Polyhedral Homotopy Methods to Solve Polynomial Systems

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IMA postdoc seminar

31 October 2006

Polyhedral Homotopies

Theorems A & B of Bernshteĭn

constructive proofs via deformations

solutions at ∞ are solutions of face systems

Numerical Tools

extrapolation to compute certificates of divergence

separate $\mathbf{z}$ from $\log(|\mathbf{z}|)$ in a numerically stable simplicial solver

Applications from Mechanism Design

design of serial chains, systems of H.-J. Su and J.M. McCarthy

Solving Systems with Homotopies

Concerns *(of anyone who tries to use <u>numerical</u> homotopies)*

1. efficiency: #paths = bound on #solutions;

   how can we find good bounds on #solutions?

2. validation: how can we be sure to have **all** solutions?

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Answers *(why we should consider polyhedral methods)*

1. generically sharp root counts,

   which can be computed by fully automatic blackboxes
2. certificates for diverging paths,

   which are cheap by-products of continuation

Geometric Root Counting

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}}$$

$$c_{i\mathbf{a}} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

$$f = (f_1, f_2, \ldots, f_n)$$

$$P_i = \mathrm{conv}(A_i)$$

Newton polytope

$$\mathcal{P} = (P_1, P_2, \ldots, P_n)$$

$L(f)$ root count in $(\mathbb{C}^*)^n$	desired properties
$L(f) = L(f_2, f_1, \ldots, f_n)$	invariant under permutations
$L(f) = L(f_1 \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	shift invariant
$L(f) \leq L(f_1 + \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	monotone increasing
$L(f) = L(f_1(\mathbf{x}^{U\mathbf{a}}), \ldots, f_n(\mathbf{x}^{U\mathbf{a}}))$	unimodular invariant
$L(f_{11} f_{12}, \ldots, f_n)$ $= L(f_{11}, \ldots, f_n) + L(f_{12}, \ldots, f_n)$	root count of product is sum of root counts

Geometric Root Counting

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$$\mathcal{P} = (P_1, P_2, \ldots, P_n)$$

$$f = (f_1, f_2, \ldots, f_n)$$

properties of $L(f)$	$V(\mathcal{P})$ mixed volume
invariant under permutations	$V(P_2, P_1, \ldots, P_n) = V(\mathcal{P})$
shift invariant	$V(P_1 + \mathbf{a}, \ldots, P_n) = V(\mathcal{P})$
monotone increasing	$V(\mathrm{conv}(P_1 + \mathbf{a}), \ldots, P_n) \geq V(\mathcal{P})$
unimodular invariant	$V(UP_1, \ldots, UP_n) = V(\mathcal{P})$
root count of product is sum of root counts	$V(P_{11} + P_{12}, \ldots, P_n) = V(P_{11}, \ldots, P_n) + V(P_{12}, \ldots, P_n)$

Geometric Root Counting

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$L(f) = L(f_2, f_1, \ldots, f_n)$	$V(P_2, P_1, \ldots, P_n) = V(\mathcal{P})$
$L(f) = L(f_1 \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	$V(P_1 + \mathbf{a}, \ldots, P_n) = V(\mathcal{P})$
$L(f) \leq L(f_1 + \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	$V(\mathrm{conv}(P_1 + \mathbf{a}), \ldots, P_n) \geq V(\mathcal{P})$
$L(f) = L(f_1(\mathbf{x}^{U\mathbf{a}}), \ldots, f_n(\mathbf{x}^{U\mathbf{a}}))$	$V(UP_1, \ldots, UP_n) = V(\mathcal{P})$
$L(f_{11} f_{12}, \ldots, f_n)$ $= L(f_{11}, \ldots, f_n) + L(f_{12}, \ldots, f_n)$	$V(P_{11} + P_{12}, \ldots, P_n)$ $= V(P_{11}, \ldots, P_n) + V(P_{12}, \ldots, P_n)$

exploit sparsity $\boxed{L(f) = V(\mathcal{P})}$ 1st theorem of Bernshteĭn

The Theorems of Bernshteĭn

Theorem A: The number of roots of a generic system equals the mixed volume of its Newton polytopes.

Theorem B: Solutions at infinity are solutions of systems supported on faces of the Newton polytopes.

D.N. Bernshteĭn: **The number of roots of a system of equations.** *Functional Anal. Appl.*, 9(3):183–185, 1975.

