Polynomial Homotopies on Multicore Workstations

Jan Verschelde

University of Illinois at Chicago
Department of Mathematics, Statistics, and Computer Science
http://www.math.uic.edu/~jan
jan@math.uic.edu

14th SIAM Conference on Parallel Processing for Scientific Computing
MS46 High-Performance Symbolic Computing
Seattle, Washington, 26 February 2010.

Outline

1. Homotopy Continuation Methods
   - solving polynomial systems
   - pleasingly parallel computations
2. Multicore Workstations
   - applying tasking
   - MPI versus threads
   - results for polyhedral blackbox solver
3. Numerical Irreducible Decomposition
   - using monodromy to factor

Solving Polynomial Systems

On input is a polynomial system $f(\mathbf{x}) = \mathbf{0}$.

A homotopy is a family of systems:

$$h(\mathbf{x}, t) = (1 - t)g(\mathbf{x}) + t\, f(\mathbf{x}) = \mathbf{0}.$$

At $t = 1$, we have the system $f(\mathbf{x}) = \mathbf{0}$ we want to solve.
At $t = 0$, we have a good system $g(\mathbf{x}) = \mathbf{0}$:

- solutions are known or easier to solve; and
- all solutions of $g(\mathbf{x}) = \mathbf{0}$ are regular.

Tracking all solution paths is pleasingly parallel,
although not every path requires the same amount of work.

Homotopy Continuation Methods

Types of homotopies h:

- $h(\mathbf{x}, t) = (1-t)g(\mathbf{x}) + t\, f(\mathbf{x}) = \mathbf{0}$, from start to target.
- $h(\mathbf{x}, t) = f(\mathbf{c}_0(1-t) + \mathbf{c}_1 t, \mathbf{x}) = \mathbf{0}$, cheater's homotopy.
- $h(\mathbf{x}, t) = f(\mathbf{c}, \mathbf{x}(t)) = \mathbf{0}$, moving basis coordinates.

Homotopies are often used in combination, or in cascades.

1. **Tien-Yien Li.** Numerical solution of polynomial systems by homotopy continuation methods. In Volume XI of *Handbook of Numerical Analysis*, pages 209–304, 2003.
2. **Andrew J. Sommese and Charles W. Wampler.** *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science.* World Scientific, 2005.

Software Systems

Starring in alphabetical order:

- **Bertini**, first released in Fall 2006, by D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. MPI executables available.
- **HOM4PS-2.0para** by T.Y. Li and C.H. Tsai (2009) is a parallel version of **HOM4PS-2.0** by T.L. Lee, T.Y. Li, and C.H. Tsai (2007); extends **HOM4PS** by T. Gao and T.Y. Li.
- **PHoMpara** by T. Gunji, S. Kim, K. Fujisawa, and M. Kojima (2006) is a parallel version of **PHoM** by T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa and T. Mizutani (2004).
- **POLSYS_GLP** is Algorithm 857 of ACM TOMS (2006) by H.-J. Su, J.M. McCarthy, M. Sosonkina, and L.T. Watson extends **HOMPACK90** by L.T. Watson, M. Sosonkina, R.C. Melville, A.P. Morgan, and H.F. Walker (1997) and **HOMPACK** by L.T. Watson, S.C. Billups, and A.P. Morgan (1987).

Anton Leykin is developing homotopy continuation in Macaulay2.
`http://www.math.uic.edu/~leykin/NAG4M2/index.html`.

Parallel PHCpack

parallel implementation of polynomial homotopy continuation methods

PHC = Polynomial Homotopy Continuation

- Version 1.0 archived as Algorithm 795 by ACM TOMS (1999)
- Pleasingly parallel implementations
  + **Yusong Wang** of Pieri homotopies (HPSEC'04)
  + **Anton Leykin** of monodromy factorization (HPSEC'05)
  + **Yan Zhuang** of polyhedral homotopies (HPSEC'06)
  + **Yun Guan** of diagonal homotopies (HPCS'08)
- Interactive Parallel Computing:
  + **Yun Guan:** PHClab, experiments with MPITB in Octave
  + **Kathy Piret:** bindings with Python, use of sockets

Release v2.3.42 extends **phcpy** and a preliminary **PHCwulf.py**.

Hardware and Software

Running on a modern workstation (*not* a supercomputer):

- Hardware: Mac Pro with 2 Quad-Core Intel Xeons at 3.2 Ghz
  Total Number of Cores: 8 1.6 GHz Bus Speed
  12 MB L2 Cache per processor, 8 GB Memory

- PHCpack is written in Ada, compiled with gnu-ada compiler
  gcc version 4.3.4 20090511 for GNAT GPL 2009 (20090511)
  Target: x86_64-apple-darwin9.6.0
  Thread model: posix

Also compiled for Linux and Windows (win32 thread model).

