(\mathbf{x}) = 0 & \text{to get } \deg(f) \text{ generic points} \end{cases}$$

$$L(\mathbf{x}(\lambda)) = \mathbf{0} \Leftrightarrow \mathbf{x}(\lambda) = \mathbf{b} + \lambda\mathbf{v}, \quad \mathbf{b}, \mathbf{v} \in \mathbb{C}^n$$

now reduced to $f(\mathbf{x}(\lambda)) = 0$, one equation in **one** unknown λ

in general: sample k-dimensional algebraic set in $\mathbb{C}^n$

using $n - k$ intrinsic coordinates

Embedding in Intrinsic Coordinates

$$\mathcal{E}_i(\mathbf{u},\mathbf{v},\mathbf{z}) = \begin{bmatrix} \mathcal{F}(\mathbf{u},\mathbf{v}) \\ \mathbf{A}\mathbf{w} + \mathbf{B}\pi_i\mathbf{z} \\ \mathbf{z} - \pi_i\left(\mathbf{C}\mathbf{w} + \mathbf{d}\right) \end{bmatrix} = \mathbf{0}, \qquad \begin{array}{c} \mathbf{w} = (\mathbf{u}\ \mathbf{v})^T, \mathbf{u},\mathbf{v} \in \mathbb{C}^k \\ \pi_i \text{ projects to } \mathbb{C}^i \\ \mathbf{z} \in \mathbb{C}^h, h = \text{top dim} \end{array}$$

$$\mathbf{A} = [\mathbb{A}\ \ -\mathbb{A}] \in \mathbb{C}^{(a+b)\times 2k}, \mathbf{B} \in \mathbb{C}^{(a+b)\times k}, \mathbf{C} \in \mathbb{C}^{k\times 2k}, \mathbf{d} \in \mathbb{C}^{k\times 1}$$

Eliminate $\mathbf{z}$ ($\pi_i^2 = \pi_i$): $L(\mathbf{w}) = \mathbf{A}\mathbf{w} + \mathbf{B}\pi_i(\mathbf{C}\mathbf{w} + \mathbf{d}) = \mathbf{0}$.

$$L(\mathbf{w}(\mathbf{y})) = \mathbf{0} \Leftrightarrow \mathbf{w}(\mathbf{y}) = \mathbf{b} + V\mathbf{y} : \boxed{\mathcal{E}_i(\mathbf{y}) = \mathcal{F}(\mathbf{b} + V\mathbf{y}) = \mathbf{0}}$$

nonsolution if $\mathbf{z} = \pi_i(\mathbf{C}\mathbf{w} + \mathbf{d}) \neq \mathbf{0}$

avoids doubling of variables

Cascade in Intrinsic Coordinates

Homotopy between $\mathcal{E}_i$ and $\mathcal{E}_{i-1}$:

$$H_i(\mathbf{u}, \mathbf{v}, \tau) = \left[\begin{array}{c} \mathcal{F}(\mathbf{u}, \mathbf{v}) \\ \mathbf{A}\mathbf{w} + \mathbf{B}\left(\tau\pi_i + (1-\tau)\pi_{i-1}\right)(\mathbf{C}\mathbf{w} + \mathbf{d}) \end{array} \right] = \mathbf{0},$$

for τ from 1 to 0. Define $\tau = \dfrac{t}{t + \gamma(1-t)}$, $\gamma \in \mathbb{C}$ random.

Let $\mathbf{Y}_i = \mathbf{A}\mathbf{w} + \mathbf{B}\pi_i\mathbf{C}\mathbf{w} = \mathbf{0}$, $L_i(\mathbf{w}) = \mathbf{A}\mathbf{w} + \mathbf{B}\pi_i(\mathbf{C}\mathbf{w} + \mathbf{d}) = \mathbf{0}$.