Structure of proofs: First show Theorem B, looking at power series expansions of diverging paths defined by a linear homotopy starting at a generic system. Then show Theorem A, using Theorem B with a homotopy defined by *lifting* the polytopes.

Some References on Polyhedral Methods

I.M. Gel'fand, M.M. Kapranov, and A.V. Zelevinsky: **Discriminants, Resultants and Multidimensional Determinants**. Birkhäuser, 1994.

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B. Sturmfels: **Polynomial equations and convex polytopes**. *Amer. Math. Monthly* 105(10):907–922, 1998.

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Bernshteǐn's second theorem

- Face $\partial_\omega f = (\partial_\omega f_1, \partial_\omega f_2, \ldots, \partial_\omega f_n)$ of system $f = (f_1, f_2, \ldots, f_n)$ with Newton polytopes $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ and mixed volume $V(\mathcal{P})$.

$$\partial_\omega f_i(\mathbf{x}) = \sum_{\mathbf{a} \in \partial_\omega A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}} \qquad \underset{\text{face of Newton polytope}}{\partial_\omega P_i = \mathrm{conv}(\partial_\omega A_i)}$$

Theorem: **If $\forall \omega \neq \mathbf{0}$, $\partial_\omega f(\mathbf{x}) = \mathbf{0}$ has no solutions in $(\mathbb{C}^*)^n$, then $V(\mathcal{P})$ is exact and all solutions are isolated.**

Otherwise, for $V(\mathcal{P}) \neq 0$: $V(\mathcal{P}) > \#$isolated solutions.

- Newton polytopes ***in general position***: $V(\mathcal{P})$ is **exact for every nonzero choice of the coefficients**.

Newton polytopes in general position

Consider $f(\mathbf{x}) = \begin{cases} c_{111}x_1x_2 + c_{110}x_1 + c_{101}x_2 + c_{100} = 0 \\ c_{222}x_1^2x_2^2 + c_{210}x_1 + c_{201}x_2 = 0 \end{cases}$

The Newton polytopes:

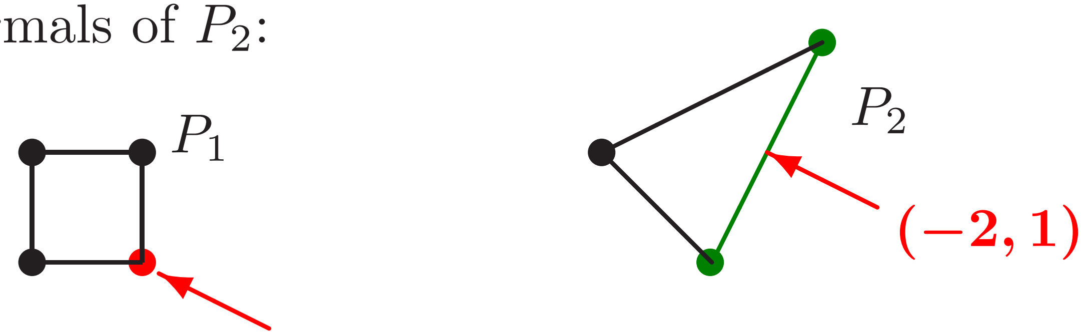
$A_1 = \{(1,1),(1,0),(0,1),(0,0)\}$ $\quad$ $A_2 = \{(2,2),(1,0),(0,1)\}$

$P_1 = \mathrm{conv}(A_1)$ $\quad$ $P_2 = \mathrm{conv}(A_2)$

Newton polytopes in general position

Consider $f(\mathbf{x}) = \begin{cases} c_{111}x_1x_2 + c_{110}x_1 + c_{101}x_2 + c_{100} = 0 \\ c_{222}x_1^2x_2^2 + c_{210}x_1 + c_{201}x_2 = 0 \end{cases}$

Look at the inner normals of P_2:



$\longrightarrow$ the corresponding face system $\begin{cases} c_{110}x_1 = 0 \\ c_{222}x_1^2x_2^2 + c_{210}x_1 = 0 \end{cases}$

does not have a solution in $(\mathbb{C}^*)^2$.