Starting Worker Tasks

procedure `Workers` is instantiated with a `Job` procedure, executing code based on the `id` number.

```
procedure Workers ( n : in natural ) is
   task type Worker ( id,n : natural );
   task body Worker is
   begin
     Job(id,n);
   end Worker;
   procedure Launch_Workers ( i,n : in natural ) is
     w : Worker(i,n);
   begin
     if i < n
      then Launch_Workers(i+1,n);
     end if;
   end Launch_Workers;
begin
  Launch_Workers(1,n);
end Workers;
```

MPI versus Threads

- MPI = Message Passing Interface

  The manager/worker paradigm:
  - worker nodes perform path tracking jobs,
  - manager maintains job queue, serves workers.

  Manager must be available to serve jobs.

- Threads are lightweight processes

  Collaborative workers launched by master thread:
  - communication overhead replaced by memory sharing,
  - job queue updated in critical section using locks.

- With MPI, we worry about communication overhead.
  With threads, memory (de)allocation must be in critical sections.

Load Balancing and Granularity Issues

We assume: # solution paths $\gg$ # cores.

Granularity Issues:

- coarse: one job = track one solution path
- fine: polynomial evaluation, linear algebra

Dynamic load balancing:

- not all jobs take the same amount of work

An academic Benchmark: cyclic n-roots

The system

$$f(\mathbf{x}) = \begin{cases} f_i = \sum_{j=0}^{n=1} \prod_{k=1}^{i} x_{(k+j) \bmod n} = 0, \quad i = 1, 2, \ldots, n-1 \\ f_n = x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0 \end{cases}$$

appeared in

- G. Björck: **Functions of modulus one on Z_p whose Fourier transforms have constant modulus**. In *Proceedings of the Alfred Haar Memorial Conference, Budapest*, pages 193–197, 1985.
- J. Backelin and R. Fröberg: **How we proved that there are exactly 924 cyclic 7-roots**. In ISSAC'91 proceedings, pages 101-111, ACM, 1991.

very sparse, well suited for polyhedral methods

First Preliminary Results

Using version 2.3.45 of PHCpack:

```
$ time phc -p -t8 < /tmp/input8
```

#worker tasks = number following the `-t`

running a cheater's homotopy on cyclic 7-roots (924 paths).

#workers	real	user	sys	speedup
1	15.478s	15.457s	0.010s	1
2	7.790s	15.483s	0.010s	1.987
4	3.926s	15.445s	0.011s	3.942
8	1.992s	15.424s	0.015s	7.770

Since version 2.3.46 of PHCpack:

```
$ phc -b -t8
```

blackbox solver (`phc -b`) uses multitasking

3 stages to solve a polynomial system $f(\mathbf{x}) = \mathbf{0}$

1. Compute the mixed volume MV (aka the BKK bound) of the Newton polytopes spanned by the supports A of f via a **regular mixed-cell configuration** Δ_ω.
2. Given Δ_ω, solve a generic system $g(\mathbf{x}) = \mathbf{0}$, using polyhedral homotopies. Every cell $C \in \Delta_\omega$ defines one homotopy

$$h_C(\mathbf{x}, s) = \sum_{\mathbf{a} \in C} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} + \sum_{\mathbf{a} \in A \setminus C} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} s^{\nu_{\mathbf{a}}}, \quad \nu_{\mathbf{a}} > 0,$$

   tracking as many paths as the mixed volume of the cell C, as s goes from 0 to 1.
3. Use $(1-t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}$ to solve $f(\mathbf{x}) = \mathbf{0}$.

Stages 2 and 3 are **computationally most intensive** ($1 \ll 2 < 3$),

e.g.: cyclic 10-roots (MV = 35940): stage 1: 21 secs, stage 2: 39 min.

A Static Distribution of the Workload

used in `mpi2cell_s` with Yan Zhuang

manager	worker 1	worker 2	worker 3
Vol(cell 1) = 5	#paths(cell 1) : 5		
Vol(cell 2) = 4	#paths(cell 2) : 4		
Vol(cell 3) = 4	#paths(cell 3) : 4		
Vol(cell 4) = 6	#paths(cell 4) : 1	#paths(cell 4) : 5	
Vol(cell 5) = 7		#paths(cell 5) : 7	
Vol(cell 6) = 3		#paths(cell 6) : 2	#paths(cell 6) : 1
Vol(cell 7) = 4			#paths(cell 7) : 4
Vol(cell 8) = 8			#paths(cell 8) : 8
total #paths : 41	#paths : 14	#paths : 14	#paths : 13

Since polyhedral homotopies solve a **generic** system $g(\mathbf{x}) = \mathbf{0}$,
we **expect** every path to take the same amount of work...

