Solving Schubert Problems with Littlewood-Richardson Homotopies

Jan Verschelde
joint work with Frank Sottile and Ravi Vakil

University of Illinois at Chicago
Department of Mathematics, Statistics, and Computer Science
`http://www.math.uic.edu/~jan`
`jan@math.uic.edu`

ISSAC 2010 35th International Symposium on Symbolic and Algebraic Computation. T.U. München, 25-28 July 2010.

Outline

1. Numerical Schubert Calculus
   - Schubert varieties and Schubert problems
   - homotopies for enumerative geometry
2. The Moving Flag in Littlewood-Richardson Homotopies
   - specialization and generalization
   - encoding the moves by black checkers
3. 2 Lines meeting 4 given general Lines in 3-space
   - checker games encode moving flag and coordinates for solutions
   - coordinate transformations and moving coordinates in homotopy
4. Localization Patterns and Checker Movements

Schubert Varieties

A Schubert variety is defined by an n-dimensional flag F:

$$F = [\mathbf{f}_1 \mathbf{f}_2 \cdots \mathbf{f}_n] \in \mathbb{C}^{n \times n} \quad \langle \mathbf{f}_1 \rangle \subset \langle \mathbf{f}_1, \mathbf{f}_2 \rangle \subset \cdots \subset \langle \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n \rangle$$

and a k-dimensional bracket $\omega \in \mathbb{N}^k$, $1 \leq \omega_1 < \omega_2 < \cdots < \omega_k \leq n$:

$$\Omega_\omega(F) = \left\{ X \in \mathbb{C}^{n \times k} \mid \dim(X \cap \langle \mathbf{f}_1, \ldots, \mathbf{f}_{\omega_i} \rangle) = i, i = 1, 2, \ldots, k \right\}.$$

For example: for $F \in \mathbb{C}^{6 \times 6}$, $\Omega_{[2\ 4\ 6]}(F)$ contains

$$X = \begin{bmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \\ x_{41} & x_{42} & x_{43} \\ 0 & x_{52} & x_{53} \\ 0 & 0 & x_{63} \end{bmatrix}$$

$$\begin{aligned} \dim(X \cap \langle \mathbf{f}_1, \mathbf{f}_2 \rangle) &= 1 \\ \dim(X \cap \langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4 \rangle) &= 2 \\ \dim(X \cap \langle \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4, \mathbf{f}_5, \mathbf{f}_6 \rangle) &= 3 \end{aligned}$$

expressed via conditions on minors $\rightarrow$
system of 13 polynomials in 9 variables

Schubert Problems

A triple intersection $[2\ 4\ 6]^3 = [2\ 4\ 6][2\ 4\ 6][2\ 4\ 6]$ means

$$\Omega_{[2\ 4\ 6]}(I) \cap \Omega_{[2\ 4\ 6]}(M) \cap \Omega_{[2\ 4\ 6]}(F)$$

where	I	:	the identity matrix represents the standard flag,
	M	:	a matrix represents the moving flag,
	F	:	another matrix represents the fixed flag.

The Littlewood-Richardson rule computes the number of solutions:

$$\begin{aligned} [2\ 4\ 6]^3 &= ([2\ 4\ 6][2\ 4\ 6])[2\ 4\ 6] \\ &= ([2\ 3\ 4] + 2[1\ 3\ 5] + [1\ 2\ 6])[2\ 4\ 6] \\ &= [2\ 3\ 4][2\ 4\ 6] + 2[1\ 3\ 5][2\ 4\ 6] + [1\ 2\ 6][2\ 4\ 6] \\ &= 0 + 2[1\ 2\ 3] + 0 \end{aligned}$$

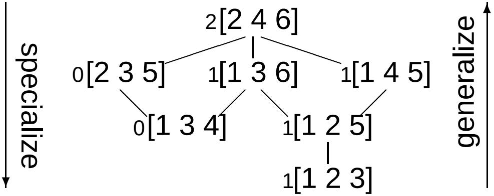
$\rightarrow$ there are 2 isolated 3-planes in $\Omega_{[2\ 4\ 6]}(I) \cap \Omega_{[2\ 4\ 6]}(M) \cap \Omega_{[2\ 4\ 6]}(F)$.

poset to resolve general problems

Resolving $[2\ 4\ 6][2\ 5\ 6]^3$:

$$
\begin{aligned}
[2\,4\,6][2\,5\,6]^3 &= (1[2\,3\,5] + 1[1\,4\,5] + 1[1\,3\,6])[2\,5\,6]^2 \\
&= (2[1\,3\,4] + 2[1\,2\,5])[2\,5\,6] \\
&= 2[1\,2\,3].
\end{aligned}
$$

Using a poset:



a Geometric Littlewood-Richardson Rule

William Fulton: *Young Tableau. With Applications to Representation Theory and Geometry*. Cambridge University Press, 1997.
The first geometric proof and interpretation was given by
Ravi Vakil: *a geometric Littlewood-Richardson rule*. Ann of Math, 2006.

A combinatorial checker game for the Littlewood-Richardson coefficients implies that we can

- count (enumerate) the solutions to Schubert problems,
- compute these solutions via explicit deformations.

