Parallel Implementation of a Subsystem-by-Subsystem Solver

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Outline

Problems we want to solve

Homotopy methods to solve polynomial systems
are "pleasingly parallel":

- the solution paths can be tracked independently;
- scale very well for a large number of processors.

$\rightarrow$ enumerate solutions one after the other

What are we solving?

- large systems in families of benchmark problems such as katsura, economics, adjacent minors;
- systems with more than 100,000 solutions;
- optimal case (no diverging paths).

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The Total Degree Homotopy

Solve by considering a simpler system in a homotopy

$$\underbrace{\left(\left\{ \begin{array}{r} x_1^2 + x_2 - 3 = 0 \\ x_1 + 0.125 x_2^2 - 1.5 = 0 \end{array} \right. \right)}_{\text{target system}} t + \gamma \underbrace{\left(\left\{ \begin{array}{r} x_1^2 - 1 = 0 \\ x_2^2 - 1 = 0 \end{array} \right. \right)}_{\text{start system}} (1-t) = \mathbf{0}$$

where t goes from 0 to 1, and $\gamma \in \mathbb{C}$ is a random constant.

For almost all choices of $\gamma \in \mathbb{C}$, every isolated solution of multiplicity *m* is reached by exactly *m* solution paths.

also called "the gamma trick"

If we take $\gamma = 1$, then at $t \approx 0.92$ singular solutions occur.

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running total degree homotpies

Katsura systems: n quadrics and one linear equation, #solutions is 2^n. Running on the NCSA machine Tungsten, using p processors, for $n = 20$: 1,048,576 paths:

p	time	min	max
16	22h47m	68,088	70,966
32	9h22m	33,329	34,113
64	4h44m	16,269	16,917
128	2h25m	7,199	8,639
256	1h16m	4,525	3,731

The table lists the total time and the minimum and maximum number of paths tracked by each worker node.
Different homotopy constants cause fluctuations.

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Witness Sets

- *Numerical representations* of positive dimensional solution sets of polynomial systems.
- A k-dimensional solution set of degree d is represented by
  1. k general hyperplanes; and
  2. d isolated solutions on those k hyperplanes.
- Witness sets are computed either
  1. *top down*: via a cascade of homotopies; or
  2. *bottom up*: diagonal homotopies intersect witness sets.
- Once solution sets of different dimensions are separated as different witness sets, with monodromy and traces we compute ***a numerical irreducible decomposition***.

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- A.J. Sommese and C.W. Wampler:
  Numerical algebraic geometry. In *The Mathematics of Numerical Analysis*, pages 749–763, AMS 1996.
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References for Diagonal Homotopies

- A.J. Sommese, J. Verschelde, and C.W. Wampler: **Homotopies for intersecting solution components of polynomial systems**. *SIAM J. Numerical Anal.* 42(4):1552-1571, 2004.
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- A.J. Sommese and C.W. Wampler: **The Numerical Solution of Systems of Polynomials Arising in Engineering and Science**. World Scientific Press, 2005.

What does a diagonal homotopy do?

input/output specification

Input: two irreducible components A and B
given by two witness sets:

Witness Set for A	Witness Set for B
$\begin{cases} f_A(x) = 0 \\ L_A(x) = 0 \end{cases}$	$\begin{cases} f_B(x) = 0 \\ L_B(x) = 0 \end{cases}$
$\sharp L_A = dim$(A)=a	$\sharp L_B = dim$(B)=b
$\{\alpha_1, \alpha_2, \ldots, \alpha_{deg(A)}\}$	$\{\beta_1, \beta_2, \ldots, \beta_{deg(B)}\}$

Output: witness sets for all pure dimensional components of A∩B

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What does diagonal homotopy do?

a special case

1. Solution pairs start a cascade of homotopies.
2. Hyperplanes are removed one by one in the cascade.

Special case: A and B are complete intersections, stored as $\sharp\{\mathbf{x} \in \mathbb{C}^n | f_A(\mathbf{x}) = 0, L_A(\mathbf{x}) = 0\} = \deg(A)$, $\sharp\{\mathbf{y} \in \mathbb{C}^n | f_B(\mathbf{y}) = 0, L_B(\mathbf{y}) = 0\} = \deg(B)$, and $\dim(A \cap B) = 0$, then the diagonal homotopy is

$$h(\mathbf{x}, \mathbf{y}, t) = \begin{cases} f_A(\mathbf{x}) = 0, f_B(\mathbf{y}) = 0 \\ (1-t)\begin{pmatrix} L_A(\mathbf{x}) \\ L_B(\mathbf{y}) \end{pmatrix} + t(\mathbf{x} - \mathbf{y}) = 0, \end{cases}$$

starting at the $\deg(A) \times \deg(B)$ solutions in $A \times B \in \mathbb{C}^{n+n}$.