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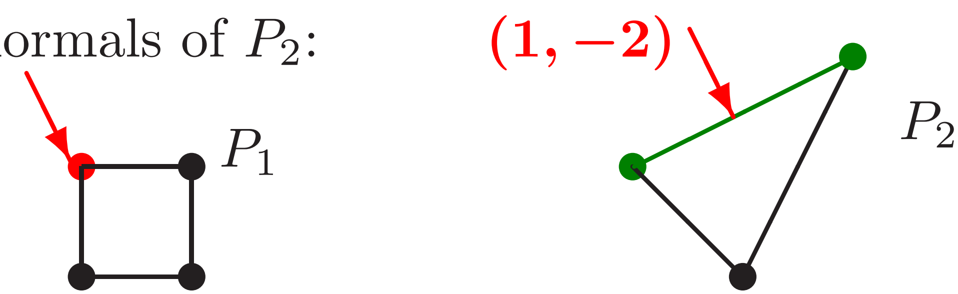

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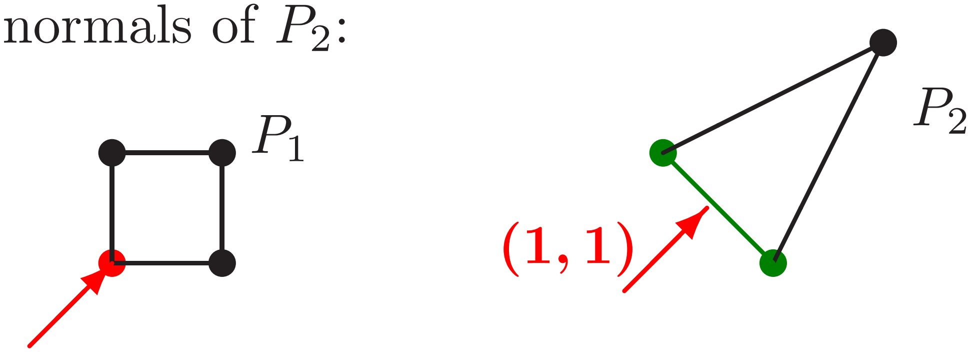

$\longrightarrow$ the corresponding face system $\begin{cases} c_{100} = 0 \\ c_{210}x_1 + c_{201}x_2 = 0 \end{cases}$

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Newton polytopes in general position

Consider $f(\mathbf{x}) = \begin{cases} c_{111}x_1x_2 + c_{110}x_1 + c_{101}x_2 + c_{100} = 0 \\ c_{222}x_1^2x_2^2 + c_{210}x_1 + c_{201}x_2 = 0 \end{cases}$

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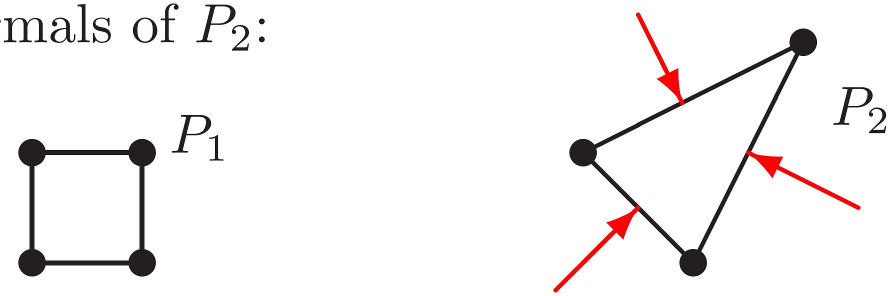

$\forall \omega \neq \mathbf{0} : \partial_\omega A_1 + \partial_\omega A_2 \leq 3 \quad \Rightarrow \quad V(P_1, P_2) = 4$ **always exact**

for all nonzero coefficients

Power Series

Theorem: $\forall \mathbf{x}(t)$, $H(\mathbf{x}(t), t) = (1-t)g(\mathbf{x}(t)) + tf(\mathbf{x}(t)) = \mathbf{0}$,

$\exists s > 0$, $m \in \mathbb{N} \setminus \{0\}$, $\omega \in \mathbb{Z}^n$:

$$\begin{cases} x_i(s) = b_i s^{\omega_i}(1 + O(s)), & i = 1, 2, \ldots, n \\ t(s) = 1 - s^m & \text{for } t \approx 1, s \approx 0 \end{cases}$$

$$\lim_{t \to 1} x_i(t) \in \mathbb{C}^*? \qquad x_i(t) \begin{cases} \to \infty \\ \in \mathbb{C}^* \\ \to 0 \end{cases} \Leftrightarrow \quad \omega_i \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases}$$

m is the ***winding number***, i.e.: the smallest number so that

$$\mathbf{z}(2\pi m) = \mathbf{z}(0), \quad H(\mathbf{z}(\theta), t(\theta)) = \mathbf{0}, \quad t = 1 + (t_0 - 1)e^{i\theta}, \quad t_0 \approx 1.$$