$\rightarrow$ Littlewood-Richardson homotopies

Motivation: experimental study of reality conjectures
`http://www.math.tamu.edu/~secant`
Christopher Hillar, Luis Garcia-Puente, Abraham Martin del Campo, James Ruffo, Zach Teitler, Stephen L. Johnson, Frank Sottile: *Experimentation at the Frontiers of Reality in Schubert Calculus.* Contemporary Math. AMS 2010.

Homotopies for Enumerative Geometry

- B. Huber, F. Sottile, and B. Sturmfels: Numerical Schubert calculus. *J. of Symbolic Computation*, 26(6):767–788, 1998.
- J. Verschelde: Numerical evidence for a conjecture in real algebraic geometry. *Experimental Mathematics* 9(2): 183–196, 2000.
- B. Huber and J. Verschelde: Pieri homotopies for problems in enumerative geometry applied to pole placement in linear systems control. *SIAM J. Control Optim.* 38(4):1265–1287, 2000.
- F. Sottile and B. Sturmfels: A sagbi basis for the quantum Grassmannian. *J. Pure and Appl. Algebra* 158(2-3): 347–366, 2001.
- T.Y. Li, X. Wang, and M. Wu: Numerical Schubert calculus by the Pieri homotopy algorithm. *SIAM J. Numer. Anal.* 20(2):578–600, 2002.
- J. Verschelde and Y. Wang: Computing dynamic output feedback laws. *IEEE Trans. Automat. Control.* 49(8):1393–1397, 2004.
- A. Leykin and F. Sottile: Galois group of Schubert problems via homotopy continuation. *Math. Comp.* 78(267): 1749–1765, 2009.

Degenerating the moving Flag

Given	I	:	the identity matrix represents the standard flag,
	M	:	a matrix represents the moving flag,
	F	:	another matrix represents the fixed flag,

we consider a triple intersection for some bracket ω:

$$
\begin{array}{ll}
\text{general problem:} & \Omega_\omega(I) \quad \cap \quad \Omega_\omega(M) \quad \cap \quad \Omega_\omega(F) \\
\downarrow\uparrow & \qquad\qquad \textit{degeneration} \ \downarrow\uparrow \ \textit{generalization} \\
\text{degenerate problem:} & \Omega_\omega(I) \quad \cap \quad \Omega_\omega(I) \quad \cap \quad \Omega_\omega(F)
\end{array}
$$

The degeneration $M \rightarrow I$ allows to satisfy the intersection condition by solving some linear systems.

Littlewood-Richardson homotopies generalize I to M via invertible transformations involving a parameter t.

Generalizing the moving Flag

first three moves for $n = 4$, random $\gamma_{ij} \in \mathbb{C}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{31}t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{31}t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{31} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{21}t & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{21}t & 1 & 0 \\ 0 & \gamma_{31} & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{21} & 1 & 0 \\ 0 & \gamma_{31} & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{11}t & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma_{11}t & 1 & 0 & 0 \\ \gamma_{21} & 0 & 1 & 0 \\ \gamma_{31} & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Generalizing the moving Flag

last three moves for $n = 4$, random $\gamma_{ij} \in \mathbb{C}$

$$\begin{bmatrix} \gamma_{11} & 1 & 0 & 0 \\ \gamma_{21} & 0 & 1 & 0 \\ \gamma_{31} & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{22}t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \gamma_{11} & 1 & 0 & 0 \\ \gamma_{21} & 0 & \gamma_{22}t & 1 \\ \gamma_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \gamma_{11} & 1 & 0 & 0 \\ \gamma_{21} & 0 & \gamma_{22} & 1 \\ \gamma_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{21}t & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{21}t & 1 & 0 \\ \gamma_{21} & \gamma_{22} & 0 & 1 \\ \gamma_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

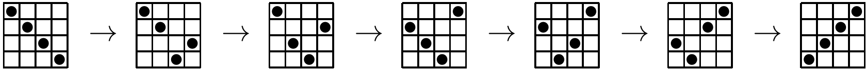
$$\begin{bmatrix} \gamma_{11} & \gamma_{21} & 1 & 0 \\ \gamma_{21} & \gamma_{22} & 0 & 1 \\ \gamma_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{13}t & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{21} & \gamma_{13}t & 1 \\ \gamma_{21} & \gamma_{22} & 1 & 0 \\ \gamma_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Encoding the Moves