Running Polyhedral Homotopies

Running polyhedral homotopies on a random coefficient system, distributing mixed cells, for the cyclic *n*-roots problems.

Tracking MV (MV = mixed volume) many solution paths:

		#tasks, times in seconds			
n	MV	1	2	4	8
7	924	12	6	3	2
8	2560	58	29	15	8
9	11016	417	209	104	52
10	35940	2156	1068	534	270

Comparison with MPI (`mpi2cell_d`) on cyclic 10-roots:

- `mpirun -n 9`: total wall time = 270.5 seconds.
- on same random coefficient system and same tolerances: elapsed wall clock time is 233 seconds.

Absolution Factorization

Factor a polynomial in several variables over $\mathbb{C}$.

Deciding irreducibility in polynomial time in degree and coefficient size is a challenge problem in symbolic computing (Kaltofen, 2000).

Symbolic-numeric algorithms:

- C. Bajaj, J. Canny, T. Garrity, J. Warren (1989)
- T. Sasaki, T. Saito, T. Hilano (1992); T. Sasaki (2001)
- A. Galligo, D. Rupprecht (2001); G. Chèze, A. Galligo (2003)
- R.M. Corless, M.W. Giesbrecht, M. van Hoeij, I.S. Kotsireas, S.M. Watt (2001)
- A.J. Sommese, J. Verschelde, C.W. Wampler (2001, 2002, 2004)
- S. Gao, E. Kaltofen, J. May, Z. Yang, L. Zhi (2004)
- A. Galligo, A. Poteaux (2009)

Numerical Irreducible Decomposition

Given a pure dimensional solution set, represented by

- a system with as many random hyperplanes as the dimension;
- as many points on the planes as the degree of the set.

Algebraic curves are Riemann surfaces...

1. loops to connect points on same irreducible factor (problem is more topologic than algebraic);
2. certify a decomposition with linear trace.

with Anton Leykin: **Decomposing Solution Sets of Polynomial Systems: A New Parallel Monodromy Breakup Algorithm.** *The International Journal of Computational Science and Engineering* 4(2):94-101, 2009.

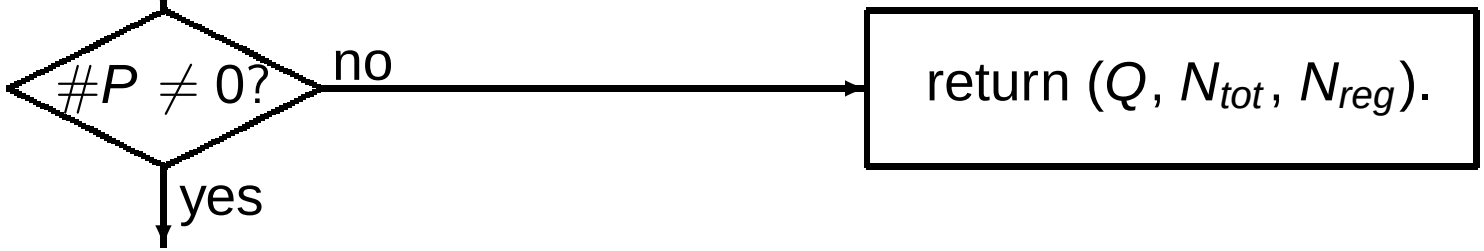
Using Monodromy to Factor

$P := \{\{a\} \mid 1 \leq a \leq d,\ \{a\}$ is not a component$\}$;
$Q := \{f \mid f \subset \{1, 2, \ldots, d\}$ is a certified irreducible factor$\}$;
construct s witness sets using s random slices;
construct the trace grid, for 2 parallel slices;
$N_{tot} := s \times d + 2 \times d$; $\quad N_{reg} := 0$;