Face Systems and Power Series

assume $\lim_{t \to 1} x_i(t) \notin \mathbb{C}^*$, thus $\omega_i \neq 0$, a **diverging** path

- $H(\mathbf{x}, t) = (1-t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}$ $\quad \left\{ \begin{array}{c} x_i(s) = b_i s^{\omega_i}(1 + O(s)) \\ t(s) = 1 - s^m, s \approx 0 \end{array} \right.$

  substitute power series

$$H(\mathbf{x}(s), t(s)) = \underbrace{f(\mathbf{x}(s))}_{\text{dominant as } s \to 0} + s^m(g(\mathbf{x}(s)) - f(\mathbf{x}(s))) = \mathbf{0}$$

only f matters

Face Systems and Power Series

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- $f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}} \to f_i(\mathbf{x}(s)) = \underbrace{\sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \prod_{i=1}^{n} b_i^{a_i} s^{\langle \mathbf{a}, \omega \rangle}}_{\partial_\omega f_i(\mathbf{x}(s)) \text{ dominant}} (1 + O(s))$

  face $\partial_\omega A_i := \{ \ \mathbf{a} \in A_i \mid \langle \mathbf{a}, \omega \rangle = \min_{\mathbf{a}' \in A_i} \langle \mathbf{a}', \omega \rangle \ \}$

  $\Rightarrow \partial_\omega f(\mathbf{b}) = \mathbf{0}, \mathbf{b} \in (\mathbb{C}^*)^n$

solution at ∞ is a solution of a face system

Richardson Extrapolation for ω and m

$$\left\{ \begin{array}{rcl} x_i(s) & = & b_i s^{\omega_i}(1 + O(s)) \\ t(s) & = & 1 - s^m \end{array} \right.$$

$$x_i(s_k) = b_i h^{k\omega_i/m} s_0 (1 + O(h^{k/m} s_0))$$

Geometric sampling $0 < h < 1$

$$1 - t_k = h(1 - t_k) = \cdots = h^k(1 - t_0)$$

$$s_k = h^{1/m} s_{k-1} = \cdots = h^{k/m} s_0$$

Input: $(\mathbf{x}(s), t(s))$ solutions along a path, $H(\mathbf{x}(s), t(s)) = \mathbf{0}$.

Output: approximations for ω and m.

Richardson Extrapolation for ω and m

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Take logarithms to find exponents of power series:

- $\log|x_i(s_k)| = \log|b_i| + \frac{k\omega_i}{m}\log(h) + \omega_i \log(s_0)$

  $+\log(1+\sum_{j=0}^{\infty} b'_j(h^{k/m}s_0)^j)$

  $v_{kk+1} := \log|x_i(s_k+1)| - \log|x_i(s_k)|$

Extrapolation on samples

$$v_{k..l} = v_{k..l-1} + \frac{v_{k+1..l} - v_{k..l-1}}{1-h}$$

$$\omega_i = m\frac{v_{0..r}}{\log(h)} + O(s_0^r)$$

$\longrightarrow$ first-order approximation for ω

... is okay for $m = 1$

Richardson Extrapolation for ω and m

$$\begin{cases} x_i(s) = b_i s^{\omega_i}(1+O(s)) \\ t(s) = 1-s^m \end{cases}$$

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$$\omega_i = m\frac{v_{0..r}}{\log(h)} + O(s_0^r)$$

- $e_i^{(k)} = (\log|x_i(s_k)| - \log|x_i(s_{k+1})|)$

$$-(\log|x_i(s_{k+1})| - \log|x_i(s_{k+2})|)$$

$$= c_1 h^{k/m} s_0(1+0(h^{k/m}))$$

$$e_i^{(kk+1)} := \log(e_i^{(k+1)}) - \log(e_i^{(k)})$$

Extrapolation on errors

$$e_i^{(k..l)} = e_i^{(k+1..l)} + \frac{e_i^{(k..l-1)} - e_i^{(k+1..l)}}{1-h_{k..l}}$$