bubble sort on $n\ n-1\ \cdots\ 2\ 1$

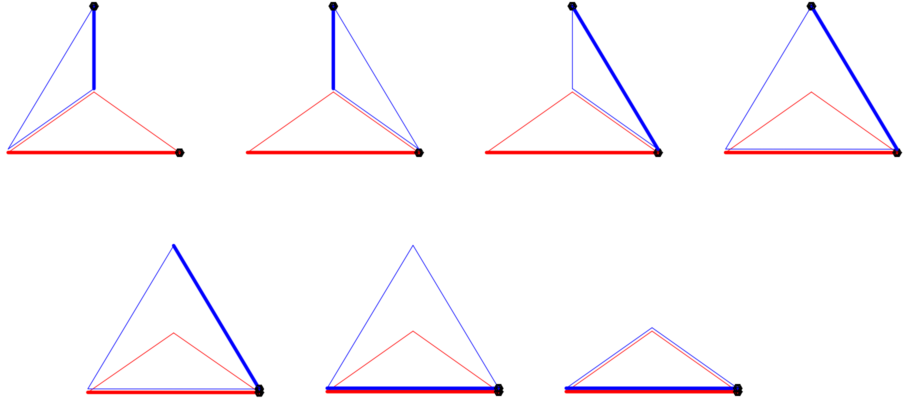
$$I \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{31} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{21} & 1 & 0 \\ 0 & \gamma_{31} & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \gamma_{11} & 1 & 0 & 0 \\ \gamma_{21} & 0 & 1 & 0 \\ \gamma_{31} & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

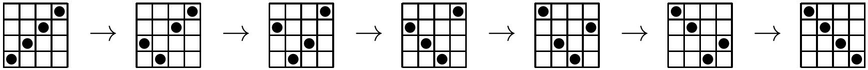
$$\to \begin{bmatrix} \gamma_{11} & 1 & 0 & 0 \\ \gamma_{21} & 0 & \gamma_{22} & 1 \\ \gamma_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \gamma_{11} & \gamma_{21} & 1 & 0 \\ \gamma_{21} & \gamma_{22} & 0 & 1 \\ \gamma_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \gamma_{11} & \gamma_{21} & \gamma_{13} & 1 \\ \gamma_{21} & \gamma_{22} & 1 & 0 \\ \gamma_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



$4\,3\,2\,1 \to 3\,4\,2\,1 \to 2\,4\,3\,1 \to 1\,4\,3\,2 \to 1\,3\,4\,2 \to 1\,2\,4\,3 \to 1\,2\,3\,4$

Specialization in $\mathbb{P}^3$





1 2 3 4 $\rightarrow$ 1 2 4 3 $\rightarrow$ 1 3 4 2 $\rightarrow$ 1 4 3 2 $\rightarrow$ 2 4 3 1 $\rightarrow$ 3 4 2 1 $\rightarrow$ 4 3 2 1

Littlewood-Richardson Homotopies

Degeneration of general flag from M to I in $\left(\begin{array}{c} n \\ 2 \end{array}\right)$ moves.

Three flag intersection condition $\Omega_\omega(I) \cap \Omega_\omega(M) \cap \Omega_\omega(F)$ is at the special position for $M = I$ reduced to the equations imposed on

$$X \in \Omega_\omega(F): \quad P(X) = 0.$$

Generalizing the moving flag M leads to homotopies of the form

$$P(M(t)X) = 0, \quad t \in [0,1].$$

The solution k-plane X is represented in this moving basis $M(t)$ in suitable local coordinates, via a localization pattern.

The Problem of Four General Lines

Classical problem in projective 3-space $\mathbb{P}^3$:

Given four general lines in $\mathbb{P}^3$ (no triplet is coplanar), find all lines that meet those four given lines nontrivially.

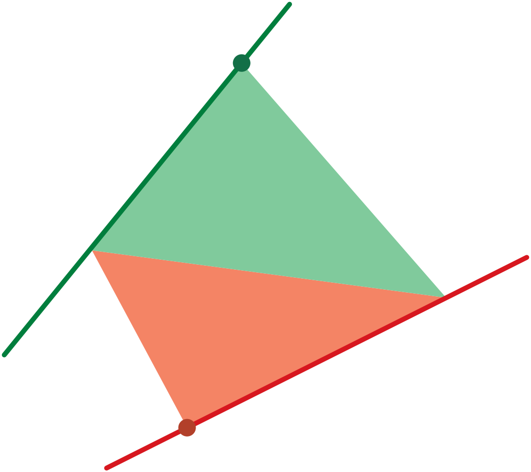
Identify $\mathbb{P}^3$ with $\mathbb{C}^4$ with natural basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$.
Let $L_1 = \mathrm{span}(\mathbf{e}_1, \mathbf{e}_2)$ and $L_2 = \mathrm{span}(\mathbf{e}_3, \mathbf{e}_4)$, consider

$$X = \begin{bmatrix} 1 & 0 \\ x_{2,1} & 0 \\ 0 & 1 \\ 0 & x_{4,2} \end{bmatrix} \quad P(X) = \begin{cases} \det([X|L_3]) = 0 \\ \det([X|L_4]) = 0 \end{cases}$$

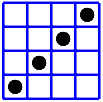
where X meets L_1 and L_2 by its pattern.

Specializing L_2 to coincide with L_1 makes X change pattern so $P(X) = \mathbf{0}$ becomes a linear system.

