solving polynomial systems with Puiseux series

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Outline

1. Introduction
   - solving sparse polynomial systems
   - space curves and initial forms
2. Solving Binomial Systems
   - unimodular coordinate transformations
   - computation of the degree and affine sets
3. Polyhedral Methods for Algebraic Sets
   - computing pretropisms with the Cayley embedding
   - Puiseux series for algebraic sets
   - application to the cyclic *n*-roots problem

polynomial systems

Consider $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, a system of equations defined by

- N polynomials $\mathbf{f} = (f_0, f_1, \ldots, f_{N-1})$,
- in n variables $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$.

A polynomial in n variables consists of a vector of nonzero complex coefficients with corresponding exponents in A:

$$f_k(\mathbf{x}) = \sum_{\mathbf{a} \in A_k} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C} \setminus \{0\}, \quad \mathbf{x}^{\mathbf{a}} = x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}.$$

Input data:

1. $A = (A_0, A_1, \ldots, A_{N-1})$ are sets of exponents, the *supports*. For $\mathbf{a} \in \mathbb{Z}^n$, we consider *Laurent* polynomials, $f_k \in \mathbb{C}[\mathbf{x}^{\pm 1}]$ $\Rightarrow$ only solutions with coordinates in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ matter.
2. $\mathbf{c}_A = (\mathbf{c}_{A_0}, \mathbf{c}_{A_1}, \ldots, \mathbf{c}_{A_{N-1}})$ are vectors of complex coefficients. Although A is exact, the coefficients may be approximate.

the cyclic 4-roots system

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0 \end{cases}$$

Cyclic 4-roots $\mathbf{x} = (x_0, x_1, x_2, x_3)$ correspond to complex circulant Hadamard matrices:

$$H = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ x_3 & x_0 & x_1 & x_2 \\ x_2 & x_3 & x_0 & x_1 \\ x_1 & x_2 & x_3 & x_0 \end{bmatrix}, \qquad |x_k| = 1, k = 1,2,3,4 \qquad H^*H = 4I_4.$$

- Haagerup: for prime p, there are $\begin{pmatrix} 2p-2 \\ p-1 \end{pmatrix}$ isolated roots.
- Backelin: for $n = \ell m^2$, there are infinitely many cyclic n-roots.

solving polynomial systems

Systems like cyclic n-roots are

- Sparse: relative to the degrees of the polynomials,

  few monomials appear with nonzero coefficients
  $\Rightarrow$ fewer roots than the Bézout bounds.

- Symmetric: solutions are invariant under permutations, $n = 4$:
  $(x_0, x_1, x_2, x_3) \rightarrow (x_1, x_2, x_3, x_0)$ and $(x_0, x_1, x_2, x_3) \rightarrow (x_3, x_2, x_1, x_0)$
  generate the permutation group.
  In addition: $(x_0, x_1, x_2, x_3) \rightarrow (x_0^{-1}, x_1^{-1}, x_2^{-1}, x_3^{-1})$.

- Not pure dimensional, for prime n, all solutions are isolated,
  but for $n = \ell m^2$, we have positive dimensional solution sets.

Our solution is to apply a hybrid symbolic-numeric approach.

Puiseux series

The Newton polygon of $f(x_0, x_1)$ is the convex hull, spanned by the exponents (a_0, a_1) of monomials $x_0^{a_0} x_1^{a_1}$ that occur in f with $c_{(a_0,a_1)} \neq 0$.

Theorem (the theorem of Puiseux)

Let $f(x_0, x_1) \in \mathbb{C}(x_0)[x_1]$: f is a polynomial in the variable x_1 and its coefficients are fractional power series in x_0.

The polynomial f has as many series solutions as the degree of f.

Every series solution has the following form:

$$\left\{ \begin{array}{l} x_0 = t^u \\ x_1 = ct^v(1 + O(t)), \quad c \in \mathbb{C}^* \end{array} \right.$$

where (u, v) is an inner normal to an edge of the lower hull of the Newton polygon of f.

The series are computed with the polyhedral Newton-Puiseux method.

limits of space curves

Assume $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ has a solution curve C,
which intersects $x_0 = 0$ at a regular point.

For $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{Z}^n$, consider $\mathbf{x} = \mathbf{z}t^{\mathbf{v}}(1 + O(t))$:

- $x_0 = z_0 t^{v_0}$, for t close to zero, $z_0 \neq 0$ and
- for $k = 1, 2, \ldots, n-1$: $x_k = z_k t^{v_k}(1 + O(t)), z_k \neq 0$.