$$h_{k..l} = h^{(l-k-1)/m_{k..l}}$$

$$m_{k..l} = \frac{\log(h)}{e_i^{(k..l)}} + O(h^{(l-k)k/m})$$

the system of Cassou-Noguès

$$
f(b,c,d,e) = \left\{
\begin{array}{rcl}
15b^4cd^2 + 6b^4c^3 + 21b^4c^2d - 144b^2c - 8b^2c^2e & & \\
-28b^2cde - 648b^2d + 36b^2d^2e + 9b^4d^3 - 120 & = & 0 \\
30c^3b^4d - 32de^2c - 720db^2c - 24c^3b^2e - 432c^2b^2 + 576ec & & \\
-576de + 16cb^2d^2e + 16d^2e^2 + 16e^2c^2 + 9c^4b^4 + 5184 & & \\
+39d^2b^4c^2 + 18d^3b^4c - 432d^2b^2 + 24d^3b^2e - 16c^2b^2de - 240c & = & 0 \\
216db^2c - 162d^2b^2 - 81c^2b^2 + 5184 + 1008ec - 1008de & & \\
+15c^2b^2de - 15c^3b^2e - 80de^2c + 40d^2e^2 + 40e^2c^2 & = & 0 \\
261 + 4db^2c - 3d^2b^2 - 4c^2b^2 + 22ec - 22de & = & 0
\end{array}
\right.
$$

Root counts: $D = 1344$, $B = 312$, $V(\mathcal{P}) = 24 > 16$ finite roots.

$$
\begin{array}{l}
\partial_{(0,0,0,-1)} f(b,c,d,e) = \\
m = 2
\end{array}
\left\{
\begin{array}{rcccl}
-8b^2c^2e - 28b^2cde + 36b^2d^2e & = & 0 & = & -2c^2 - 7cd + 9d^2 \\
-32de^2c + 16d^2e^2 + 16e^2c^2 & = & 0 & = & -2dc + d^2 + c^2 \\
-80de^2c + 40d^2e^2 + 40e^2c^2 & = & 0 & = & -2dc + d^2 + c^2 \\
22ec - 22de & = & 0 & = & c - d
\end{array}
\right.
$$

More References on Polyhedral Methods

J.M. Rojas: **Toric Laminations, Sparse Generalized Characteristic Polynomials, and a Refinement of Hilbert's Tenth Problem**. In F. Cucker and M. Shub, editors, *Foundations of Computational Mathematics*, pages 369–381, Springer-Verlag 1997.

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Bernshteĭn's first theorem

Let $g(\mathbf{x}) = \mathbf{0}$ have the same Newton polytopes $\mathcal{P}$ as $f(\mathbf{x}) = \mathbf{0}$, but with randomly choosen complex coefficients.

I. Compute $V_n(\mathcal{P})$:			II. Solve $g(\mathbf{x}) = \mathbf{0}$:	
I.1	lift polytopes	$\Leftrightarrow$	II.1	introduce parameter t
I.2	mixed cells	$\Leftrightarrow$	II.2	start systems
I.3	volume of mixed cell	$\Leftrightarrow$	II.3	path following

III. Coefficient-parameter continuation to solve $f(\mathbf{x}) = \mathbf{0}$:

$$H(\mathbf{x}, t) = (1 - t)g(\mathbf{x}) + t f(\mathbf{x}) = \mathbf{0}, \quad \text{for } t \text{ from 0 to 1.}$$

#isolated solutions in $(\mathbb{C}^*)^n$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, of $f(\mathbf{x}) = \mathbf{0}$ is bounded by the mixed volume of the Newton polytopes of f.

Finding Mixed Cells

$$g(x_1, x_2, t) = \left\{ \begin{array}{r} c_{111} x_1 x_2 t^2 + c_{110} x_1 t^7 + c_{101} x_2 t^3 + c_{100} t^3 = 0 \\ c_{222} x_1^2 x_2^2 t^5 + c_{210} x_1 t^3 + c_{201} x_2 t^2 = 0 \end{array} \right.$$

- At $t = 1$: the system $g(\mathbf{x}, 1) = g(\mathbf{x}) = \mathbf{0}$ we want to solve.
- Where to start?

$\longrightarrow$ look for inner normals $\omega \in \mathbb{Z}^3$, $\omega_3 > 0$, such that after

$$x_1 = y_1 s^{\omega_1}, \quad x_2 = y_2 s^{\omega_2}, \quad t = s^{\omega_3},$$

the system $g(\mathbf{y}, s) = \mathbf{0}$ has solutions in $(\mathbb{C}^*)^2$ at $s = 0$.