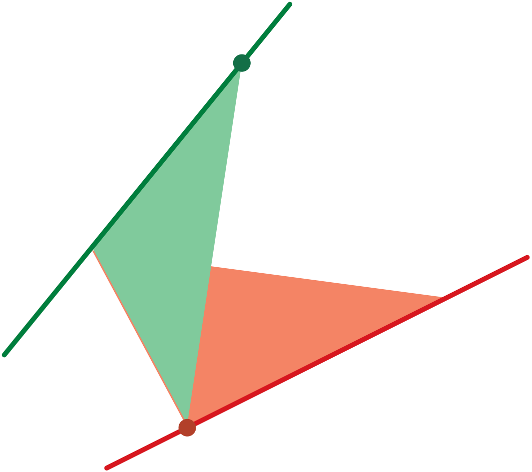
Specializing Moving Flag to Fixed Flag – stage 0



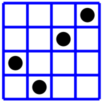
$$\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$



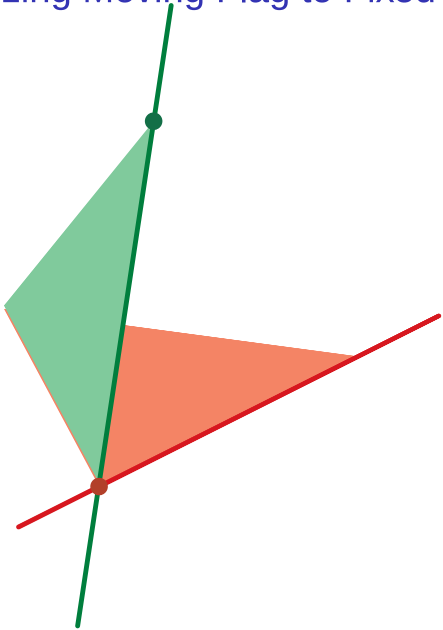
Specializing Moving Flag to Fixed Flag – stage 1



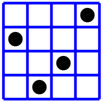
1 1 1 0
1 1 0 1
1 1 0 0
1 0 0 0



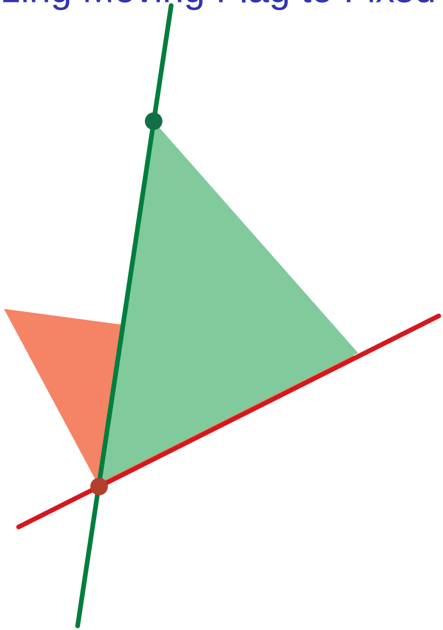
Specializing Moving Flag to Fixed Flag – stage 2



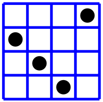
$$\begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$



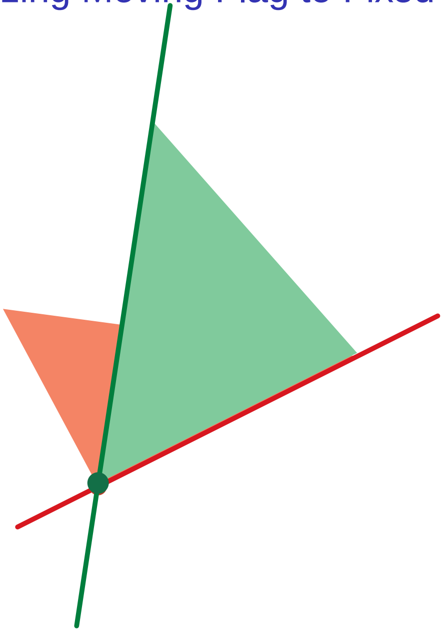
Specializing Moving Flag to Fixed Flag – stage 3



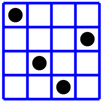
$$\begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix}$$



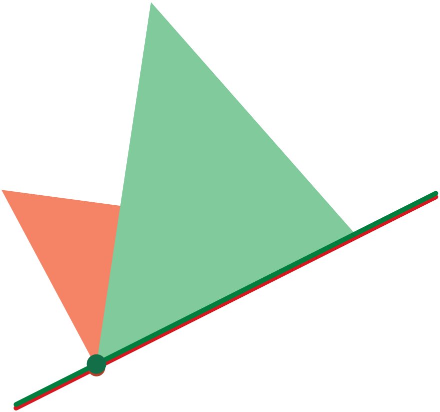
Specializing Moving Flag to Fixed Flag – stage 4



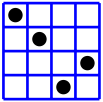
$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{matrix}$$



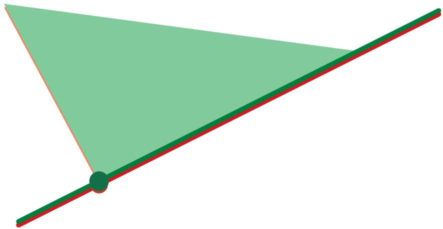
Specializing Moving Flag to Fixed Flag – stage 5



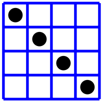
$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}$$



