Solving Schubert Problems with Littlewood-Richardson Homotopies

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Schubert Varieties

A Schubert variety is defined by an n-dimensional flag F:

$$F = [\mathbf{f}_1 \mathbf{f}_2 \cdots \mathbf{f}_n] \in \mathbb{C}^{n \times n} \quad \langle \mathbf{f}_1 \rangle \subset \langle \mathbf{f}_1, \mathbf{f}_2 \rangle \subset \cdots \subset \langle \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n \rangle$$

and a k-dimensional bracket $\omega \in \mathbb{N}^k$, $1 \leq \omega_1 < \omega_2 < \cdots < \omega_k \leq n$:

$$\Omega_\omega(F) = \left\{ X \in \mathbb{C}^{n \times k} \mid \dim(X \cap \langle \mathbf{f}_1, \ldots, \mathbf{f}_{\omega_i} \rangle) = i, i = 1, 2, \ldots, k \right\}.$$

For example: for $F \in \mathbb{C}^{6 \times 6}$, $\Omega_{[2\ 4\ 6]}(F)$ contains

$$X = \begin{bmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \\ x_{41} & x_{42} & x_{43} \\ 0 & x_{52} & x_{53} \\ 0 & 0 & x_{63} \end{bmatrix}$$

$$\dim(X \cap \langle \mathbf{f}_1, \mathbf{f}_2 \rangle) = 1$$
$$\dim(X \cap \langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4 \rangle) = 2$$
$$\dim(X \cap \langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4, \mathbf{f}_5, \mathbf{f}_6 \rangle) = 3$$

expressed via conditions on minors $\rightarrow$
system of 13 polynomials in 9 variables

Schubert Problems

A triple intersection $[2\ 4\ 6]^3 = [2\ 4\ 6][2\ 4\ 6][2\ 4\ 6]$ means

$$\Omega_{[2\ 4\ 6]}(I) \cap \Omega_{[2\ 4\ 6]}(M) \cap \Omega_{[2\ 4\ 6]}(F)$$

where

where	I	:	the identity matrix represents the standard flag,
	M	:	a matrix represents the moving flag,
	F	:	another matrix represents the fixed flag.

The Littlewood-Richardson rule computes the number of solutions:

$$\begin{aligned} [2\ 4\ 6]^3 &= ([2\ 4\ 6][2\ 4\ 6])[2\ 4\ 6] \\ &= ([2\ 3\ 4] + 2[1\ 3\ 5] + [1\ 2\ 6])[2\ 4\ 6] \\ &= [2\ 3\ 4][2\ 4\ 6] + 2[1\ 3\ 5][2\ 4\ 6] + [1\ 2\ 6][2\ 4\ 6] \\ &= 0 + 2[1\ 2\ 3] + 0 \end{aligned}$$

$\rightarrow$ there are 2 isolated 3-planes in $\Omega_{[2\ 4\ 6]}(I) \cap \Omega_{[2\ 4\ 6]}(M) \cap \Omega_{[2\ 4\ 6]}(F)$.

a Geometric Littlewood-Richardson Rule

William Fulton: *Young Tableau. With Applications to Representation Theory and Geometry*. Cambridge University Press, 1997.
The first geometric proof and interpretation was given by
Ravi Vakil: *a geometric Littlewood-Richardson rule*. Ann of Math, 2006.

A combinatorial checker game for the Littlewood-Richardson coefficients implies that we can

- count (enumerate) the solutions to Schubert problems,
- compute these solutions via explicit deformations.

$\longrightarrow$ Littlewood-Richardson homotopies

Motivation: experimental study of reality conjectures
`http://www.math.tamu.edu/~secant/phpfiles/monitor.php`
Christopher Hillar, Luis Garcia-Puente, Abraham Martin del Campo, James Ruffo, Zach Teitler, Stephen L. Johnson, Frank Sottile: *Experimentation at the Frontiers of Reality in Schubert Calculus.* `arXiv:0906.2497`.

Homotopies for Enumerative Geometry

- B. Huber, F. Sottile, and B. Sturmfels: Numerical Schubert calculus. *J. of Symbolic Computation*, 26(6):767–788, 1998.
- J. Verschelde: Numerical evidence for a conjecture in real algebraic geometry. *Experimental Mathematics* 9(2): 183–196, 2000.
- B. Huber and J. Verschelde: Pieri homotopies for problems in enumerative geometry applied to pole placement in linear systems control. *SIAM J. Control Optim.* 38(4):1265–1287, 2000.
- F. Sottile and B. Sturmfels: A sagbi basis for the quantum Grassmannian. *J. Pure and Appl. Algebra* 158(2-3): 347–366, 2001.
- T.Y. Li, X. Wang, and M. Wu: Numerical Schubert calculus by the Pieri homotopy algorithm. *SIAM J. Numer. Anal.* 20(2):578–600, 2002.
- J. Verschelde and Y. Wang: Computing dynamic output feedback laws. *IEEE Trans. Automat. Control.* 49(8):1393–1397, 2004.
- A. Leykin and F. Sottile: Galois group of Schubert problems via homotopy continuation. *Math. Comp.* 78(267): 1749–1765, 2009.