$\mathbf{Y}_i$ and $\mathbf{Y}_{i-1}$ share a common null space of dimension $2k - a - b + 1$:

$$\mathrm{Null}\mathbf{Y}_{i-1} = [E \ F] \quad \text{and} \quad \mathrm{Null}\mathbf{Y}_i = [E \ G].$$

With $\mathbf{b}$ a particular solution common to $L_i(\mathbf{w}) = \mathbf{0}$ and $L_{i-1}(\mathbf{w}) = \mathbf{0}$, the homotopy in intrinsic coordinates is

$$H_i(\mathbf{y}, t) = \mathcal{F}(\mathbf{b} + [E \ tF + \gamma(1-t)G]\mathbf{y}) = \mathbf{0}, \quad \text{for } t \text{ from 0 to 1.}$$

Numerical Experiment

adding again a leg to a moving Griffis-Duffy platform

extrinsic coordinates:	34.7 CPU seconds
intrinsic coordinates:	15.8 CPU seconds

Avoiding the doubling of variables saves here about 50% computational time.

Save **more** if intersecting **higher** dimensional sets.

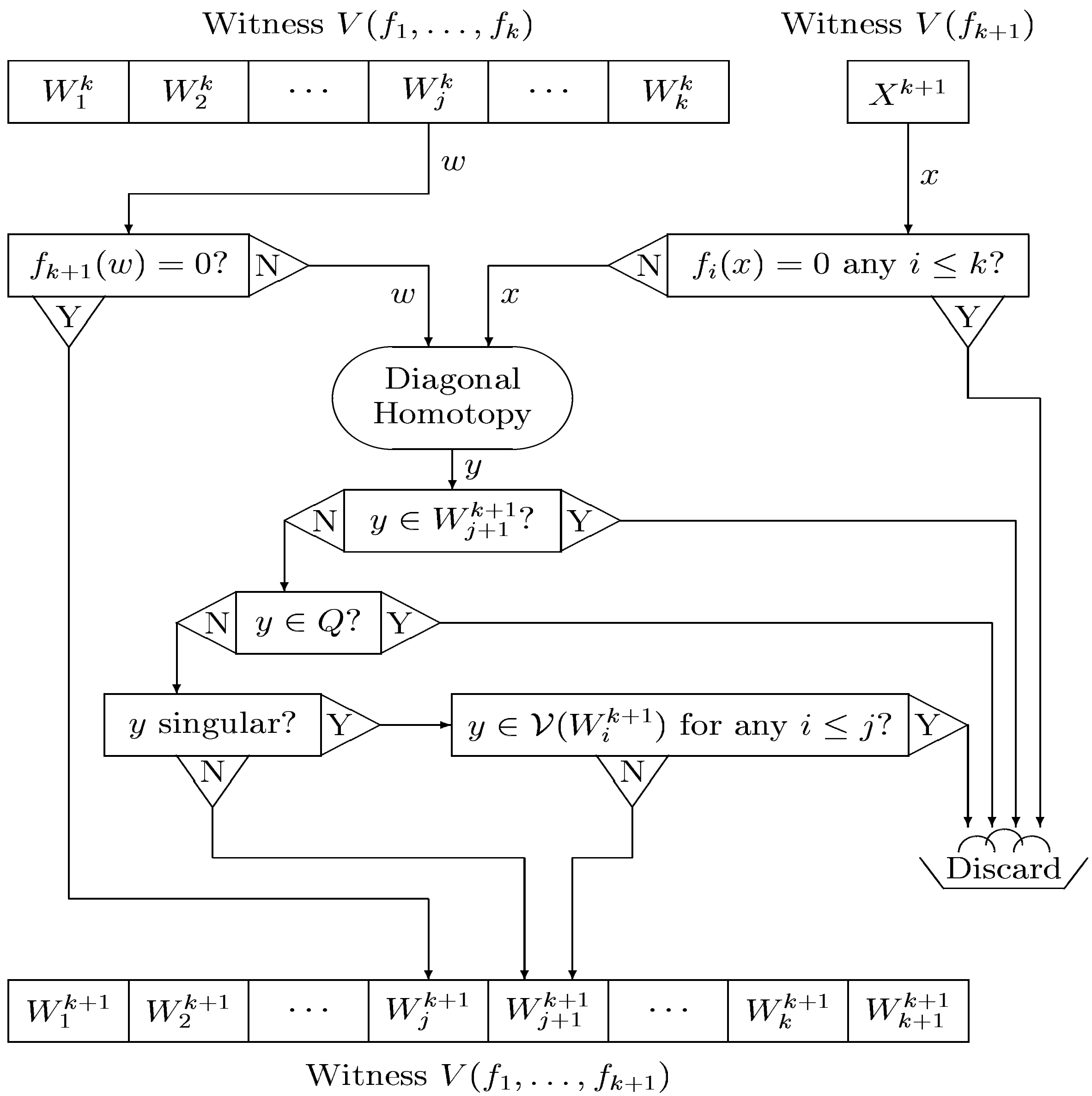
Save **less** if intersecting **lower** dimensional sets.