Substitute $x_0 = z_0 t^{v_0}, x_k = z_k t^{v_k}(1 + O(t))$ in $f_\ell(\mathbf{x}) = \displaystyle\sum_{\mathbf{a} \in A_\ell} c_\ell \mathbf{x}^{\mathbf{a}}$:

$$
\begin{aligned}
f_\ell(\mathbf{x} = \mathbf{z}t^{\mathbf{v}}(1 + O(t)) &= \sum_{\mathbf{a} \in A_\ell} c_{\mathbf{a}} z_0^{a_0} t^{a_0 v_0} \prod_{k=1}^{n-1} z_k t^{a_k v_k}(1 + O(t)) \\
&= \sum_{\mathbf{a} \in A_\ell} c_{\mathbf{a}} z^{\mathbf{a}} t^{a_0 v_0 + a_1 v_1 + \cdots + a_{n-1} v_{n-1}}(1 + O(t)).
\end{aligned}
$$

Because $\mathbf{z} \in (\mathbb{C}^*)^n$, there must be at least two terms in f_ℓ left as $t \to 0$.

initial forms and tropisms

Denote the inner product of vectors **u** and **v** as $\langle \mathbf{u}, \mathbf{v} \rangle$.

Definition

Let $\mathbf{v} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ be a direction vector. Consider $f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_\mathbf{a} \mathbf{x}^\mathbf{a}$.

The ***initial form of f in the direction v*** is

$$\mathrm{in}_\mathbf{v}(f) = \sum_{\substack{\mathbf{a} \in A \\ \langle \mathbf{a}, \mathbf{v} \rangle = m}} c_\mathbf{a} \mathbf{x}^\mathbf{a}, \quad \text{where } m = \min\{ \langle \mathbf{a}, \mathbf{v} \rangle \mid \mathbf{a} \in A \}.$$

Definition

Let the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ define a curve. A ***tropism*** consists of the leading powers $(v_0, v_1, \ldots, v_{n-1})$ of a Puiseux series of the curve.

The leading coefficients of the Puiseux series satisfy $\mathrm{in}_\mathbf{v}(\mathbf{f})(\mathbf{x}) = \mathbf{0}$.

curves of cyclic 4-roots

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0 \end{cases}$$

One tropism $\mathbf{v} = (+1, -1, +1, -1)$ with $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{z}) = \mathbf{0}$:

$$\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x}) = \begin{cases} x_1 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_1x_2x_3 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0. \end{cases}$$

We look for solutions of the form

$$(x_0 = t^{+1}, x_1 = z_1t^{-1}, x_2 = z_2t^{+1}, x_3 = z_3t^{-1}).$$

solving the initial form system

Substitute $(x_0 = t^{+1}, x_1 = z_1 t^{-1}, x_2 = z_2 t^{+1}, x_3 = z_3 t^{-1})$:

$\mathrm{in}_{\mathbf{v}}(\mathbf{f})(x_0 = t^{+1}, x_1 = z_1 t^{-1}, x_2 = z_2 t^{+1}, x_3 = z_3 t^{-1})$

$$= \left\{ \begin{array}{c} (1 + z_2)t^{+1} = 0 \\ z_1 + z_1 z_2 + z_2 z_3 + z_3 = 0 \\ (z_1 z_2 + z_3 z_1)t^{+1} = 0 \\ z_1 z_2 z_3 - 1 = 0. \end{array} \right.$$

We find two solutions: $(z_1 = -1, z_2 = -1, z_3 = +1)$ and $(z_1 = +1, z_2 = -1, z_3 = -1)$.

Two space curves $\left(t, -t^{-1}, -t, t^{-1}\right)$ and $\left(t, t^{-1}, -t, -t^{-1}\right)$ satisfy the entire cyclic 4-roots system.

overview of our polyhedral method

- finding pretropisms and solving initial forms

  Initial forms with at least two monomials in every equation define the intersection points of the solution set with the coordinate hyperplanes.

- unimodular coordinate transformations

  Via the Smith normal form we obtain nice representations for solutions at infinity.
  Solutions of binomial systems are monomial maps.

- computing terms of Puiseux series

  Although solutions to any initial forms may be monomial maps, in general we need a second term in the Puiseux series expansion to distinguish between

  - a positive dimensional solution set, and
  - an isolated solution at infinity.