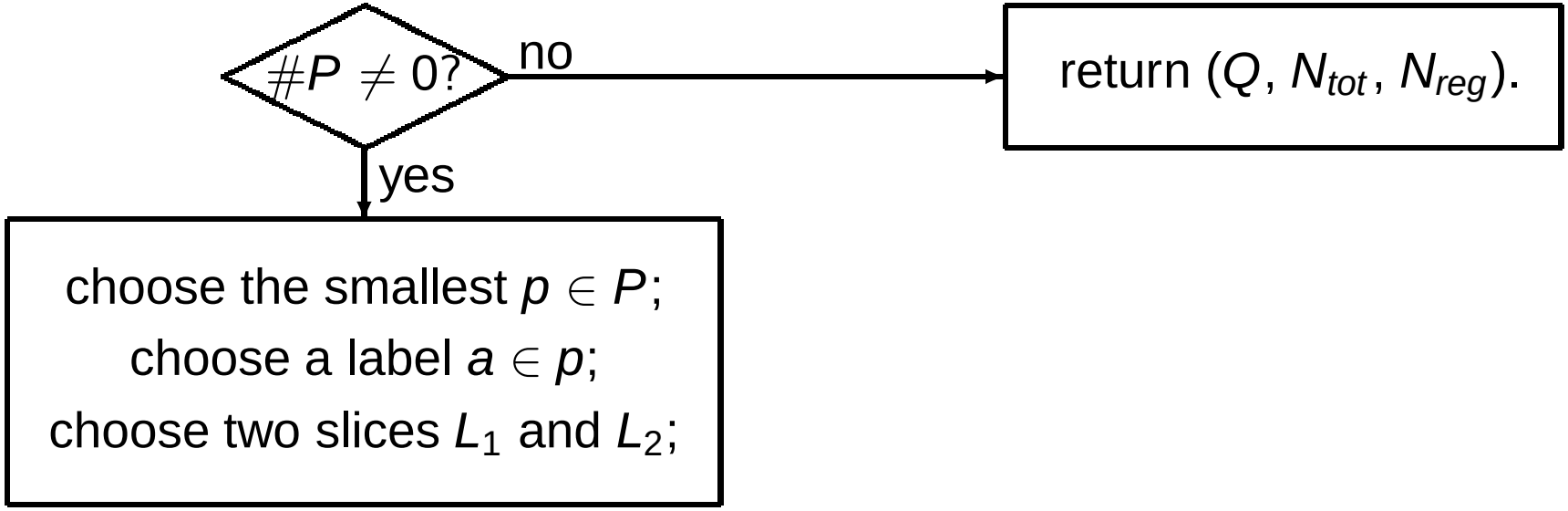
The initialization requires $(s+2)d$ paths

Using Monodromy to Factor

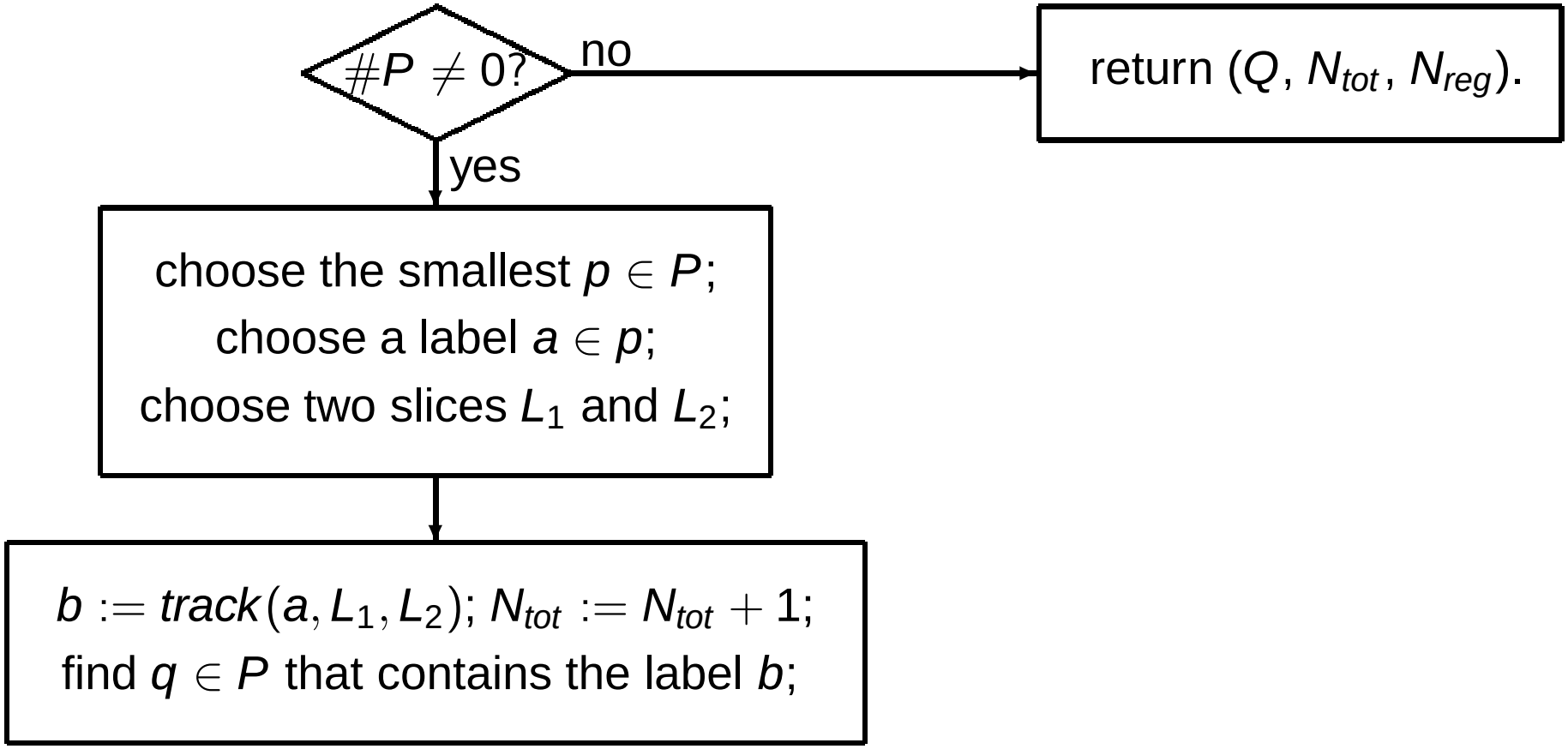
$P := \{\{a\} \mid 1 \leq a \leq d,\ \{a\} \text{ is not a component}\}$;
$Q := \{f \mid f \subset \{1, 2, \ldots, d\} \text{ is a certified irreducible factor}\}$;
construct s witness sets using s random slices;
construct the trace grid, for 2 parallel slices;
$N_{tot} := s \times d + 2 \times d$; $N_{reg} := 0$;