4. An Equation-by-Equation Solver

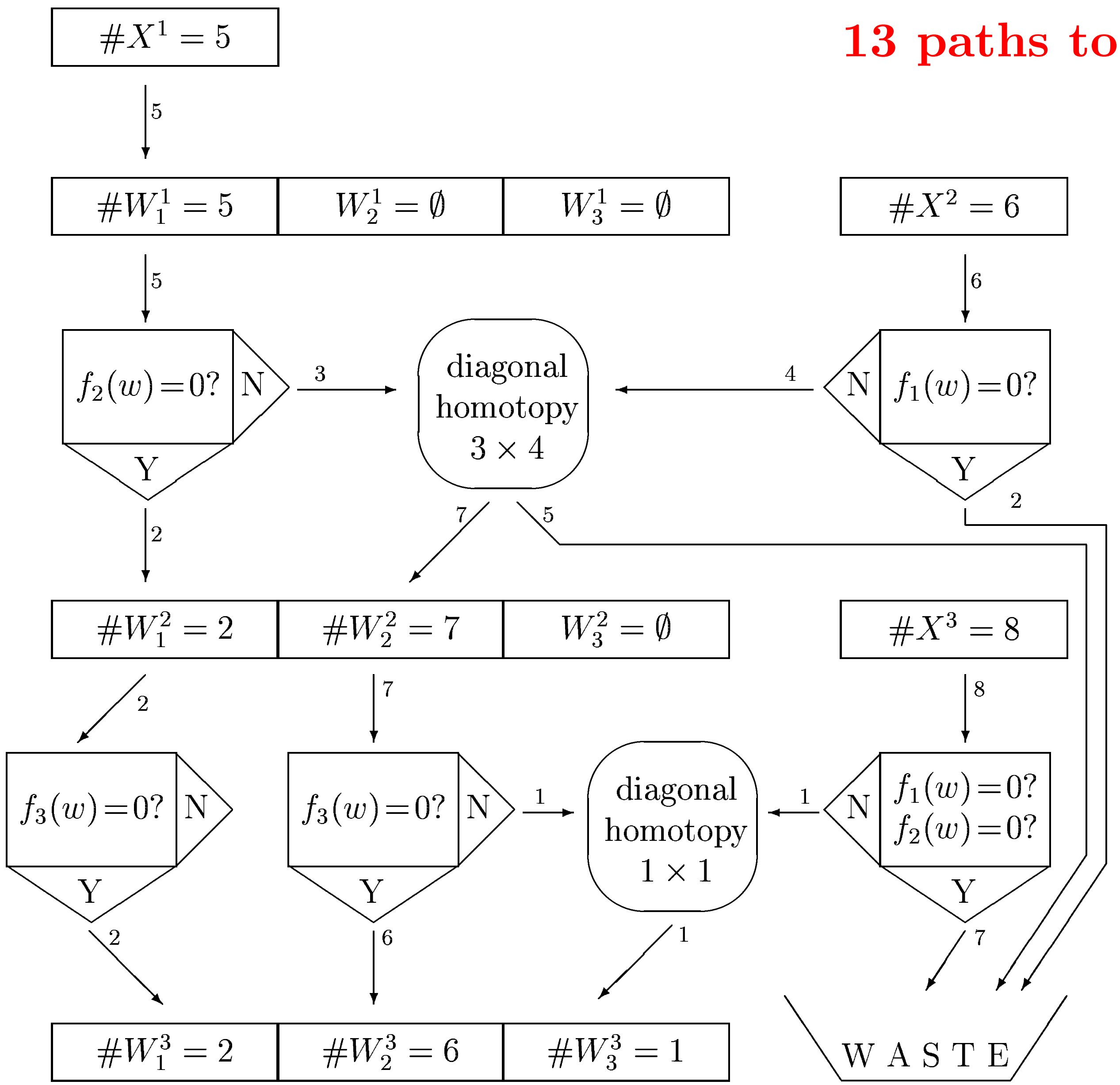
repeatedly apply diagonal homotopies to intersect with hypersurfaces