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binomial systems

Definition

A ***binomial system*** has exactly two monomials with nonzero coefficient in every equation.

The binomial equation $c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}} - c_{\mathbf{b}}\mathbf{x}^{\mathbf{b}} = 0$, $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, $c_{\mathbf{a}}, c_{\mathbf{b}} \in \mathbb{C} \setminus \{0\}$, has normal representation $\mathbf{x}^{\mathbf{a}-\mathbf{b}} = c_{\mathbf{b}}/c_{\mathbf{a}}$.

A binomial system of N equations in n variables is then defined by an exponent matrix $A \in \mathbb{Z}^{N \times n}$ and a coefficient vector $\mathbf{c} \in (\mathbb{C}^*)^N$: $\mathbf{x}^A = \mathbf{c}$.

Motivations to study binomial systems:

1. A unimodular coordinate transformation provides a monomial parametrization for the solution set.
2. The leading coefficients of a Puiseux series satisfy a system of binomial equations.
3. Finding all solutions with zero coordinates can happen via a generalized permanent calculation.

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an example

Consider as an example for $\mathbf{x}^A = \mathbf{c}$ the system

$$\left\{ \begin{array}{r} x_0^2 x_1 x_2^4 x_3^3 - 1 = 0 \\ x_0 x_1 x_2 x_3 - 1 = 0 \end{array} \right. \quad A = \left[\begin{array}{cccc} 2 & 1 & 4 & 3 \\ 1 & 1 & 1 & 1 \end{array} \right]^T \quad \mathbf{c} = \left[\begin{array}{c} 1 \\ 1 \end{array} \right].$$

As basis of the null space of A we can for example take $\mathbf{u} = (-3, 2, 1, 0)$ and $\mathbf{v} = (-2, 1, 0, 1)$.

The vectors **u** and **v** are tropisms for a two dimensional algebraic set.

Placing **u** and **v** in the first two rows of a matrix M, extended so $\det(M) = 1$, we obtain a coordinate transformation, $\mathbf{x} = \mathbf{y}^M$:

$$M = \left[\begin{array}{rrrr} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \quad \left\{ \begin{array}{l} x_0 = y_0^{-3} y_1^{-2} y_2 \\ x_1 = y_0^2 y_1 y_3 \\ x_2 = y_0 \\ x_3 = y_1. \end{array} \right.$$

monomial transformations

By construction, as $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$:

$$MA = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} = B.$$

The corresponding monomial transformation $\mathbf{x} = \mathbf{y}^M$ performed on $\mathbf{x}^A = \mathbf{c}$ yields $\mathbf{y}^{MA} = \mathbf{y}^B = \mathbf{c}$, eliminating the first two variables:

$$\begin{cases} y_2^2 y_3 - 1 = 0 \\ y_2 y_3 - 1 = 0. \end{cases}$$

Solving this reduced system gives values z_2 and z_3 for y_2 and y_3. Leaving y_0 and y_1 as parameters t_0 and t_1 we find as solution

$$(x_0 = z_2 t_0^{-3} t_1^{-2}, x_1 = z_3 t_0^2 t_1, x_2 = t_0, x_3 = t_1).$$

unimodular coordinate transformations

Definition

A ***unimodular coordinate transformation*** $\mathbf{x} = \mathbf{y}^M$ is determined by an invertible matrix $M \in \mathbb{Z}^{n \times n}$: $\det(M) = \pm 1$.

For a d dimensional solution set of a binomial system:

1. The null space of A gives d tropisms, stored in the rows of a d-by-n-matrix B.
2. Compute the Smith normal form S of B: $UBV = S$.
3. There are three cases:
   1. $U = I \Rightarrow M = V^{-1}$
   2. If $U \neq I$ and S has ones on its diagonal, then extend U^{-1} with an identity matrix to form M.
   3. Compute the Hermite normal form H of B and let D be the diagonal elements of H, then $M = \begin{bmatrix} D^{-1}B \\ \mathbf{0} \quad I \end{bmatrix}$.

