Software for Numerical Algebraic Geometry

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Joint work with Andrew J. Sommese and Charles W. Wampler.

Numerical Factorization of Multivariate Complex Polynomials

Outline of Talk:

1. We factor in three stages:
   (a) monodromy grouping of witness points;
   (b) certification of grouping with linear traces;
   (c) interpolation to get polynomials for the factors.
2. We remove multiplicies by differentation and use a theorem of Marden and Walsh for bound on precision.
3. Application: study singularities of Stewart-Gough platforms.

Problem Statement

- Input: $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

  coefficients known approximately, work with limited precision

- Wanted: write f as product of irreducible factors, as

$$f(\mathbf{x}) = \prod_{i=1}^{N} q_i(\mathbf{x})^{\mu_i}, \quad \sum_{i=1}^{N} \mu_i \deg(q_i) = \deg(f),$$

  every irreducible factor q_i occurs with multiplicity μ_i.

E. Kaltofen: **Challenges of symbolic computation: my favorite open problems**. *J. Symbolic Computation* 29(6): 891–919, 2000.

Related Work

Y. Huang, W. Wu, H.J. Stetter, and L. Zhi: **Pseudofactors of multivariate polynomials.** In *Proceedings of ISSAC 2000*, ed. by C. Traverso, pages 161–168, ACM 2000.

R.M. Corless, M.W. Giesbrecht, M. van Hoeij, I.S. Kotsireas and S.M. Watt: **Towards factoring bivariate approximate polynomials.** In *Proceedings of ISSAC 2001*, ed. by B. Mourrain, pages 85–92, ACM 2001.

A. Galligo and D. Rupprecht: **Semi-numerical determination of irreducible branches of a reduced space curve.** In *Proceedings of ISSAC 2001*, ed. by B. Mourrain, pages 137–142, ACM 2001.

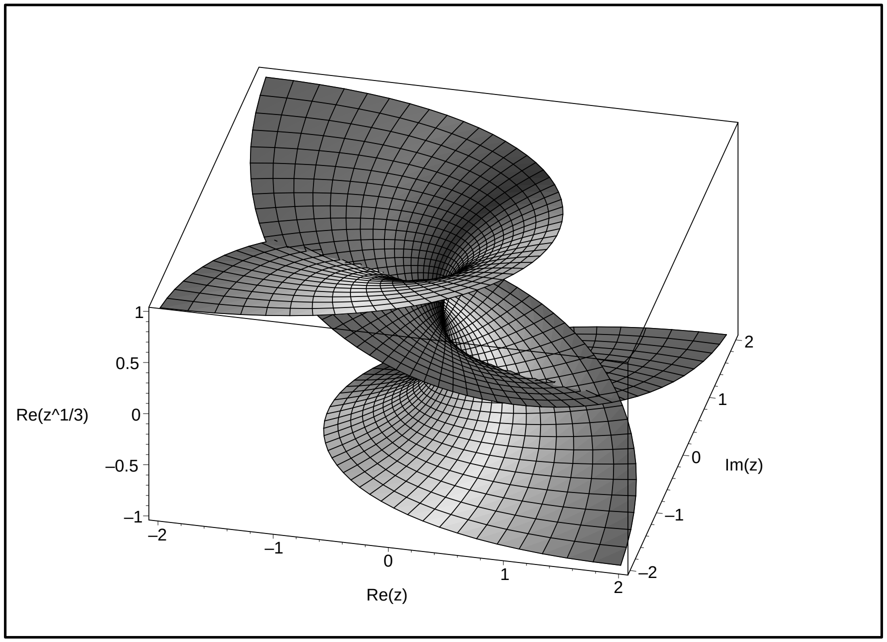
A. Galligo and D. Rupprecht: **Irreducible decomposition of curves.** *J. Symbolic Computation* 33(5):661–677, 2002.

T. Sasaki: **Approximate multivariate polynomial factorization based on zero-sum relations.** In *Proceedings of ISSAC 2001*, ed. by B. Mourrain, pages 284–291, ACM 2001.