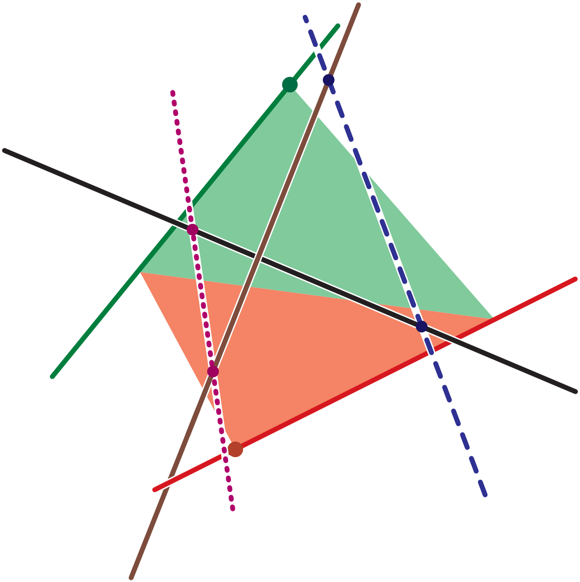
Specializing Moving Flag to Fixed Flag – stage 6



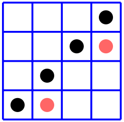
$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$



Resolving the Problem of Four Lines – stage 0

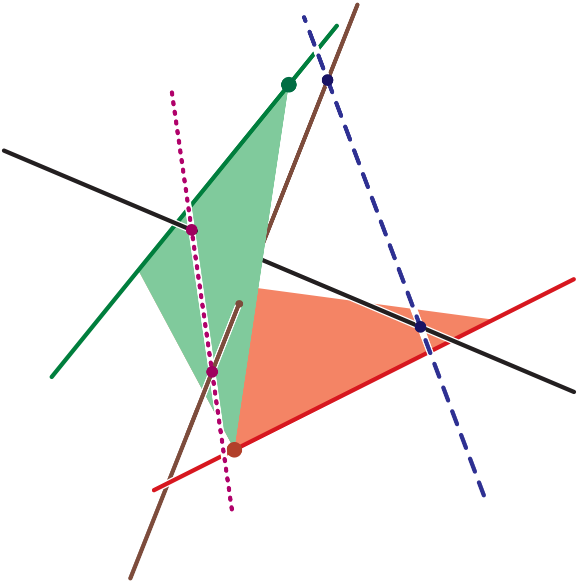


$$\begin{matrix} * & 0 \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{matrix}$$

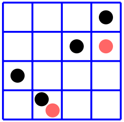


$$\begin{matrix} * & 0 \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{matrix}$$

Resolving the Problem of Four Lines – stage 1

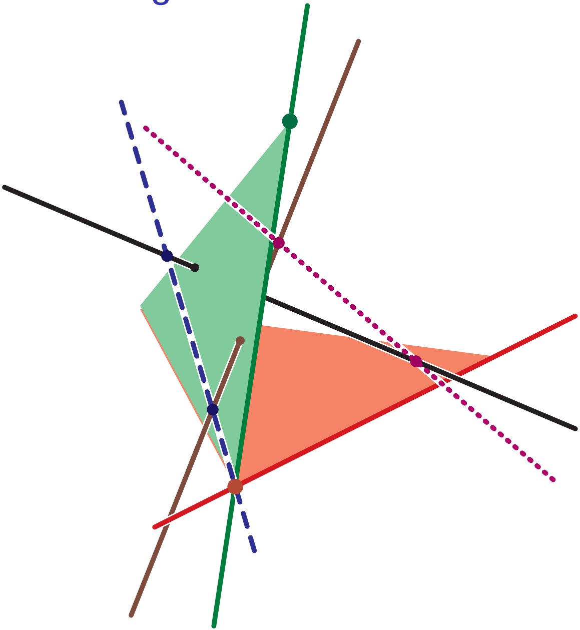


$$\begin{array}{cc} * & 0 \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array}$$

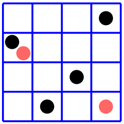


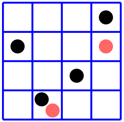
$$\begin{array}{cc} * & 0 \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array}$$

Resolving the Problem of Four Lines – stage 2



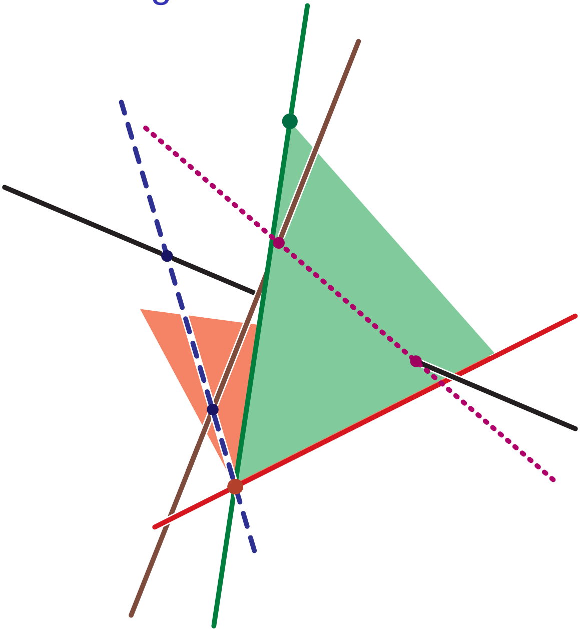
$$\begin{array}{cc} 0 & * \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array}$$