Running the Pieri Homotopies

One of the largest cases in the experimental study of reality conjectures is $[3\ 5\ 6]^9 = 42$, at $30,814.808$ seconds per instance.

$[3\ 5\ 6]^9$ imposes 9 hypersurface conditions on a 3-plane in $\mathbb{C}^6$, expressed as 9 cubic equations on a 3-plane in 9 variables:

$$X = \begin{bmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \\ x_{41} & x_{42} & x_{43} \\ 0 & x_{52} & x_{53} \\ 0 & 0 & x_{63} \end{bmatrix} \quad f(X) = \left\{ \begin{array}{r} \det([X|L_i]) = 0 \\ L_i \in \mathbb{C}^{6\times 3} \\ i = 1, 2, \ldots, 9. \end{array} \right.$$

Verifying Shapiro^2 conjecture on Mac OS X 2.2 Ghz Intel:

1. Pieri homotopies on a generic complex instance: 2.702 seconds.
2. Computing all 42 real 3-planes that meet 9 given 3-planes which osculate a rational normal curves takes 7.185 seconds.

The mixed volume for $f(X) = \mathbf{0}$ is $809 \gg 42$. Solves a static output placement problem of a linear system with 3 inputs and 3 outputs.

Degenerating the moving Flag

Given I : the identity matrix represents the standard flag,
M : a matrix represents the moving flag,
F : another matrix represents the fixed flag,

we consider a triple intersection for some bracket ω:

$$
\begin{array}{ll}
\text{general problem:} & \Omega_\omega(I) \cap \Omega_\omega(M) \cap \Omega_\omega(F) \\
\downarrow\uparrow & \text{degeneration} \downarrow\uparrow \text{generalization} \\
\text{degenerate problem:} & \Omega_\omega(I) \cap \Omega_\omega(I) \cap \Omega_\omega(F)
\end{array}
$$

The degeneration $M \rightarrow I$ allows to satisfy the intersection condition by solving some linear systems.

Littlewood-Richardson homotopies generalize I to M
via invertible transformations involving a parameter t.

Generalizing the moving Flag

first three moves for $n = 4$, random $\gamma_{ij} \in \mathbb{C}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{31}t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{31}t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{31} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{21}t & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{21}t & 1 & 0 \\ 0 & \gamma_{31} & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{21} & 1 & 0 \\ 0 & \gamma_{31} & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{11}t & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma_{11}t & 1 & 0 & 0 \\ \gamma_{21} & 0 & 1 & 0 \\ \gamma_{31} & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Generalizing the moving Flag

last three moves for $n = 4$, random $\gamma_{ij} \in \mathbb{C}$

$$\begin{bmatrix} \gamma_{11} & 1 & 0 & 0 \\ \gamma_{21} & 0 & 1 & 0 \\ \gamma_{31} & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{22}t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \gamma_{11} & 1 & 0 & 0 \\ \gamma_{21} & 0 & \gamma_{22}t & 1 \\ \gamma_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \gamma_{11} & 1 & 0 & 0 \\ \gamma_{21} & 0 & \gamma_{22} & 1 \\ \gamma_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{21}t & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{21}t & 1 & 0 \\ \gamma_{21} & \gamma_{22} & 0 & 1 \\ \gamma_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

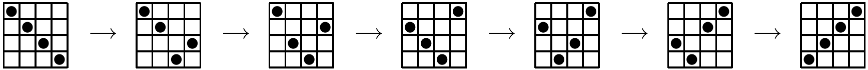
$$\begin{bmatrix} \gamma_{11} & \gamma_{21} & 1 & 0 \\ \gamma_{21} & \gamma_{22} & 0 & 1 \\ \gamma_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{13}t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{21} & \gamma_{13}t & 1 \\ \gamma_{21} & \gamma_{22} & 1 & 0 \\ \gamma_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Encoding the Moves