R.M. Corless, A. Galligo, I.S. Kotsireas, and S.M. Watt: **A geometric-numeric algorithm for absolute factorization of multivariate polynomials.** In *Proceedings of ISSAC 2002*, ed. by T. Mora, pages 37–45, ACM 2002.

E. Kaltofen and J. May: **On approximate irreducibility of polynomials in several variables.** To appear in *Proceedings of ISSAC 2003*.

The Riemann Surface of $z^3 - w = 0$:



R.M. Corless and D.J. Jeffrey: **Graphing elementary Riemann surfaces.** *SIGSAM Bulletin* 32(1):11–17, 1998.

Monodromy to Decompose Solution Components

Given: a system $f(\mathbf{x}) = \mathbf{0}$; and $W = (Z, L)$:

$$\text{for all } \mathbf{w} \in Z : f(\mathbf{w}) = \mathbf{0} \text{ and } L(\mathbf{w}) = \mathbf{0}.$$

Wanted: partition of Z so that all points in a subset of Z lie on the same irreducible factor.

Example: does $f(x, y) = xy - 1 = 0$ factor?

Consider $H(x, y, \theta) = \begin{cases} xy - 1 = 0 \\ x + y = 4e^{i\theta} \end{cases}$ for $\theta \in [0, 2\pi]$.

For $\theta = 0$, we start with two real solutions. When $\theta > 0$, the solutions turn complex, real again at $\theta = \pi$, then complex until at $\theta = 2\pi$. Back at $\theta = 2\pi$, we have again two real solutions, but their order is permuted $\Rightarrow$ irreducible.

Connecting Witness Points

1. For two sets of hyperplanes K and L, and a random $\gamma \in \mathbb{C}$

$$H(\mathbf{x}, t, K, L, \gamma) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ \gamma K(\mathbf{x})(1-t) + L(\mathbf{x})t = \mathbf{0} \end{cases}$$

   We start paths at $t = 0$ and end at $t = 1$.

2. For $\alpha \in \mathbb{C}$, trace the paths defined by $H(\mathbf{x}, t, K, L, \alpha) = \mathbf{0}$.

   For $\beta \in \mathbb{C}$, trace the paths defined by $H(\mathbf{x}, t, L, K, \beta) = \mathbf{0}$.

   Compare start points of first path tracking with end points of second path tracking. Points which are permuted belong to the same irreducible factor.

3. Repeat the loop with other values of α and β.

Finding Witness Points

Instead of $n-1$ random hyperplanes $L(\mathbf{x}) = \mathbf{0}$ to cut the surface $f(\mathbf{x}) = 0$, consider the random line $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$; in particular:

Input : $f(\mathbf{x})$ polynomial in n variables with complex coefficients;
$\mathbf{x}_0$ and $\mathbf{v}$ represent a random line $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$.

Output : $W = \{W_1, W_2, \ldots, W_m\}$, $m = \max_{i=1}^{d} \mu_i$,
for all $X \in W_i$: $\#X = \mu_i$.

To solve $f(\mathbf{x}(t)) = 0$ we use the method of Weierstrass.

(also known as Durand-Kerner)

Linear Traces

$$\text{Consider} \quad f(x, y(x)) = (y - y_1(x))(y - y_2(x))(y - y_3(x))$$
$$= y^3 - t_1(x)y^2 + t_2(x)y - t_3(x)$$

We are interested in the linear trace: $t_1(x) = c_1 x + c_0$.

Sample the cubic at $x = x_0$ and $x = x_1$. The samples are $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$.

$$\text{Solve} \quad \begin{cases} y_{00} + y_{01} + y_{02} = c_1 x_0 + c_0 \\ y_{10} + y_{11} + y_{12} = c_1 x_1 + c_0 \end{cases} \quad \text{to find} \quad c_0, c_1.$$

With t_1 we can predict the sum of the y's for a fixed choice of x. For example, samples at $x = x_2$ are $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$. Then, $t_1(x_2) = c_1 x_2 + c_0 = y_{20} + y_{21} + y_{22}$.