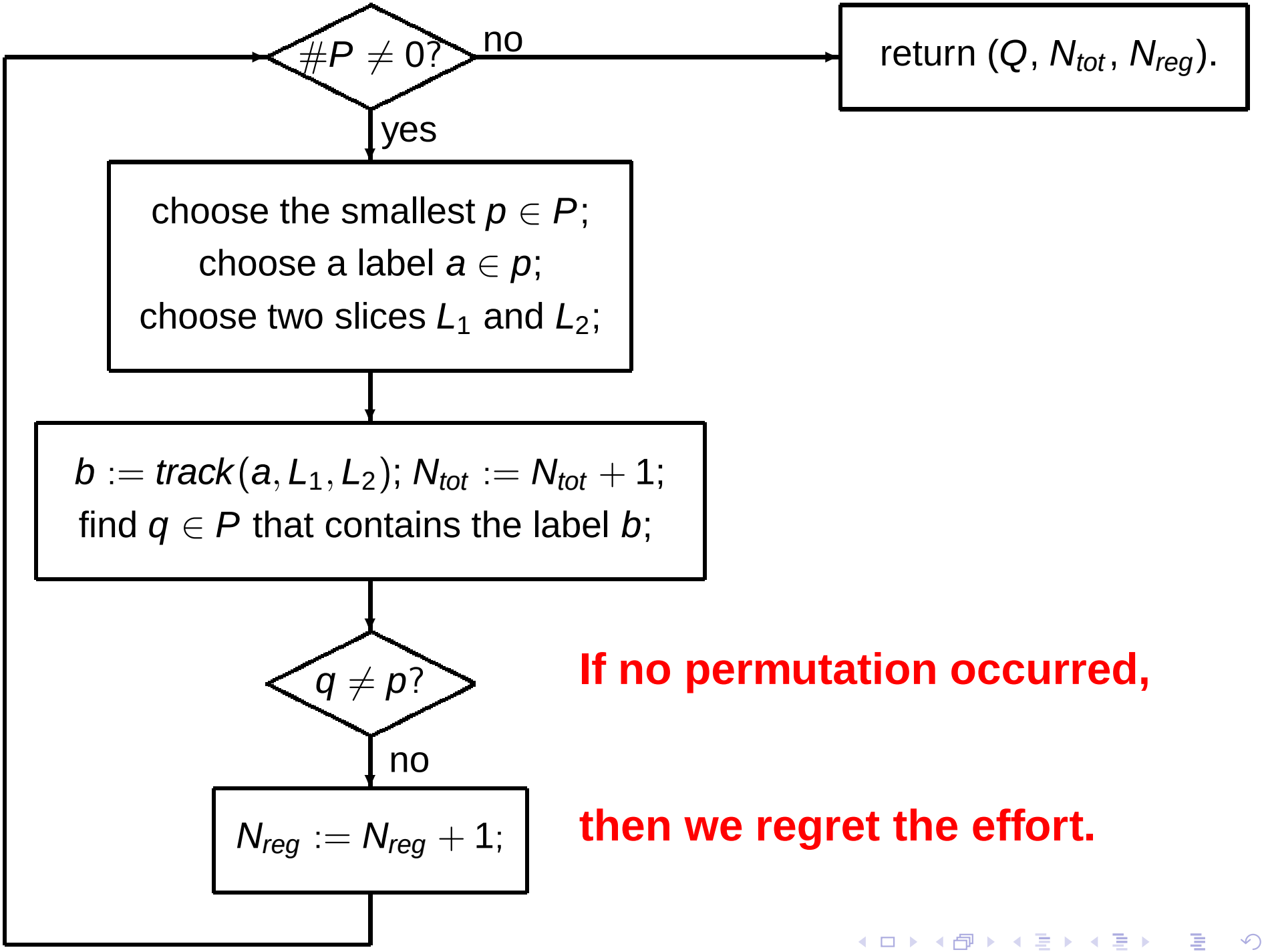
On return is a certified factorization, and path tracking statistics.