bubble sort on $n\ n-1\ \cdots\ 2\ 1$

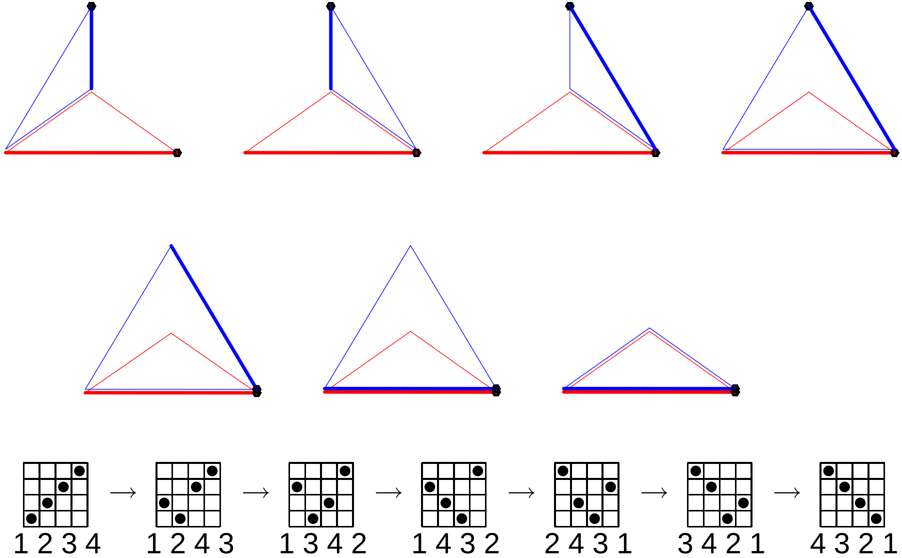
$$I \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{31} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{21} & 1 & 0 \\ 0 & \gamma_{31} & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \gamma_{11} & 1 & 0 & 0 \\ \gamma_{21} & 0 & 1 & 0 \\ \gamma_{31} & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \gamma_{11} & 1 & 0 & 0 \\ \gamma_{21} & 0 & \gamma_{22} & 1 \\ \gamma_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \gamma_{11} & \gamma_{21} & 1 & 0 \\ \gamma_{21} & \gamma_{22} & 0 & 1 \\ \gamma_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \gamma_{11} & \gamma_{21} & \gamma_{13} & 1 \\ \gamma_{21} & \gamma_{22} & 1 & 0 \\ \gamma_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



4 3 2 1 $\rightarrow$ 3 4 2 1 $\rightarrow$ 2 4 3 1 $\rightarrow$ 1 4 3 2 $\rightarrow$ 1 3 4 2 $\rightarrow$ 1 2 4 3 $\rightarrow$ 1 2 3 4

Specialization in $\mathbb{P}^3$



Littlewood-Richardson Homotopies

Degeneration of general flag from M to I in $\left(\begin{array}{c} n \\ 2 \end{array} \right)$ moves.

Three flag intersection condition $\Omega_\omega(I) \cap \Omega_\omega(M) \cap \Omega_\omega(F)$ is at the special position for $M = I$ reduced to the equations imposed on

$$X \in \Omega_\omega(F) : \quad P(X) = 0.$$

Generalizing the moving flag M leads to homotopies of the form

$$P(M(t)X) = 0, \quad t \in [0,1].$$

The solution k-plane X is represented in this moving basis $M(t)$ in suitable local coordinates, via a localization pattern.

Localization Patterns

For $X \in \Omega_{[2\ 4\ 6]}(F)$:

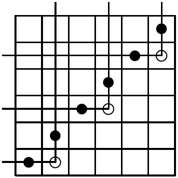
$$X = \begin{bmatrix} x_{11} & 0 & 0 \\ 1 & 0 & 0 \\ 0 & x_{32} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_{53} \\ 0 & 0 & 1 \end{bmatrix}$$

$$I = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6]$$

for any x_{11}, x_{32}, and x_{53} :

$$\dim(X \cap \langle \mathbf{e}_1, \mathbf{e}_2 \rangle) = 1$$
$$\dim(X \cap \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle) = 2$$
$$\dim(X \cap \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6 \rangle) = 3$$

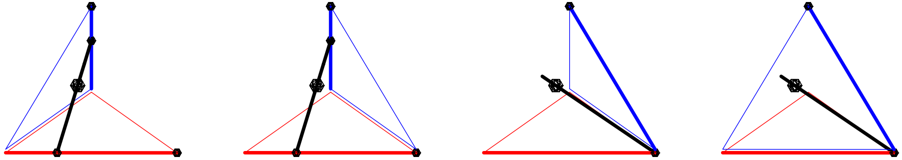
Localization patterns are encoded by white checkers:



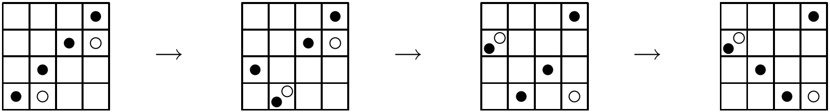
*		
1		
	*	
	1	
		*
		1

$$\begin{matrix} * & * & * & * & * & 1 \\ * & * & * & * & 1 & 0 \\ * & * & * & 1 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

a line meeting 2 lines and a fixed point in $\mathbb{P}^3$

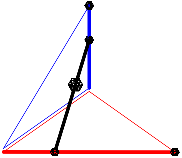


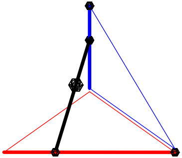
$$\begin{array}{cc} * & 0 \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array} \qquad \begin{array}{cc} * & 0 \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array} \qquad \begin{array}{cc} 0 & * \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array} \qquad \begin{array}{cc} 0 & * \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array}$$