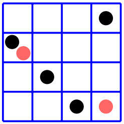


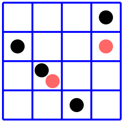
$$\begin{array}{cc} * & 0 \\ 1 & * \\ 0 & 0 \\ 0 & 1 \end{array}$$

Resolving the Problem of Four Lines – stage 3



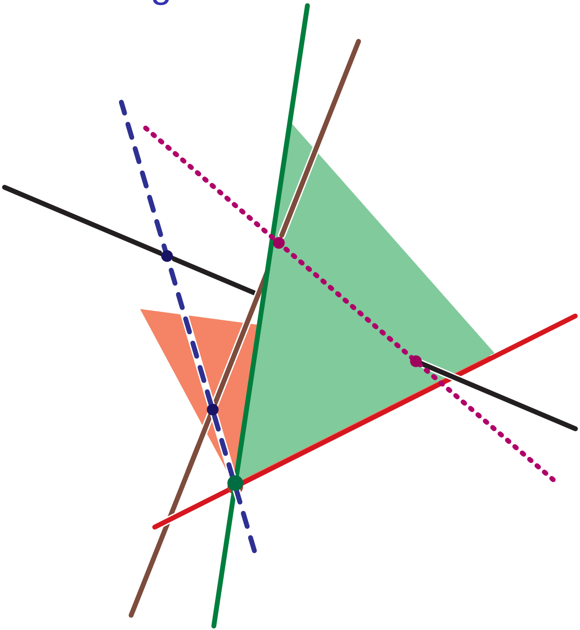
$$\begin{array}{cc} 0 & * \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array}$$



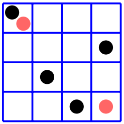


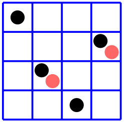
$$\begin{array}{cc} * & 0 \\ 1 & * \\ 0 & 1 \\ 0 & 0 \end{array}$$

Resolving the Problem of Four Lines – stage 4



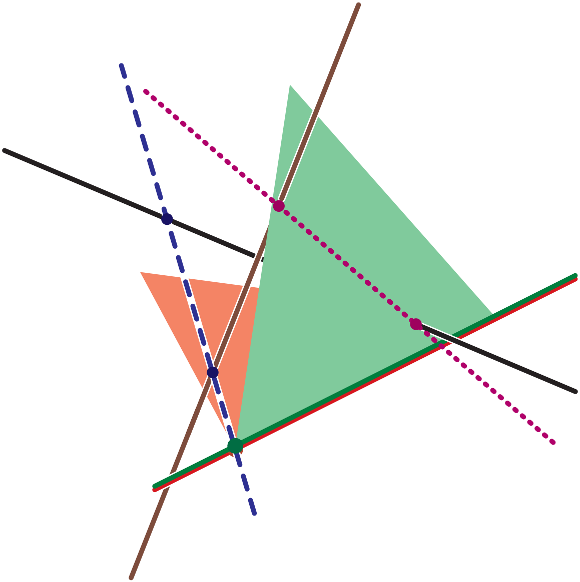
$$\begin{matrix} 1 & 0 \\ 0 & * \\ 0 & * \\ 0 & 1 \end{matrix}$$



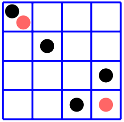


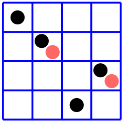
$$\begin{matrix} * & * \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}$$

Resolving the Problem of Four Lines – stage 5



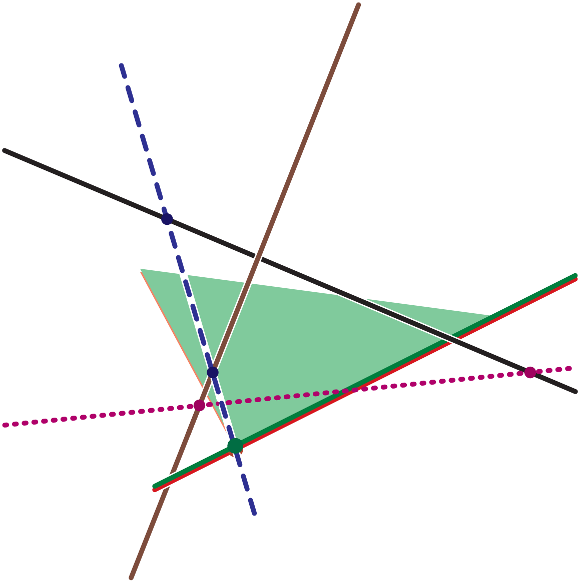
$$\begin{array}{cc} 1 & 0 \\ 0 & * \\ 0 & * \\ 0 & 1 \end{array}$$



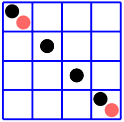


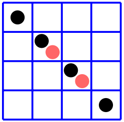
$$\begin{array}{cc} * & * \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}$$