Validation of Breakup with Linear Trace

Do we have enough witness points on a factor?

- We may not have enough monodromy loops to connect all witness points on the same irreducible component.
- We have enough witness points on the curve if the value at the linear trace can predict the sum of one coordinate of all points in the set.

Notice: Instead of monodromy, we may enumerate all possible factors and use linear traces to certify. While the complexity of this enumeration is exponential, it works well for low degrees.

Dealing with Multiplicities

On a factor of degree d and multiplicity μ,

we find d clusters, each of μ witness points.

Choose $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and compute

$$g(\mathbf{x}) := \left(v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + \cdots + v_n \frac{\partial}{\partial x_n} \right)^{\mu - 1} f(\mathbf{x}).$$

Then apply the techniques to the multiplicity one roots of $g(\mathbf{x})$ corresponding to the clusters.

Using a theorem of Marden and Walsh

Assume d is the degree of $f(z)$, $f \in \mathbb{C}[z]$;

μ is the multiplicity of a root of f;

z_0 is the center of the cluster around the multiple root;

$\Delta_r(z_0) = \{ \ z \in \mathbb{C} \mid |z - z_0| \leq r \ \}$ contains the cluster;

r is the radius of the disk $\Delta_r(z_0)$;

R is largest such that $\{ \ z \in \mathbb{C} \mid |z - z_0| \geq R \ \}$ contains all other $d - \mu$ roots of f.

If $\frac{R}{r} \geq \frac{2\binom{d}{\mu}}{d-\mu+1}$, then $f^{(k)}$ has exactly $\mu - k$ roots in $\Delta_r(z_0)$, for $k = 1, 2, \ldots, \mu - 1$.

Applying the bound for R/r

Given a cluster of μ roots (and $d - \mu$ other roots), compute

- z_0 as the average of the roots in the cluster;
- r as the largest distance of the roots in the cluster to z_0;
- R as the smallest distance of the other $d - \mu$ roots to z_0.

$$\frac{R}{r} \geq \frac{2\binom{d}{\mu}}{d - \mu + 1} \quad \Rightarrow \quad r \leq R\left(\frac{d - \mu + 1}{2\binom{d}{\mu}}\right)$$

We obtain a bound on r, the precision of the roots in the cluster, in order for the successive derivatives of f to be safe.

Numerical Limitations

- Evaluation of high degree polynomials is numerically unstable:

$$f(x) = (x_0 + tv)^d = \sum_{k=0}^{d} \binom{d}{k} x_0^{d-k} v^k t^k = 0,$$

  for example, $d = 30$ and $k = 15$: nine decimal places in $\binom{d}{k}$.

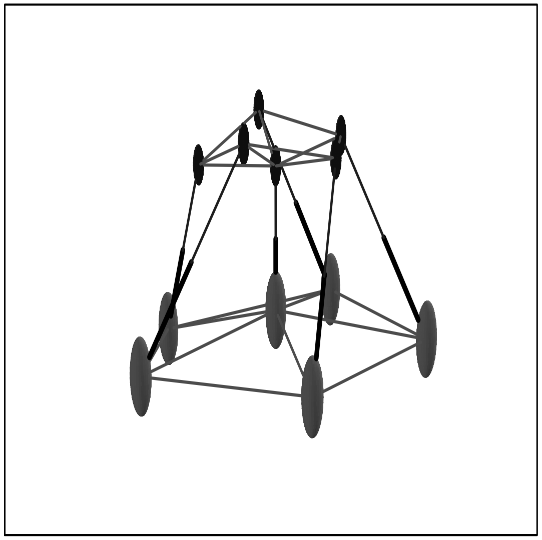
- Working precision determines accuracy of factorization:

$$f(x, y) = xy + 10^{-16}$$

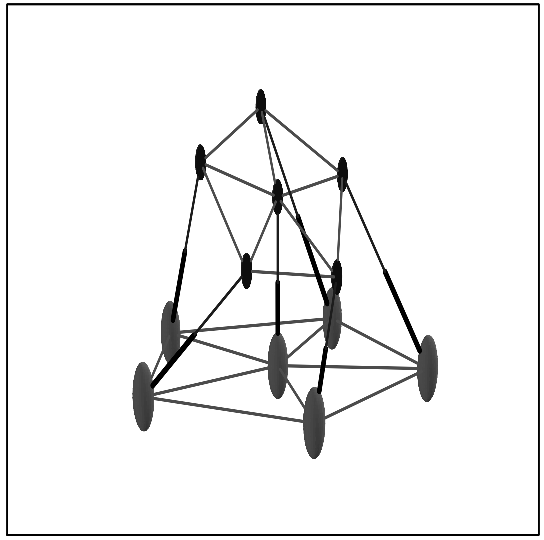
  – will factor when working with double precision floats;

  – will not factor as soon as precision is high enough.

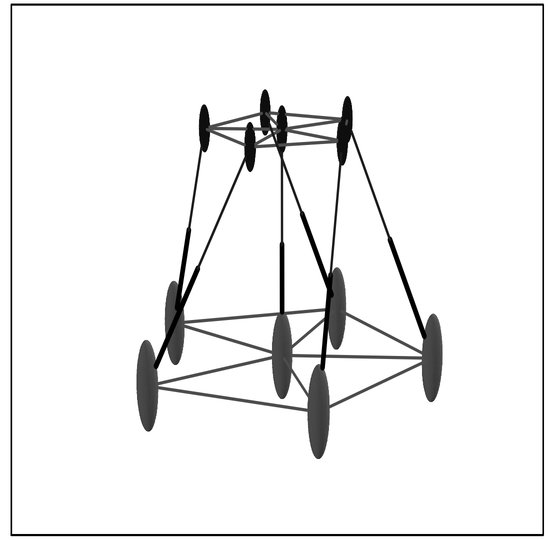
three cases of Stewart-Gough platforms



General platform,
fixed position



Planar base,
planar platform



Parallel base
and platform

J.P. Merlet: **Parallel Robots**. Kluwer Academic Publishers, 2000.

Singularities of Stewart-Gough Platforms

At singularity, rigidity of device is lost, allowing finite motion which cannot be controlled by leg lengths (*disaster!*).

Denote	$\mathbf{p} \in \mathbb{C}^3$	position of platform;
	$\mathbf{q} \in \mathbb{P}^3$	quaternion defines a rotation;
	$\mathbf{a}_i, \mathbf{b}_i \in \mathbb{C}^3$	ball joints at platform and base, $i = 1, 2, \ldots, 6$;
	$\mathbf{J} \in \mathbb{C}^{6\times 6}$	Jacobian matrix of mapping from platform motion to leg lengths.

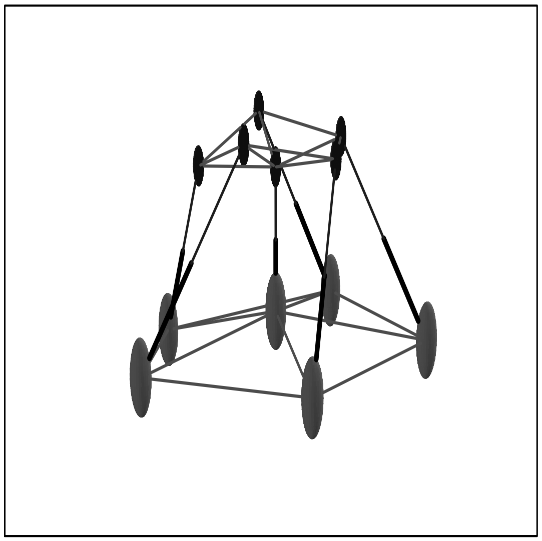
Then the condition on a singular configuration is $\det \mathbf{J} = 0$.