Use path tracking statistics to discriminate against nonproductive slices.

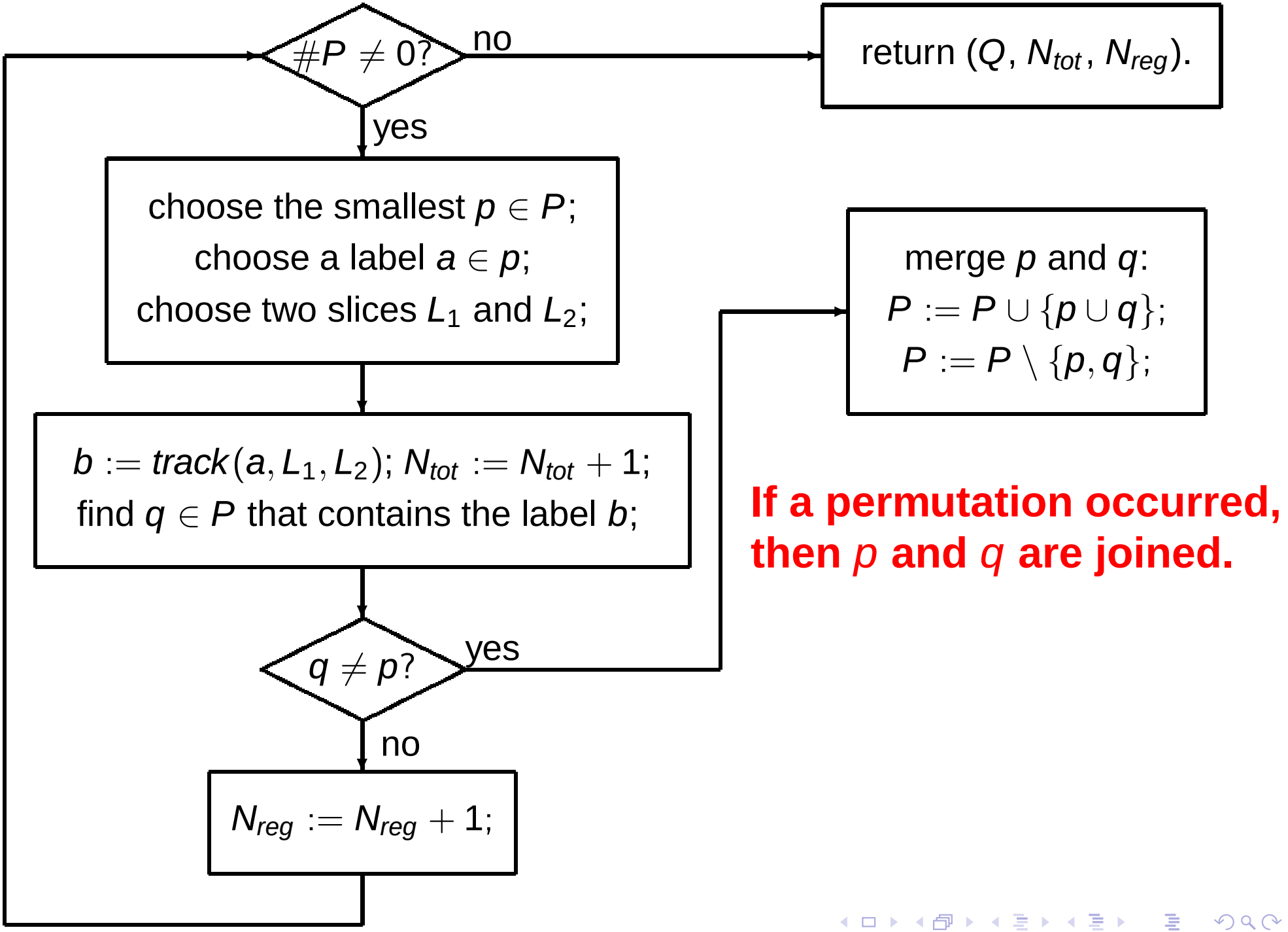


Track a path to close a loop.