Resolving the Problem of Four Lines – stage 6



$$\begin{matrix} 1 & 0 \\ 0 & * \\ 0 & * \\ 0 & 1 \end{matrix}$$

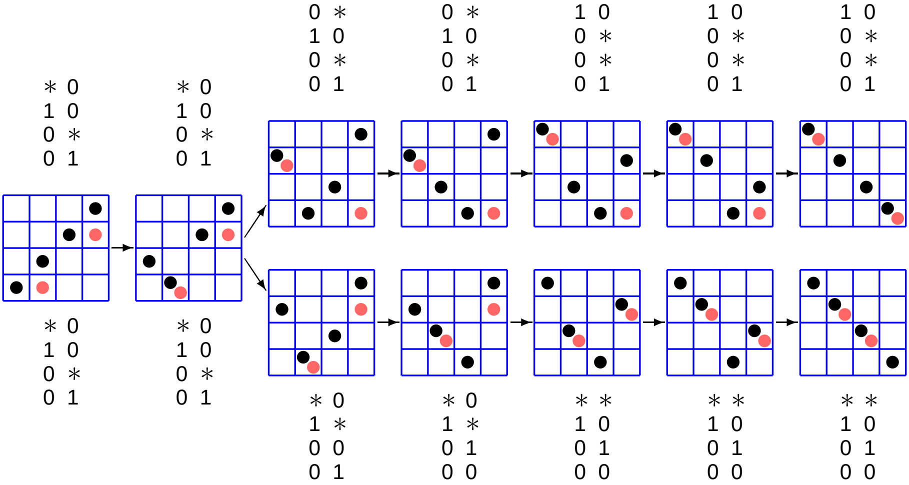




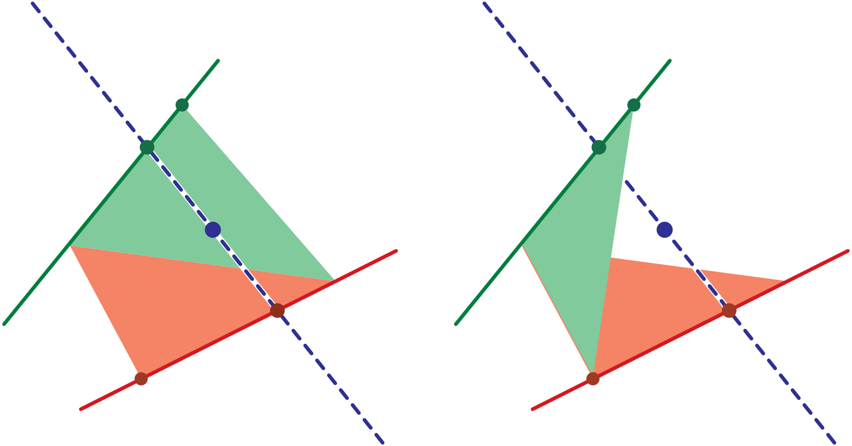
$$\begin{matrix} * & * \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}$$

Checker Games

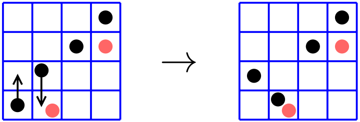
resolving [2 4][2 4][2 4][2 4]



no homotopy, only change of coordinates



$$\begin{array}{cc} * & 0 \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array}$$

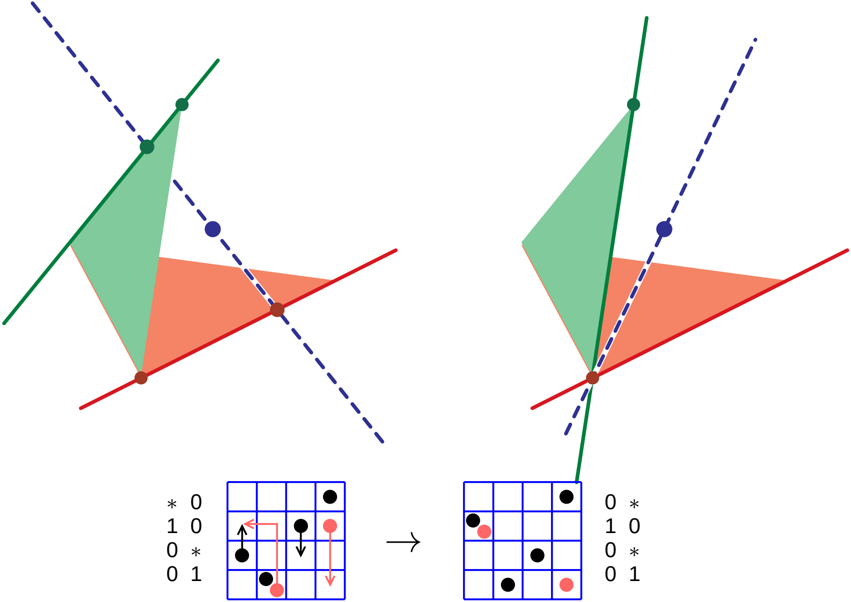


$$\begin{array}{cc} * & 0 \\ 1 & 0 \\ 0 & * \\ 0 & 1 \end{array}$$

corresponding coordinate transformation

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}
\begin{bmatrix} x_{11} & 0 \\ 1 & 0 \\ 0 & x_{32} \\ 0 & 1 \end{bmatrix}
= \begin{bmatrix} x_{11} & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & x_{32}-1 \end{bmatrix}
\equiv \begin{bmatrix} x_{11} & 0 \\ 1 & 0 \\ 0 & 1/(x_{32}-1) \\ 0 & 1 \end{bmatrix}
$$