$\det \mathbf{J}$ is a polynomial of degree 1728 in 43 variables: $\mathbf{p}$, $\mathbf{q}$, $\mathbf{a}_i$, $\mathbf{b}_i$.

Merlet. *Int. J. Robotics Research* 8(5):45–56, 1989.

Bayer St-Onge and Gosselin. *Int. J. Robotics Research* 19(3):271–288, 2000.

first general case of a Stewart-Gough platform



General platform,
fixed position

- case of *almost all* manipulators

  $\mathbf{p}, \mathbf{a}_i$, and $\mathbf{b}_i$ are randomly chosen

- $\deg(\det \mathbf{J}) = 12$, homogeneous in $\mathbf{q}$

  the expanded $\det \mathbf{J}$ has 910 terms

- $\det \mathbf{J} = F_1(\mathbf{q})(F_2(\mathbf{q}))^3$

  $\mathbf{q} = (q_0, q_1, q_2, q_3)$ quaternion

  $\deg(F_1) = 6$

  $F_2(\mathbf{q}) = q_0^2 + q_1^2 + q_2^2 + q_3^2$

  F_2 has no physical significance

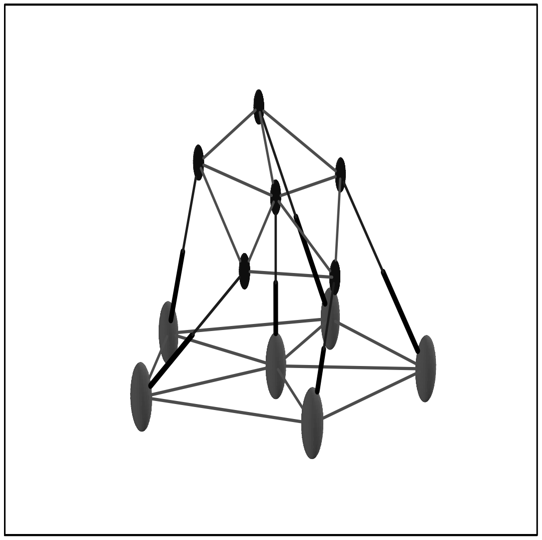
Computational results for first platform

cluster	r	R	R/r
one	1.7E-05	3.4E-01	2.0E+04
two	4.9E-06	1.7E-01	3.6E+04

Lower bound on R/r evaluates to 44.

Elapsed user CPU times on 2.4Ghz WindowsXP			
1.	monodromy grouping	:	0h 6m 40s 469ms
2.	linear traces certification	:	0h 0m 30s 672ms
3.	interpolation at factors	:	1h 41m 53s 78ms
4.	multiplication validation	:	0h 0m 8s 156ms
	total time for all 4 stages	:	1h 49m 12s 391ms

second case: planar base and platform



Planar base and platform

- ball joints $\mathbf{a}_i$ lie in planar platform
  ball joints $\mathbf{b}_i$ lie in planar base
- $\deg(\det \mathbf{J}) = 12$, homogeneous in $\mathbf{q}$
  the expanded $\det \mathbf{J}$ has 910 terms
- $\det \mathbf{J} = F_1(\mathbf{q})(F_2(\mathbf{q}))^3$
  $\mathbf{q} = (q_0, q_1, q_2, q_3)$ quaternion
  $\deg(F_1) = 6 \quad \deg(F_2) = 2$

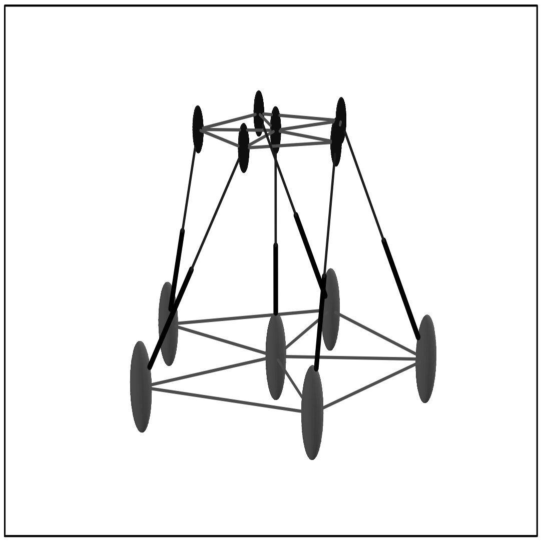
Computational results for second platform

cluster	r	R	R/r
one	6.2E-05	2.4E-01	3.8E+04
two	4.8E-05	6.0E-01	1.2E+04

Lower bound on R/r evaluates to 44.