At $t = 1$, we find solutions at the diagonal $\mathbf{x} = \mathbf{y}$, in $A \cap B$.

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Parallel Diagonal Homotopy

- Runs in various stages: every stage removes one hyperplane in the cascade of homotopies.
- Currently we use the extrinsic version of the diagonal homotopy.
- For memory efficiency, *jumpstarting* homotopy:
  1. The manager computes a start solution or reads it from file "just in time" whenever a worker needs a path tracking job.
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An Illustration

Assume two witness sets are completed, each has degree 4.
Using 5 workers:

manager		**workers**
path 1 to node 1	*(1,1)*	node 1 receives path 1
path 2 to node 2	*(1,2)*	node 2 receives path 2
path 3 to node 3	*(1,3)*	node 3 receives path 3
path 4 to node 4	*(1,4)*	node 4 receives path 4
resetting file for witness set 2		
path 5 to node 1	*(2,1)*	node 1 receives path 5
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Divide and Conquer

Schematic overview of solving a system of eight quadrics.

1: 2
1,2: 4
2: 2
1,2,3,4: 16
3: 2
3,4: 4
4: 2
1,2,3,4,5,6,7,8: 256
5: 2
5,6: 4
6: 2
5,6,7,8: 16
7: 2
7,8: 4
8: 2

Assume homotopy is optimal: no diverging paths.

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2: 2
1,2: 4

3: 2
4: 2
3,4: 4

1,2,3,4: 16

5: 2
6: 2
5,6: 4

7: 2
8: 2
7,8: 4

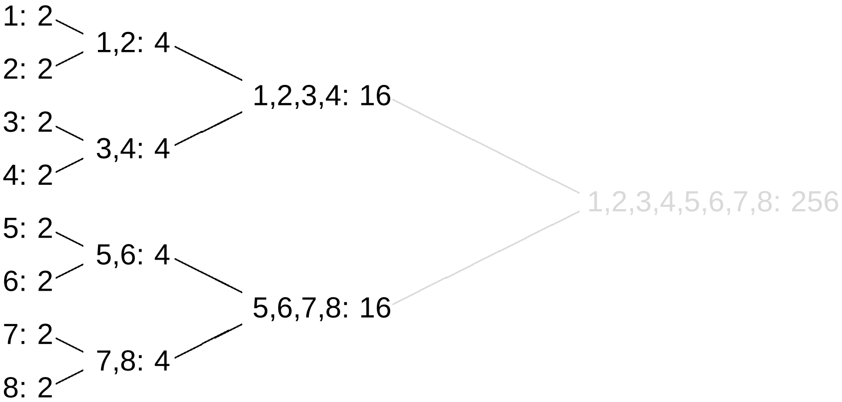
5,6,7,8: 16

1,2,3,4,5,6,7,8: 256

Assume homotopy is optimal: no diverging paths.

Divide and Conquer

Schematic overview of solving a system of eight quadrics.



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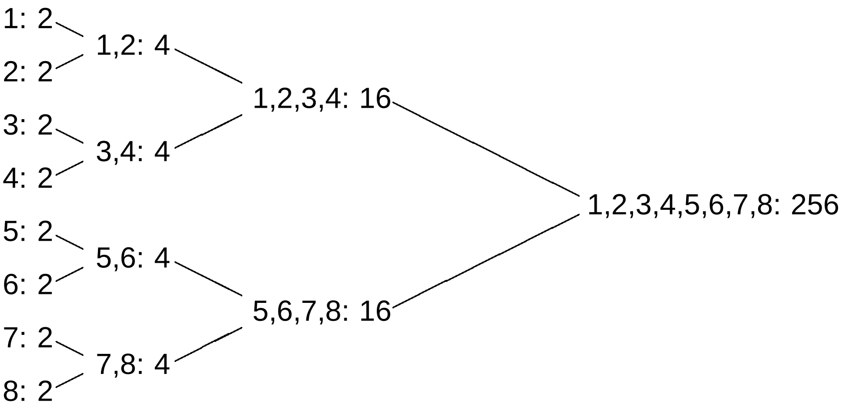

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Data Structures

The triangular state table

1	.....	.....	.....
2	.....	.....	
3	.....		
4	.....		
5			
6			
7			
8			

of completed jobs

Queue of jobs

...	job 4	job 3	job 2	job 1

Queue of idle workers

...	worker 3	worker 2	worker 1

One job

2	1	2	1	2	3	4	2

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Initial Job Distribution

manager		**worker**	
broadcast file name	$\rightarrow$	receive file name	*file with equations*
send data	$\rightarrow$	receive data solve equation write to file	*data =* *equation indices* *terminated by 0*
receive data	$\leftarrow$	send data	*synchronization*