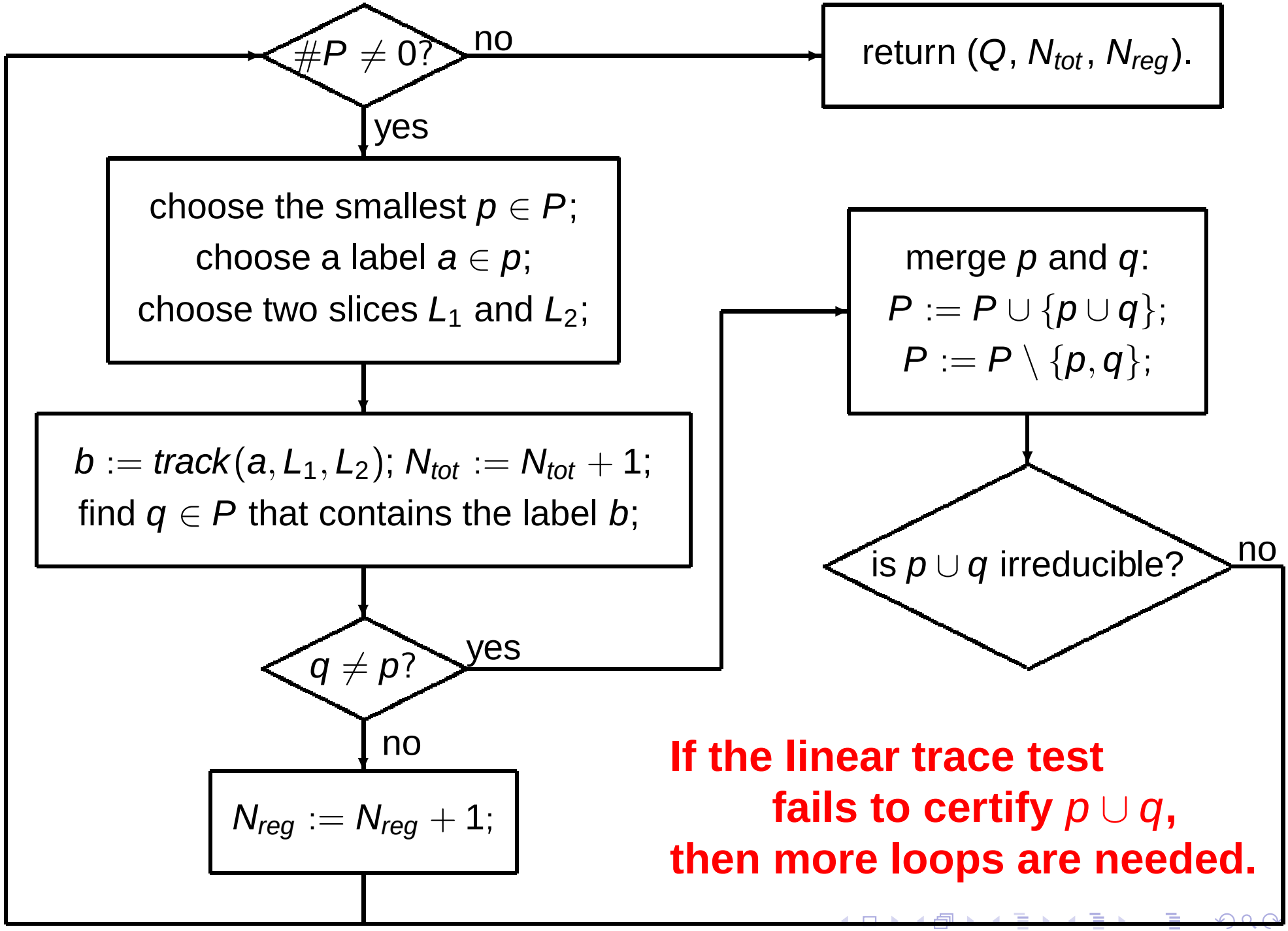


If no permutation occurred,

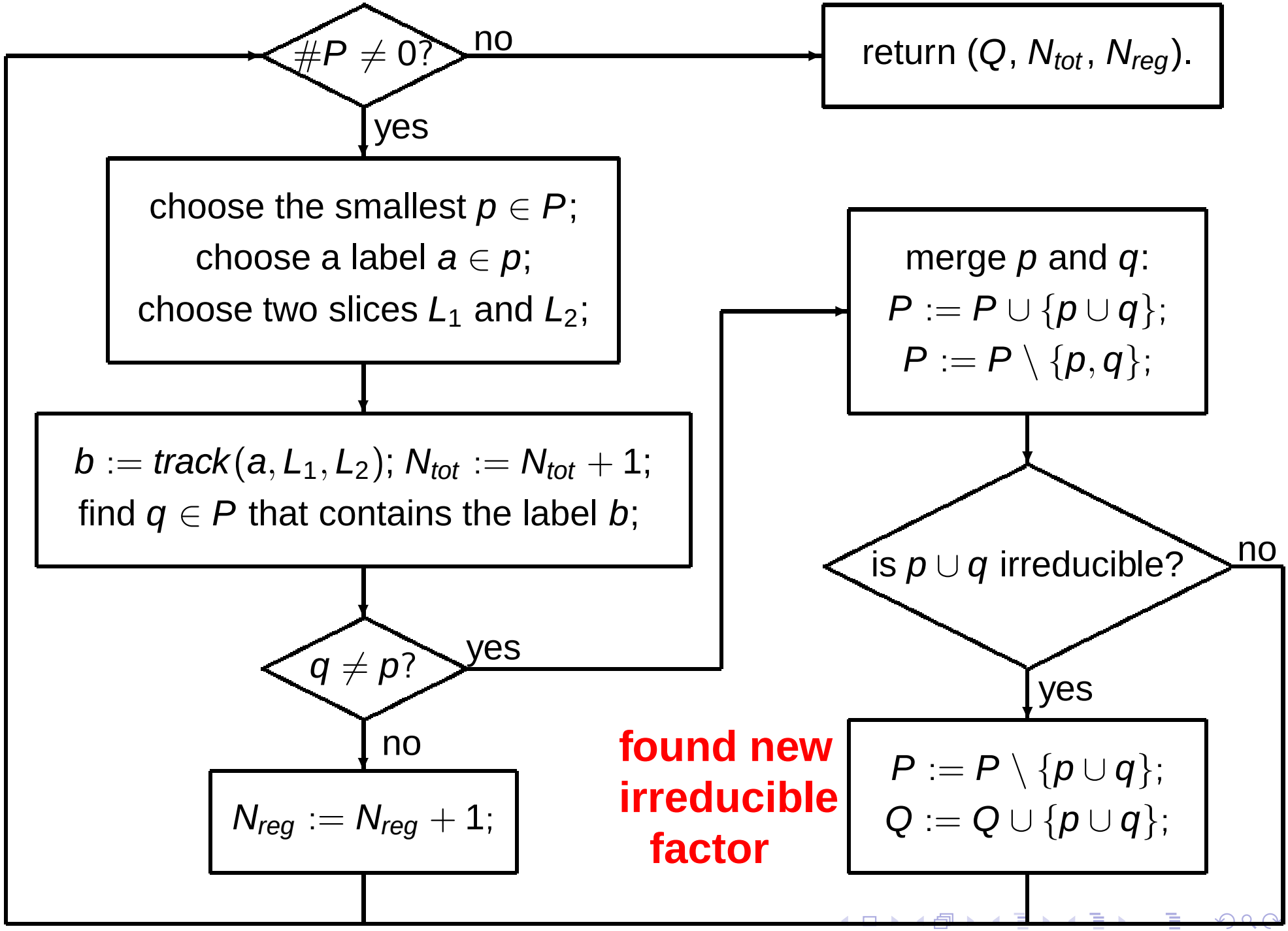
then we regret the effort.



If a permutation occurred, then p and q are joined.



If the linear trace test fails to certify $p \cup q$, then more loops are needed.


$\#P \neq 0$?
no
return (Q, N_{tot}, N_{reg}).
yes
choose the smallest $p \in P$;
choose a label $a \in p$;
choose two slices L_1 and L_2;
$b := track(a, L_1, L_2)$; $N_{tot} := N_{tot} + 1$;
find $q \in P$ that contains the label b;
merge p and q:
$P := P \cup \{p \cup q\}$;
$P := P \setminus \{p, q\}$;
is $p \cup q$ irreducible?
no
yes
$q \neq p$?
yes
no
found new irreducible factor
$P := P \setminus \{p \cup q\}$;
$Q := Q \cup \{p \cup q\}$;
$N_{reg} := N_{reg} + 1$;

The New Algorithm – Parallel Version

Using a manager/worker model:

- Initialization phase involves the distribution of d paths,
  for the s new witness sets and trace grid.
  Manager distributes jobs to path tracking nodes i, $i > 0$.
- After initialization:
  Manager keeps looking for available nodes to assign paths.
  Other nodes are either busy or ready to start new jobs.

Compared to our first parallel implementation, this algorithm interleaves the computation of the linear trace by the manager with the distribution of path tracking jobs.

Experiments with MPI

- The new parallel monodromy breakup algorithm shows an even distribution of the time spent by the nodes.
- Using fewer slices reduces initialization time at the expense of a higher running time in main loop.
- Compared to the first parallel implementation, the new algorithm shows a more predictable and regular performance.
- On larger examples, e.g. factoring a 10-dimensional surface of degree 256 in $\mathbb{C}^{18}$, the new algorithm still takes only 80% of the very best time of the first parallel implementation.

Conclusions and Future Plans

Conclusions:

- multitasking is convenient programming model
- effective speedups on modern workstation

Some future applications:

- multi-tiered parallel implementations
- investigation of granularity issues
- attention for quality up (not only speedup)
- parallel versions of Littlewood-Richardson homotopies