Elapsed user CPU times on 2.4Ghz WindowsXP			
1.	monodromy grouping	:	0h 17m 34s 735ms
2.	linear traces certification	:	0h 0m 27s 359ms
3.	interpolation at factors	:	1h 24m 45s 766ms
4.	multiplication validation	:	0h 0m 8s 172ms
	total time for all 4 stages	:	1h 42m 56s 32ms

third case: parallel base and platform



Parallel base and platform

- ball joints $\mathbf{a}_i, \mathbf{b}_i$ in parallel planes, position $\mathbf{p}$ is variable, $q_1 = q_2 = 0$
- $\deg(\det \mathbf{J}) = 15$, in $(\mathbf{p}, \mathbf{q})$
  expanded $\det \mathbf{J}$ has 24 terms,
  much sparser, as $24 << 910$
- $\det \mathbf{J} = a p_3^3 (q_0 + b q_3)(q_0 + c q_3)$
  $$(q_0 + i q_3)^5 (q_0 - i q_3)^5$$
  where the constants a, b, c
  depend on the choice of $\mathbf{a}_i, \mathbf{b}_i$

Computational results for third platform

cluster	r	R	R/r
one	5.1E-07	1.0E+00	2.0E+06
two	7.3E-04	3.4E-01	4.7E+02
three	4.0E-03	7.2E-01	1.8E+02

Lower bound on R/r evaluates to 546.

Elapsed user CPU times on 2.4Ghz WindowsXP			
1.	monodromy grouping	:	1m 13s 656ms
2.	linear traces certification	:	0m 3s 891ms
3.	interpolation at factors	:	0m 4s 734ms
4.	multiplication validation	:	0m 1s 657ms
	total time for all 4 stages	:	1m 23s 938ms

Monodromy Compared to the Enumeration Method

Enumeration of all possible factors certified by linear traces outperforms the monodromy algorithm for our application:

User CPU times on 2.4Ghz Windows XP		
case	monodromy	enumeration
1	6m 40s 460ms	40s 750ms
2	17m 34s 735ms	31s 657ms
3	1m 13s 656ms	3s 0ms

Random irreducible polynomials of five monomials:

User CPU times on 2.4Ghz Windows XP		
degree	monodromy	enumeration
10	5s 484ms	312ms
15	8s 187ms	1s 453ms
16	16s 63ms	2s 875ms

Conclusions

- general monodromy breakup with linear trace certification specialized to factorization of multivariate polynomials
- replace singular roots with nonsingular ones by differentiation, estimate precision needed using result of Marden and Walsh
- applied to study singularities of Stewart-Gough platforms
- enumeration method of Galligo and Rupprecht faster than monodromy for modest degrees

Relevant papers at www.math.uic.edu/~jan

Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation.
ACM Transactions on Mathematical Software 25(2): 251-276, 1999.

with A.J. Sommese and C.W. Wampler: **Symmetric functions applied to decomposing solution sets of polynomial systems.**
SIAM J. Numer. Anal. 40(6):2026–2046, 2002.

with A.J. Sommese and C.W. Wampler: **Numerical Irreducible Decomposition using PHCpack.**
In *Algebra, Geometry, and Software Systems*, edited by M. Joswig and N. Takayama, pages 109–130, Springer-Verlag 2003.

with A.J. Sommese and C.W. Wampler: **Numerical Factorization of Multivariate Complex Polynomials.**

with Yusong Wang: **Computing Dynamic Output Feedback Laws.**