Polynomial Homotopy Continuation with PHCpack

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PHCpack

software for Polynomial Homotopy Continuation

The name of PHCpack is modeled after the successful line of `*PACK` collections of codes in numerical analysis.

PHCpack consists of

1. open source code in Ada with interfaces to C and Python, compiles with `gcc`, available as a software package;
2. an executable program `phc` for various platforms with interfaces for Maple, MATLAB (Octave), Macaulay 2.

ACM TOMS archived version 1.0 of PHCpack as Algorithm 795. The current version is 2.3.56.

`http://www.math.uic.edu/~jan/download.html`

Related Software

- **Bertini** [D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler];
- **CONSOL** [A.P. Morgan];
- **HOM4PS-2.0** [T.L. Lee, T.Y. Li, and C.H. Tsai];
- **HomLab** [C.W. Wampler];
- **NAG4M2** [A. Leykin];
- **POLSYS_PLP** [S.M. Wise, A.J. Sommese, L.T. Watson];
- **POLSYS_GLP** [H.-J. Su, J.M. McCarthy, M. Sosonkina, and L.T. Watson] is part of HOMPACK as is POLSYS_PLP;
- **PHoM** [T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa, and T. Mizutani].

For mixed volumes: **DEMiCs** [T. Mizutani and A. Takeda];
Mixvol [I.Z. Emiris and J.F. Canny]; and
MixedVol [T. Gao, T.Y. Li, and M. Wu].

symbolic-numeric view on polynomial homotopies

Our goal is to solve a polynomial system $f(\mathbf{x}) = \mathbf{0}$.

A polynomial homotopy continuation method consists in two parts:

1. definition of a family of systems, a **homotopy**, e.g.:

   $$h(\mathbf{x}, t) = (1 - t)\, g(\mathbf{x}) + t\, f(\mathbf{x}) = \mathbf{0}, \quad t \in [0, 1],$$

   where $g(\mathbf{x}) = \mathbf{0}$ is a generic system with known solutions;
2. path tracking or **continuation** methods using predictor-corrector to approximate solution paths $\mathbf{x}(t)$ defined by $h(\mathbf{x}(t), t) = \mathbf{0}$.

A homotopy is ***optimal*** if no paths diverge.
We have optimal homotopies for several classes of systems, e.g.:

1. polyhedral homotopies for sparse systems with fixed supports;
2. Littlewood-Richardson homotopies for Schubert problems.

overview of demonstration

1. an improved blackbox solver `phc -b`
   1. faster mixed volume computation with `MixedVol`
   2. Laurent systems on input (negative exponents)
   3. parallel execution using threads mapped to cores
2. numerical algebraic geometry
   1. witness sets represent positive dimensional solution sets
   2. monodromy factorization certified by linear trace
   3. a bottom-up blackbox solver in `phc -a`
   4. demonstration with Maple worksheet
3. sweeping for singularities
   1. tracking one path defined by a natural parameter homotopy
   2. isolated real points on a complex curve are singular

an improved blackbox solver phc -b

- Assumes square system and only isolated solutions wanted.
- Polyhedral homotopies if mixed volume $<$ Bézout bounds.

At the command prompt, do **phc -b input output**
where **input** and **output** are file names.

Example of an **input** file:

```
2
 x*y + x^-1 + y^-1 - 1;
 x^-1*y + x*y^-1 + 1;
```

With negative exponents: no solutions with zero components.
Otherwise, computes stable mixed volumes with MixedVol.

To use 8 threads: **phc -b -t8 input output**
(with Genady Yoffe, see PASCO 2010 proceedings).

Representations of Polynomials and Solutions

Polynomials are mapped to file similar to Maple's `lprint` or string conversion and can be parsed in a direct manner.
Symbols not to use as variable names: `i`, `I`, `e`, `E`.

A solution is represented by

1. a value for the continuation parameter t;
2. a multiplicity flag;
3. real and imaginary parts as values for the variables;
4. three numerical attributes for an approximate zero $\mathbf{z}$:
   1. `err` $||\Delta \mathbf{z}||$, or estimate for forward error;
   2. `rco`: estimate for the inverse of the condition number;
   3. `res`: $||f(\mathbf{z})||$, or estimate for backward error.

For PHCmaple (with Anton Leykin), `phc -z` takes an output file and writes the solutions as a Maple list.
Similar conversions for MATLAB or Octave in PHClab (with Yun Guan).

Numerical Algebraic Geometry

[Sommese & Wampler, 1996]: a pun on numerical linear algebra.

Highlights of a sequence of joint papers:

- cascades of homotopies to separate pure dimensional sets;
- numerical irreducible decomposition with homotopy continuation via monodromy loops and linear trace;
- diagonal homotopies to intersect solution sets.

Joint with Anton Leykin and Ailing Zhao:

- deflation to recondition isolated singular solutions.

Andrew J. Sommese and Charles W. Wampler: *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science.* World Scientific, 2005.

witness sets

To represent an $(n-k)$-dimensional subset of $f^{-1}(\mathbf{0})$ of degree d:

1. system $f(\mathbf{x}) = \mathbf{0}$ augmented with k general hyperplanes L; and
2. d generic points on $f^{-1}(\mathbf{0}) \cap L$.

This data representation fits the output format
of a run with polynomial homotopies.

Our running example:

```
3
 (x^2 - y)*(x - 0.1);
 (x^3 - z)*(y - 0.3);
 (x*y - z)*(z - 0.5);
```

Twisted cubic: (x, x^2, x^3), (0.1,0.3,0.5) and more isolated roots.
Running `phc -b` directly will not give any witness set.

top down or bottom up

Three stages in a top-down approach:

1. `phc -c, menu option #0`: run a cascade
   Adding one hyperplane, output files are
   1. `ex1out_sw1`: three generic points on twisted cubic;
   2. `ex1out_sw0`: candidate isolated points.

   Solutions may require filtering (`_sw` = super witness set).
2. `phc -f, menu option #2`: breakup a witness set
   Input is the file `ex1out_sw1` and the output is a partition into subsets of generic points for each irreducible component.
3. `phc -f, menu option #1`: filter junk
   Nine solutions are on file in `ex1out_sw0`. To filter points on twisted cubic, we provide a witness set for the twisted cubic. The homotopy membership test shows 5 of 9 lie on twisted cubic.

In the bottom-up approach, `phc -a` is an equation-by-equation solver.

PHCmaple: a Maple interface to PHCpack

- Started in 2003 as one Maple procedure (Maple 7):
  1. calls the blackbox solver with `ssystem`;
  2. loads solutions into Maple.
- Developed and maintained by Anton Leykin.
  `http://www.math.uic.edu/~leykin/PHCmaple`
  also via
  `http://people.math.gatech.edu/~aleykin3/software.html`
- Demonstration: run `usePHCmaple.mw` in Maple 14.

Sweeping for Singular Points

Joint with Kathy Piret, a sweeping homotopy was developed.
Isolated real points on complex curves are singular.

Example of input file to `phc -p` starts with

```
4 5
 x1*x2^2 + x1*x3^2 - A*x1 + 1;
 x2*x1^2 + x2*x3^2 - A*x2 + 1;
 x3*x1^2 + x3*x2^2 - A*x3 + 1;
 (1-t)*(A-0.1) + t*(A+0.1);
```

where A is a natural parameter and `t` is an artificial parameter.

As `t` goes from 0 to 1, we sweep solutions
in the parameter range from $+0.1$ to -0.1 for A.

Focus first on detection problem: find value for `t` (and thus also A)
for which the corresponding solution is singular.
Accurate location of singularity happens via deflation.