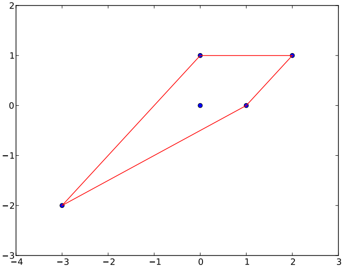
computation of the degree

To compute the degree of $(x_0 = z_2 t_0^{-3} t_1^{-2}, x_1 = z_3 t_0^2 t_1, x_2 = t_0, x_3 = t_1)$ we use two random linear equations:

$$\left\{ \begin{array}{l} c_{10} x_0 + c_{11} x_1 + c_{12} x_2 + c_{13} x_3 + c_{14} = 0 \\ c_{20} x_0 + c_{21} x_1 + c_{22} x_2 + c_{23} x_3 + c_{24} = 0 \end{array} \right.$$

after substitution:

$$\left\{ \begin{array}{l} c'_{10} t_0^{-3} t_1^{-2} + c'_{11} t_0^2 t_1 + c_{12} t_0 + c_{13} t_1 + c_{14} = 0 \\ c'_{20} t_0^{-3} t_1^{-2} + c'_{21} t_0^2 t_1 + c_{22} t_0 + c_{23} t_1 + c_{24} = 0 \end{array} \right.$$



Theorem (Koushnirenko's Theorem)

If all n polynomials in $\mathbf{f}$ *share the same Newton polytope P, then the number of isolated solutions of* $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ *in* $(\mathbb{C}^*)^n \leq$ *the volume of P.*

As the area of the Newton polygon equals 8, the surface has degree 8.

affine solution sets

An incidence matrix M of a bipartite graph:

$$\mathbf{f}(\mathbf{x}) = \left\{ \begin{array}{l} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \end{array} \right. \qquad M[\mathbf{x}^\mathbf{a}, x_k] = \left\{ \begin{array}{lll} 1 & \text{if} & a_k > 0 \\ 0 & \text{if} & a_k = 0. \end{array} \right.$$

Meaning of $M[\mathbf{x}^\mathbf{a}, x_k] = 1$: $x_k = 0 \Rightarrow \mathbf{x}^\mathbf{a} = 0$.

The matrix linking monomials to variables is

$$M[\mathbf{x}^\mathbf{a}, x_k] = \left[\begin{array}{c|cccccc} & x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \\ \hline x_{11}x_{22} & 1 & 0 & 0 & 0 & 1 & 0 \\ x_{21}x_{12} & 0 & 1 & 0 & 1 & 0 & 0 \\ x_{12}x_{23} & 0 & 1 & 0 & 0 & 0 & 1 \\ x_{22}x_{13} & 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right].$$

Observe: overlapping columns x_{12} with x_{22} gives all ones.

enumerating all candidate affine solution sets

Apply row expansion on the matrix

$$M[\mathbf{x}^{\mathbf{a}}, x_k] = \left[\begin{array}{c|cccccc} & x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \\ \hline x_{11}x_{22} & 1 & 0 & 0 & 0 & 1 & 0 \\ x_{21}x_{12} & 0 & 1 & 0 & 1 & 0 & 0 \\ x_{12}x_{23} & 0 & 1 & 0 & 0 & 0 & 1 \\ x_{22}x_{13} & 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right].$$

- Selecting 1 means setting the corresponding variable to zero.
- Monomials must be considered in pairs: if one monomial in an equation vanishes, then so must the other one.
- For all affine sets, we must skip pairs of rows, preventing from certain variables to be set to zero.
- To decide whether one candidate set C_1 belongs to another set C_2, we construct the defining equations $I(C_1)$ and $I(C_2)$ and apply $C_1 \subseteq C_2 \Leftrightarrow I(C_1) \supseteq I(C_2)$.

the Cayley embedding – an example

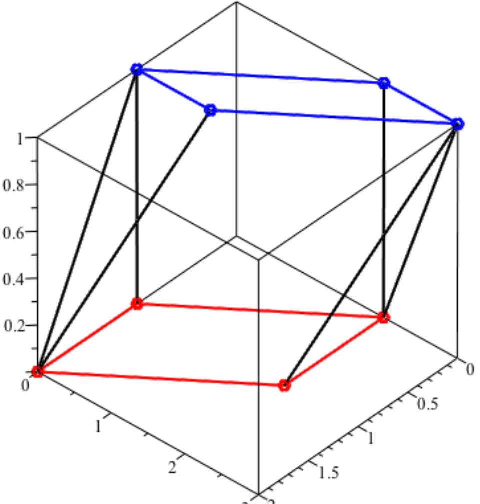
$$\begin{cases} p = (x_0 - x_1^2)(x_0 + 1) = x_0^2 + x_0 - x_1^2 x_0 - x_1^2 = 0 \\ q = (x_0 - x_1^2)(x_1 + 1) = x_0 x_1 + x_0 - x_1^3 - x_1^2 = 0 \end{cases}$$