$$f = \begin{bmatrix} (y - x^2)(x^2 + y^2 + z^2 - 1)(x - 0.5) \\ (z - x^3)(x^2 + y^2 + z^2 - 1)(y - 0.5) \\ (y - x^2)(z - x^3)(x^2 + y^2 + z^2 - 1)(z - 0.5) \end{bmatrix} = \mathbf{0}.$$

Previous approach: 197 paths to find all candidate witness points.
With the new approach we will just have to trace 13 paths.


Witness $V(f_1, \ldots, f_k)$
Witness $V(f_{k+1})$
W_1^k
W_2^k
$\cdots$
W_j^k
$\cdots$
W_k^k
X^{k+1}
w
x
$f_{k+1}(w) = 0?$
N
Y
w
x
N
$f_i(x) = 0$ any $i \leq k$?
Y
Diagonal Homotopy
y
N
$y \in W_{j+1}^{k+1}$?
Y
N
$y \in Q$?
Y
y singular?
Y
$y \in \mathcal{V}(W_i^{k+1})$ for any $i \leq j$?
Y
N
N
Discard
W_1^{k+1}
W_2^{k+1}
$\cdots$
W_j^{k+1}
W_{j+1}^{k+1}
$\cdots$
W_k^{k+1}
W_{k+1}^{k+1}
Witness $V(f_1, \ldots, f_{k+1})$

13 paths to track



Adjacent Minors of a General 2-by-8 Matrix

from algebraic statistics (Diaconis, Eisenbud, Sturmfels, 1998):

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} & x_{29} \end{bmatrix}$$

8 quadrics in 18 unknowns: 10-dimensional surface of degree 256

stage	#paths			user cpu time		
1	4	=	2×2	0.11s	=	110ms
2	8	=	4×2	0.41s	=	410ms
3	16	=	8×2	1.61s	=	1s 610ms
4	32	=	16×2	3.75s	=	3s 750ms
5	64	=	32×2	12.41s	=	12s 410ms
6	128	=	64×2	34.89s	=	34s 890ms
7	256	=	128×2	104.22s	=	1m 44s 220ms
total user cpu time				157.56s	=	2m 37s 560ms

8m 22s for direct (extrinsic) homotopy

Apple PowerBook G4 1GHz

A General 6-by-6 Eigenvalue Problem

$f(\mathbf{x}, \lambda) = \lambda \mathbf{x} - A\mathbf{x} = \mathbf{0}, \quad A \in \mathbb{C}^{6 \times 6}$, A is random matrix

6 equations in 7 unknowns: curve of degree $7 < 64 = 2^6$

stage in solver	1	2	3	4	5	total
#convergent paths	3	4	5	6	7	25
#divergent paths	1	2	3	4	5	15
#paths tracked	4	6	8	10	12	40

15 is much less than $64 - 6 = 58$ divergent paths with direct homotopy, using the plain theorem of Bézout

Conclusions

Accomplishments:

+ flexible solver
+ promising performance

Future Work:

– singularities become more common
– need to exploit structure