Coordinate Transformations give Homotopies

$$g(x_1 = y_1, x_2 = y_2 s, t = s)$$

$$= \left\{ \begin{array}{r} c_{111} y_1 (y_2 s) s^2 + c_{110} y_1 s^7 + c_{101} (y_2 s) s^3 + c_{100} s^3 = 0 \\ c_{222} y_1^2 (y_2 s)^2 s^5 + c_{210} y_1 s^3 + c_{201} (y_2 s) s^2 = 0 \end{array} \right.$$

$$= \left\{ \begin{array}{r} c_{111} y_1 y_2 s^3 + c_{110} y_1 s^7 + c_{101} y_2 s^4 + c_{100} s^3 = 0 \\ c_{222} y_1^2 y_2^2 s^7 + c_{210} y_1 s^3 + c_{201} y_2 s^3 = 0 \end{array} \right.$$

$$= \left\{ \begin{array}{r} c_{111} y_1 y_2 + c_{110} y_1 s^4 + c_{101} y_2 s + c_{100} = 0 \\ c_{222} y_1^2 y_2^2 s^4 + c_{210} y_1 + c_{201} y_2 = 0 \end{array} \right.$$

At $s = 0$ we find a binomial system which has two solutions.
The two solutions extend to solutions of $g(\mathbf{x}) = g(\mathbf{x}, s = 1) = \mathbf{0}$.

Coordinate Transformations and Inner Normals

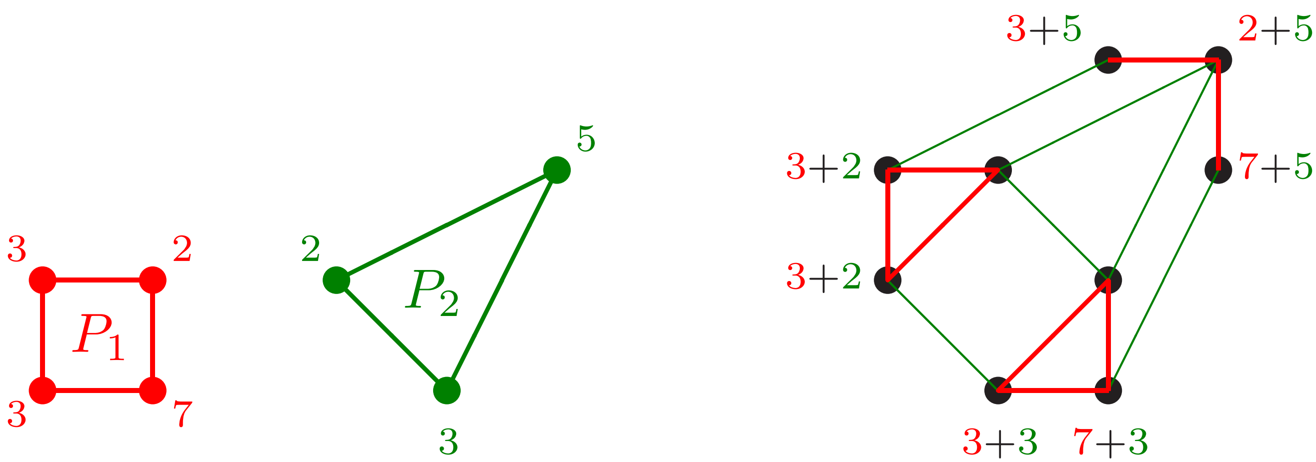
Applying the transformation $(x_1 = y_1 s^{\omega_1}, x_2 = y_2 s^{\omega_2}, t = s^{\omega_3})$, to a lifted monomial $\mathbf{x}^{\mathbf{a}} t^{l(\mathbf{a})}$ yields

$$\begin{aligned} x_1^{a_1} x_2^{a_2} t^{l(\mathbf{a})} &= (y_1 s^{\omega_1})^{a_1} (y_2 s^{\omega_2})^{a_2} (s^{\omega_3})^{l(\mathbf{a})} \\ &= y_1^{a_1} y_2^{a_2} s^{a_1\omega_1 + a_2\omega_2 + a_3 l(\mathbf{a})} \\ &= y_1^{a_1} y_2^{a_2} s^{\langle (\mathbf{a}, l(\mathbf{a})), \omega \rangle}. \end{aligned}$$