homotopy, as red checkers swap rows



homotopy as red checkers swap

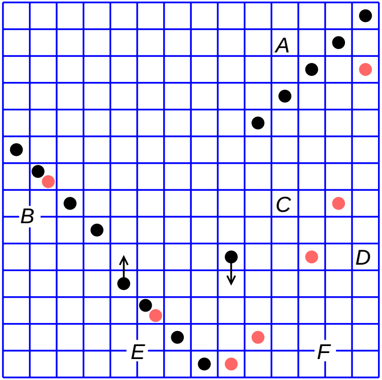
Similar to the case of a line meeting two lines and a fixed point, we use a homotopy:

$$X(t) = \begin{bmatrix} x_{12}t & x_{12} \\ x_{32} & 0 \\ x_{32}t & x_{32} \\ 0 & 1 \end{bmatrix}.$$

At $t = 0$, $X(0)$ fits the pattern.

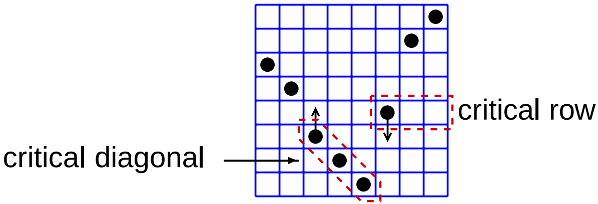
At $t = 1$, a coordinate change makes $X(1)$ to fit the pattern.

Localization Patterns



$$\begin{bmatrix}
x_{1,1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_{2,1} & \cdot & x_{2,3} & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & x_{3,3} & x_{3,4} & \cdot & \cdot & \cdot \\
\cdot & \cdot & x_{4,3} & x_{4,4} & \cdot & \cdot & \cdot \\
\cdot & \cdot & x_{5,3} & x_{5,4} & \cdot & x_{5,6} & \cdot \\
\cdot & x_{6,2} & x_{6,3} & x_{6,4} & x_{6,5} & x_{6,6} & x_{6,7} \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & x_{8,4} & x_{8,5} & x_{8,6} & x_{8,7} \\
\cdot & \cdot & \cdot & x_{9,4} & x_{9,5} & x_{9,6} & x_{9,7} \\
\cdot & \cdot & \cdot & 1 & \cdot & x_{10,6} & x_{10,7} \\
\cdot & \cdot & \cdot & \cdot & x_{11,5} & x_{11,6} & x_{11,7} \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & x_{13,7} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{bmatrix}$$

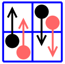
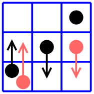
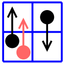
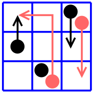
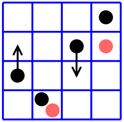
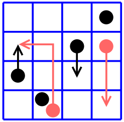
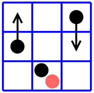
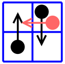
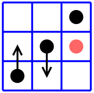
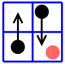
Rising and Falling Checkers



To resolve $\Omega_\omega(F) \cap \Omega_\tau(M)$, 9 cases to consider:

1. Where is the top red checker in the critical diagonal?
   - (a) In the rising checker's square.
   - (b) Elsewhere in the critical diagonal.
   - (c) There is no red checker in the critical diagonal.
2. Where is the red checker in the critical row?
   - (α) In the descending checker's square.
   - (β) Elsewhere in the critical row.
   - (γ) There is no red checker in the critical row.

Movement of Red Checkers

	α	β	γ
a			
b		 or 	
c			

An Implementation in PHCpack

Since v2.3.46 Littlewood-Richardson homotopies are in PHCpack, at `http://www.math.uic.edu/~jan/download.html`

`phc -e` option #4 allows to resolve intersection conditions,

e.g.: in $\mathbb{C}^{10}$: $[6\ 8\ 10]^7 = 720[1\ 2\ 3]$,
in $\mathbb{C}^{11}$: $[7\ 9\ 11]^8 = 3598[1\ 2\ 3]$,
in $\mathbb{C}^{12}$: $[9\ 11\ 12][8\ 11\ 12]^{13} = 860574[1\ 2\ 3]$, etc...

Solving small Schubert problems on a Mac OS X 2.2 Ghz:

- $[2\ 4]^4 = 2$ takes 5 milliseconds,
- $[2\ 4\ 6]^3 = 2$ takes 169 milliseconds,
- $[2\ 5\ 8]^2[4\ 6\ 8] = 2$ takes 2.556 seconds,
- $[2\ 4\ 6\ 8]^2[2\ 5\ 7\ 8] = 3$ takes 8.595 seconds.