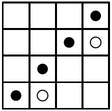
1 2 3 4 $\qquad$ 1 2 4 3 $\qquad$ 1 3 4 2 $\qquad$ 1 4 3 2

change of coordinates, no homotopy

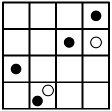




$$\begin{matrix} * & 0 \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{matrix} \qquad \begin{matrix} * & 0 \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{matrix}$$



$\longrightarrow$

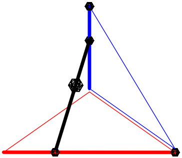


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 \\ 1 & 0 \\ 0 & x_{32} \\ 0 & 1 \end{bmatrix}$$

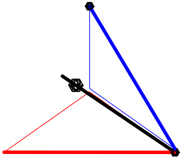
$$= \begin{bmatrix} x_{11} & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & x_{32} - 1 \end{bmatrix}$$

$$\equiv \begin{bmatrix} x_{11} & 0 \\ 1 & 0 \\ 0 & 1/(x_{32} - 1) \\ 0 & 1 \end{bmatrix}$$

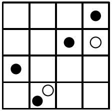
white checkers swap: use homotopy



$$\begin{array}{cc} * & 0 \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array}$$

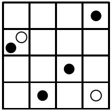


$$\begin{array}{cc} 0 & * \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array}$$



1 2 4 3

$\longrightarrow$



1 3 4 2

moving coordinates:

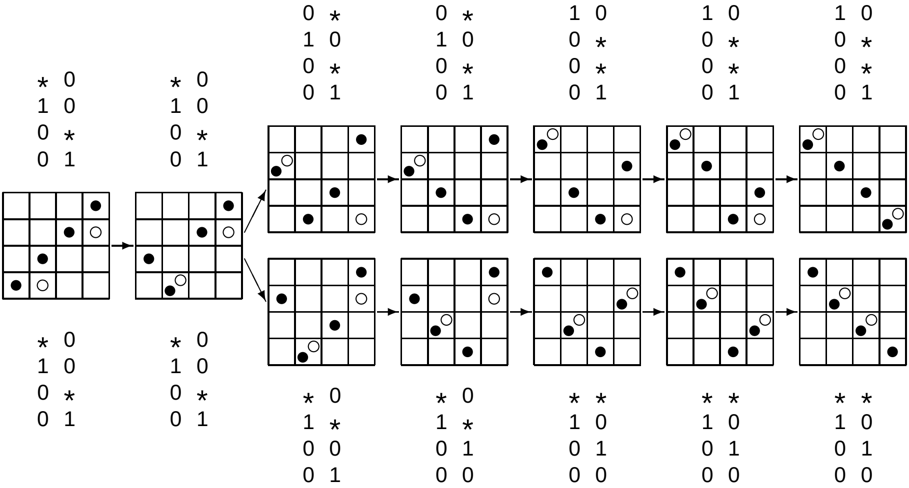
$$X(t) = \begin{bmatrix} x_{12}t & x_{12} \\ x_{32} & 0 \\ x_{32}t & x_{32} \\ 0 & 1 \end{bmatrix}$$

at $t = 0$: $X(0)$ fits right pattern

at $t = 1$: coordinate change for $X(1)$

Checker Games

resolving [2 4][2 4][2 4][2 4]



An Implementation in PHCpack

Source code and executables for PHCpack v2.3.46 are available at `http://www.math.uic.edu/~jan/download.html`

`phc -e` option #4 allows to resolve intersection conditions,

e.g.: in $\mathbb{C}^{10}$: $[6\ 8\ 10]^7 = 720[1\ 2\ 3]$,
in $\mathbb{C}^{11}$: $[7\ 9\ 11]^8 = 3598[1\ 2\ 3]$,
in $\mathbb{C}^{12}$: $[9\ 11\ 12][8\ 11\ 12]^{13} = 860574[1\ 2\ 3]$, etc...

Solving small Schubert problems on a Mac OS X 2.2 Ghz:

- $[2\ 4]^4 = 2$ takes 5 milliseconds,
- $[2\ 4\ 6]^3 = 2$ takes 169 milliseconds,
- $[2\ 5\ 8]^2[4\ 6\ 8] = 2$ takes 2.556 seconds,
- $[2\ 4\ 6\ 8]^2[2\ 5\ 7\ 8] = 3$ takes 8.595 seconds.