A binomial system contains $\mathbf{x}^{\mathbf{a}} t^{l(\mathbf{a})}$ and $\mathbf{x}^{\mathbf{b}} t^{l(\mathbf{b})}$ if there exists an inner normal $\omega \in \mathbb{Z}^3$, $\omega_3 > 0$, such that

$$\begin{cases} \langle (\mathbf{a}, l(\mathbf{a})), \omega \rangle = \langle (\mathbf{b}, l(\mathbf{b})), \omega \rangle \\ \langle (\mathbf{a}, l(\mathbf{a})), \omega \rangle < \langle (\mathbf{e}, l(\mathbf{e})), \omega \rangle, \quad \forall \mathbf{e} \in A \setminus \{\mathbf{a}, \mathbf{b}\}. \end{cases}$$

A Regular Mixed Subdivision



Three mixed cells:

$$(\{(1,1,2),(1,0,7)\},\{(2,2,5),(1,0,3)\}) \quad \omega=(-12,5,1) \quad V=1$$

$$(\{(1,1,2),(0,1,3)\},\{(2,2,5),(0,1,2)\}) \quad \omega=(1,-5,1) \quad V=1$$

$$(\{(1,1,2),(0,0,3)\},\{(1,0,3),(0,1,2)\}) \quad \omega=(0,1,1) \quad V=2$$

Algorithms and Software for Mixed Volumes

T. Gao and T.Y. Li: **Mixed volume computation for semi-mixed systems**. *Discrete Comput. Geom.* 29(2):257–277, 2003.

T. Gao and T.Y. Li and M. Wu: **Algorithm 846: MixedVol: a software package for mixed-volume computation**. *ACM Trans. Math. Softw.* 31(4):555–560, 2005.

T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa, and T. Mizutani: **PHoM – a polyhedral homotopy continuation method for polynomial systems**. *Computing* 73(4): 55–77, 2004.

T. Mizutani, A. Takeda, and M. Kojima: **Dynamic enumeration of all mixed cells**. *Discrete Comput. Geom.* to appear.

A well conditioned polynomial system

just one of the 11,417 start systems generated by polyhedral homotopies

12 equations, 13 distinct monomials (after division):

$$\left\{ \begin{array}{c} b_1 x_5 x_8 + b_2 x_6 x_9 = 0 \\ b_3 x_2^2 + b_4 = 0 \\ b_5 x_1 x_4 + b_6 x_2 x_5 = 0 \\ c_1^{(k)} x_1 x_4 x_7 x_{12} + c_2^{(k)} x_1 x_6 x_{10}^2 + c_3^{(k)} x_2 x_4 x_8 x_{10} + c_4^{(k)} x_2 x_4 x_{11}^2 \\ + c_5^{(k)} x_2 x_6 x_8 x_{11} + c_6^{(k)} x_3 x_4 x_9 x_{10} + c_7^{(k)} x_4^2 x_{12}^2 + c_8^{(k)} x_3 x_6 \\ + c_9^{(k)} x_4^2 + c_{10}^{(k)} x_9 = 0, \quad k = 1, 2, \ldots, 9 \end{array} \right.$$

Random coefficients chosen on the complex unit circle.

Despite the high degrees, only 100 well conditioned solutions.

Solve a "binomial" system $\mathbf{x}^A = \mathbf{b}$ via Hermite

Hermite normal form of A: $MA = U$, $\det(M) = \pm 1$,
U is upper triangular, $|\det(U)| = |\det(A)| = \#$solutions.

Let $\mathbf{x} = \mathbf{z}^M$, then $\mathbf{x}^A = \mathbf{z}^{MA} = \mathbf{z}^U$, so solve $\mathbf{z}^U = \mathbf{b}$.

$n = 2$:

$$[z_1 \quad z_2] \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = [b_1 \quad b_2].$$

$$\left\{ \begin{array}{rcl} z_1^{u_{11}} & = & b_1 \\ z_1^{u_{12}} \; z_2^{u_{22}} & = & b_2 \end{array} \right. \qquad \begin{array}{c} |b_k| = 1 \Rightarrow |z_i| = 1 \\ \textbf{numerically well conditioned} \end{array}$$

Reduce a "simplicial" system $C\mathbf{x}^A = \mathbf{b}$ via LU

$$
\begin{array}{c} C = LU \\ \text{assume } \det(C) \neq 0 \end{array}
\Rightarrow
\begin{array}{lll} (1) & LU\mathbf{y} = \mathbf{b} & \text{linear system} \\ (2) & \mathbf{x}^A = \mathbf{y} & \text{binomial system} \end{array}
$$

This is a numerically unstable algorithm!

Randomly chosen coefficients for C and $\mathbf{b}$ on complex unit circle,

but still, varying magnitudes in $\mathbf{y}$ do occur.