The Cayley polytope is the convex hull of

$\{(2,0,0),(1,0,0),(1,2,0),(0,2,0)\}$

$\cup$

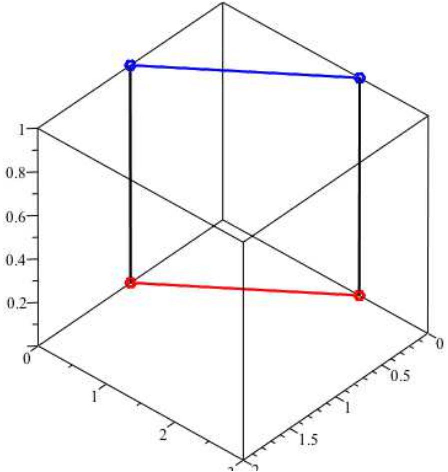
$\{(1,1,1),(1,0,1),(0,3,1),(0,2,1)\}$.



facet normals and initial forms

The Cayley polytope
has facets spanned by
one edge of the
Newton polygon of p
and
one edge of the
Newton polygon of q.

Consider $\mathbf{v} = (2, 1, 0)$.



$$\begin{cases} \text{in}_{(2,1)}(p) = \text{in}_{(2,1)}\left(x_0^2 + x_0 - x_1^2 x_0 - x_1^2\right) = x_0 - x_1^2 \\ \text{in}_{(2,1)}(q) = \text{in}_{(2,1)}\left(x_0 x_1 + x_0 - x_1^3 - x_1^2\right) = x_0 - x_1^2 \end{cases}$$

computing all pretropisms

Definition

A nonzero vector $\mathbf{v}$ is a ***pretropism*** for the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ if $\#\mathrm{in}_{\mathbf{v}}(f_k) \geq 2$ for all $k = 0, 1, \ldots, N-1$.

Application of the Cayley embedding to $(A_0, A_1, \ldots, A_{N-1})$:

$$E = \{\ (\mathbf{a}, \mathbf{0}) \mid \mathbf{a} \in A_0\ \} \cup \bigcup_{k=1}^{N-1} \{\ (\mathbf{a}, \mathbf{e}_k) \mid \mathbf{a} \in A_k\ \} \subset \mathbb{Z}^{n+N-1},$$

where $\mathbf{0}, \mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0)$, …, $\mathbf{e}_{N-1} = (0, 0, \ldots, 1)$ span the standard unit simplex in $\mathbb{R}^{N-1}$.

The set of all facet normals to the convex hull of E contains all normals to facets spanned by at least two points of each support.

We used `cddlib` to compute all pretropisms of the cyclic n-roots system, up to $n = 12$ (148.5 hours on a 3.07GHz CPU with 4GB RAM).

cones of pretropisms

Definition

A ***cone of pretropism*** is a polyhedral cone spanned by pretropisms.

If we are looking for an algebraic set of dimension d and

- if there are no cones of vectors perpendicular to edges of the Newton polytopes of $f(\mathbf{x}) = \mathbf{0}$ of dimension d, then the system $f(\mathbf{x}) = \mathbf{0}$ has no solution set of dimension d that intersects the first d coordinate planes properly; otherwise
- if a d-dimensional cone of vectors perpendicular to edges of the Newton polytopes exists, then that cone defines a part of the tropical prevariety.

For the cyclic 9-roots system,
we found a two dimensional cone of pretropisms.

the tropical prevariety of cyclic *n*-roots

All facets normals of the Cayley polytope computed with `cddlib` on a 3.07GHz Linux computer with 4Gb RAM:

n	#normals	#pretropisms	#generators	user cpu time
8	831	94	11	< 1 sec
9	4,840	276	17	1 sec
12	907,923	38,229	290	148 hours 27 min

Tropical intersections with `Gfan` on a 2.26GHz MacBook:

n	#rays	f-vector	user cpu time
8	94	1 94 108 48	15 sec
9	276	1 276 222 54	1 min 11 sec
12	5,582	1 5582 37786 66382 42540 8712	21 hours 1 min

Note that `Gfan` can exploit permutation symmetry.