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- Runs in $\lceil \log_2(n) \rceil$ stages, $n = $ #equations.
- Homotopies in stage k involve 2^k equations.
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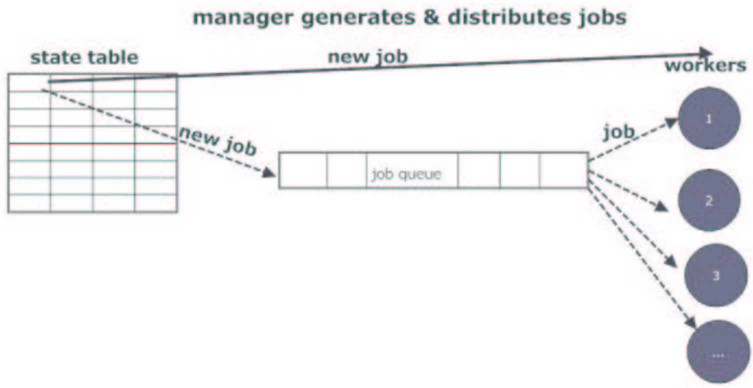
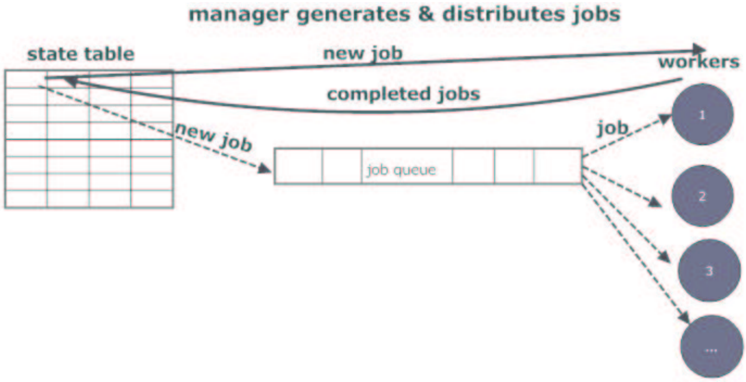
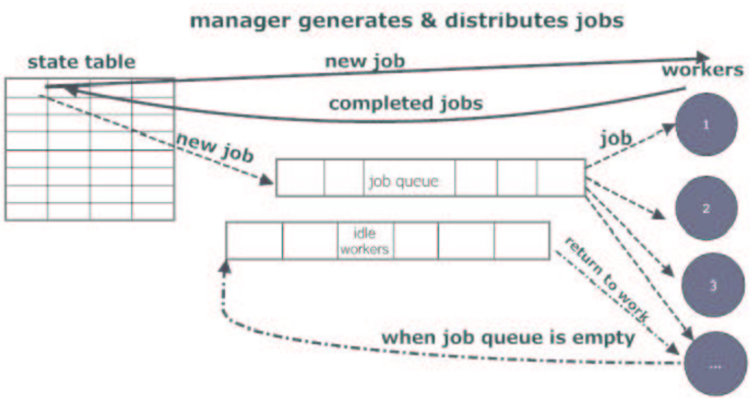
Job Scheduling: main loop, a picture



Job Scheduling: main loop, continued



Job Scheduling: main loop, finally



Software & Equipment

Diagonal homotopies are available in PHCpack.
`http://www.math.uic.edu/~jan/download.html`

1. Parallel code uses and improves sequential versions.
2. `PHClib` forms interface with PHCpack as library.
3. Main parallel programs use MPI for communication.

Computers used:

- Software development on personal cluster:
    1. One workstation with two dual 2.4Ghz processors.
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Complexity of Job Scheduling

Job scheduling uses dynamic load balancing.
Some additional concerns:

- There must be sufficient points in both witness sets in order to intersect a pair of witness sets.
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- The solutions for the second witness set are arriving much slower than those for the first witness set.

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Solving katsura8

on a 2.4Ghz Rocketcalc personal cluster

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2	459s	1,408	1,408
3	277s	787	621
4	175s	514	391
5	140s	375	289
6	104s	307	240
7	98s	251	207
8	86s	218	173
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10	81s	167	132
11	72s	152	124
12	68s	147	110

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double #workers (1,2,4,8):
$p = 2 \rightarrow 3 \rightarrow 5 \rightarrow 9$

time: 459s
$\rightarrow$ 277s
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Summary

- parallel diagonal homotopy allows jumpstarting for efficient memory management
- dynamic load balancing leads to acceptable speedup
- synchronization along stages gives overhead