Puiseux series for algebraic sets

Proposition

If $f(\mathbf{x}) = \mathbf{0}$ *is in Noether position and defines a* d*-dimensional solution set in* $\mathbb{C}^n$*, intersecting the first* d *coordinate planes in regular isolated points, then there are* d *linearly independent tropisms* $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_{d-1} \in \mathbb{Q}^n$ *so that the initial form system* $\mathrm{in}_{\mathbf{v}_0}(\mathrm{in}_{\mathbf{v}_1}(\cdots \mathrm{in}_{\mathbf{v}_{d-1}}(f)\cdots))(\mathbf{x} = \mathbf{y}^M) = \mathbf{0}$ *has a solution* $\mathbf{c} \in (\mathbb{C} \setminus \{0\})^{n-d}$*.*

This solution and the tropisms are the leading coefficients and powers of a generalized Puiseux series expansion for the algebraic set:

$$
\begin{array}{rclrcl}
x_0 & = & t_0^{v_{0,0}} & x_d & = & c_0 t_0^{v_{0,d}} t_1^{v_{1,d}} \cdots t_{d-1}^{v_{d-1,d}} + \cdots \\
x_1 & = & t_0^{v_{0,1}} t_1^{v_{1,1}} & x_{d+1} & = & c_1 t_0^{v_{0,d+1}} t_1^{v_{1,d+1}} \cdots t_{d-1}^{v_{d-1,d+1}} + \cdots \\
& \vdots & & & \vdots & \\
x_{d-1} & = & t_0^{v_{0,d-1}} t_1^{v_{1,d-1}} \cdots t_{d-1}^{v_{d-1,d-1}} & x_n & = & c_{n-d-1} t_0^{v_{0,n-1}} t_1^{v_{1,n-1}} \cdots t_{d-1}^{v_{d-1,n-1}} + \cdots
\end{array}
$$

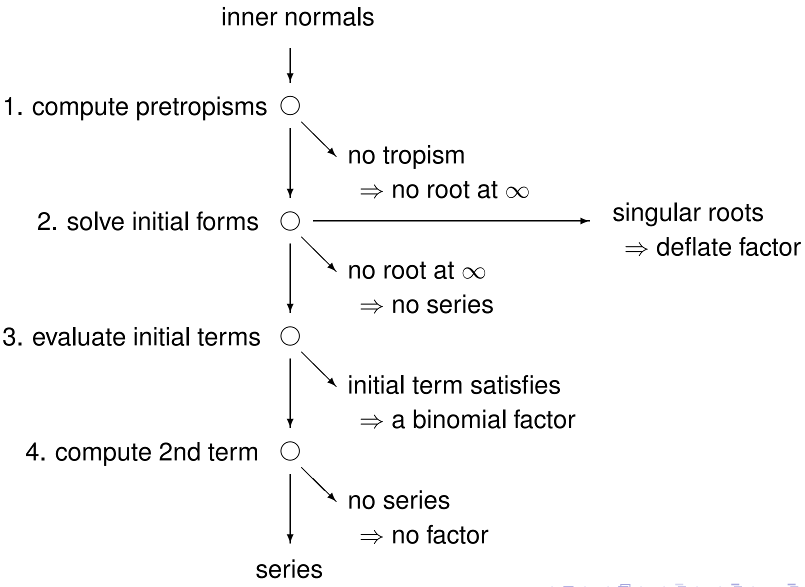
our polyhedral approach

For every d-dimensional cone C of pretropisms:

1. We select d linearly independent generators to form the d-by-n matrix A and the unimodular transformation $\mathbf{x} = \mathbf{y}^M$.
2. If $\mathrm{in}_{\mathbf{v}_0}(\mathrm{in}_{\mathbf{v}_1}(\cdots \mathrm{in}_{\mathbf{v}_{d-1}}(f) \cdots))(\mathbf{x} = \mathbf{y}^M) = \mathbf{0}$ has no solution in $(\mathbb{C}^*)^{n-d}$, then return to step 1 with the next cone C, else continue.
3. If the leading term of the Puiseux series satisfies the entire system, then we report an explicit solution of the system and return to step 1 to process the next cone C.
Otherwise, we take the current leading term to the next step.
4. If there is a second term in the Puiseux series,
then we have computed an initial development for an algebraic set and report this development in the output.

Note: to ensure the solution of the initial form system is not isolated, it suffices to compute a series development *for a curve*.

our approach depicted in stages



relevant software

- `cddlib` by Komei Fukuda and Alain Prodon implements the double description method to efficiently enumerate all extreme rays of a general polyhedral cone.
- `Gfan` by Anders Jensen to compute Gröbner fans and tropical varieties uses `cddlib`.
- The `Singular` library `tropical.lib` by Anders Jensen, Hannah Markwig and Thomas Markwig for computations in tropical geometry.
- `Macaulay2` interfaces to Gfan.
- `Sage` interfaces to Gfan.
- `PHCpack` (published as Algorithm 795 ACM TOMS) provides our numerical blackbox solver.

positive dimensional sets of cyclic n-roots

- $n = 8$: Tropisms, their cyclic permutations, and degrees:

$$
\begin{array}{cr}
(1,-1,1,-1,1,-1,1,-1) & 8 \times 2 = 16 \\
(1,-1,0,1,0,0,-1,0) \rightarrow (1,0,0,-1,0,1,-1,0) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,-1,0,0,1,0,-1) \rightarrow (1,0,-1,1,0,-1,0,0) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,-1,1,0,-1,0,0) \rightarrow (1,0,-1,0,0,1,0,-1) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,0,-1,0,1,-1,0) \rightarrow (1,-1,0,1,0,0,-1,0) & 8 \times 2 + 8 \times 2 = 32 \\
 & \text{TOTAL} = 144
\end{array}
$$

- $n = 9$: A 2-dimensional cone of tropisms spanned by $\mathbf{v}_0 = (1,1,-2,1,1,-2,1,1,-2)$ and $\mathbf{v}_1 = (0,1,-1,0,1,-1,0,1,-1)$. Denoting by $u = e^{i2\pi/3}$ the primitive third root of unity, $u^3 - 1 = 0$:

$$
\begin{array}{lll}
x_0 = t_0 & x_3 = u t_0 & x_6 = u^2 t_0 \\
x_1 = t_0 t_1 & x_4 = u t_0 t_1 & x_7 = u^2 t_0 t_1 \\
x_2 = u^2 t_0^{-2} t_1^{-1} & x_5 = t_0^{-2} t_1^{-1} & x_8 = u t_0^{-2} t_1^{-1}.
\end{array}
$$

- $n = 12$: Computed 77 quadratic space curves.

results in the literature

Our results for $n = 9$ and $n = 12$ are in agreement with

- J.C. Faugère. **Finding all the solutions of Cyclic 9 using Gröbner basis techniques.** In *Computer Mathematics - Proceedings of the Fifth Asian Symposium (ASCM 2001)*, pages 1–12. World Scientific, 2001.
- R. Sabeti. **Numerical-symbolic exact irreducible decomposition of cyclic-12.** *London Mathematical Society Journal of Computation and Mathematics*, 14:155–172, 2011.

a tropical version of Backelin's Lemma

Lemma (Tropical Version of Backelin's Lemma)

For $n = m^2\ell$, where $\ell \in \mathbb{N} \setminus \{0\}$ and ℓ is no multiple of k^2, for $k \geq 2$, there is an $(m-1)$-dimensional set of cyclic n-roots, represented exactly as

$$\begin{aligned}
x_{km+0} &= u^k t_0 \\
x_{km+1} &= u^k t_0 t_1 \\
x_{km+2} &= u^k t_0 t_1 t_2 \\
&\vdots \\
x_{km+m-2} &= u^k t_0 t_1 t_2 \cdots t_{m-2} \\
x_{km+m-1} &= \gamma u^k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1}
\end{aligned}$$

for $k = 0, 1, 2, \ldots, m-1$, free parameters $t_0, t_1, \ldots, t_{m-2}$, constants $u = e^{\frac{i2\pi}{m\ell}}$, $\gamma = e^{\frac{i\pi\beta}{m\ell}}$, with $\beta = (\alpha \mod 2)$, and $\alpha = m(m\ell - 1)$.

summary

Promising results on the cyclic *n*-roots problem give a proof of concept for a new polyhedral method to compute algebraic sets.

For the computation of pretropisms, we rely on

- `cddlib` on the Cayley embedding of the Newton polytopes, or
- `Gfan` for the tropical intersection.

To process the pretropisms, we

- use `Sage` to extract initial form systems
  and look for the second term in the Puiseux series;
- solve initial form systems with the blackbox solver of `PHCpack`.