High powers, e.g.: 50, magnify the imbalance

$\rightarrow$ numerical underflow or overflow in binomial solver.

Separate Magnitudes from Angles

Solve $\mathbf{x}^A = \mathbf{y}$ via Hermite: $MA = U \Rightarrow \mathbf{x} = \mathbf{z}^M : \mathbf{z}^U = \mathbf{y}$.

$\mathbf{z} = |\mathbf{z}|\mathbf{e}_\mathbf{z}$, $\mathbf{e}_\mathbf{z} = \exp(i\theta_\mathbf{z})$, $\mathbf{y} = |\mathbf{y}|\mathbf{e}_\mathbf{y}$, $\mathbf{e}_\mathbf{y} = \exp(i\theta_\mathbf{y})$, $i = \sqrt{-1}$.

Solve $\mathbf{z}^U = \mathbf{y}$: $|\mathbf{z}|^U \mathbf{e}_\mathbf{z}^U = |\mathbf{y}|\mathbf{e}_\mathbf{y} \Leftrightarrow \left\{ \begin{array}{ll} \mathbf{e}_\mathbf{z}^U = \mathbf{e}_\mathbf{y} & \text{well conditioned} \\ |\mathbf{z}|^U = |\mathbf{y}| & \text{find magnitudes} \end{array} \right.$

To solve $|\mathbf{z}|^U = |\mathbf{y}|$, consider: $U \log(|\mathbf{z}|) = \log(|\mathbf{y}|)$.

Even as the magnitude of the values $\mathbf{y}$ may be extreme, $\log(|\mathbf{y}|)$ will be modest in size.

a numerically stable simplicial solver

We solve $C\mathbf{x}^A = \mathbf{b}$ by

1. LU factorization of $C \rightarrow \mathbf{x}^A = \mathbf{y}$, where $C\mathbf{y} = \mathbf{b}$.
2. Use Hermite normal form of A: $MA = U$, $\det(M) = \pm 1$, to solve binomial system $\mathbf{e}_\mathbf{z}^U = \mathbf{e}_\mathbf{y}$, $\mathbf{z} = |\mathbf{z}|\mathbf{e}_\mathbf{z}$, $\mathbf{y} = |\mathbf{y}|\mathbf{e}_\mathbf{y}$.
3. Solve upper triangular linear system $U \log(|\mathbf{z}|) = \log(|\mathbf{y}|)$.
4. Compute magnitude of $\mathbf{x} = \mathbf{z}^M$ via $\log(|\mathbf{x}|) = M \log(|\mathbf{z}|)$.
5. As $|\mathbf{e}_\mathbf{z}| = 1$, let $\mathbf{e}_\mathbf{x} = \mathbf{e}_\mathbf{z}^M$.

Even as $\mathbf{z}$ may be extreme, we deal with $|\mathbf{z}|$ at a logarithmic scale and never raise small or large number to high powers.

Only at the very end do we calculate $|\mathbf{x}| = 10^{\log(|\mathbf{x}|)}$ and $\mathbf{x} = |\mathbf{x}|\mathbf{e}_\mathbf{x}$.

Design of Serial Chains I

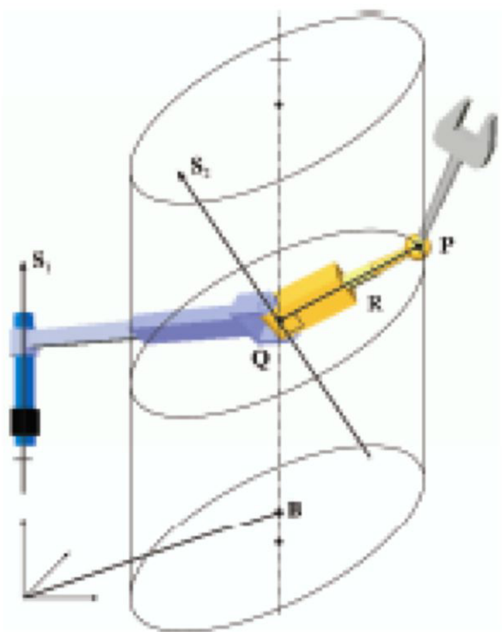


Figure 4.4: The elliptic cylinder reachable by a PRS serial chain.

H.J. Su. *Computer-Aided Constrained Robot Design Using Mechanism Synthesis Theory.* PhD thesis, University of California, Irvine, 2004.

Design of Serial Chains II

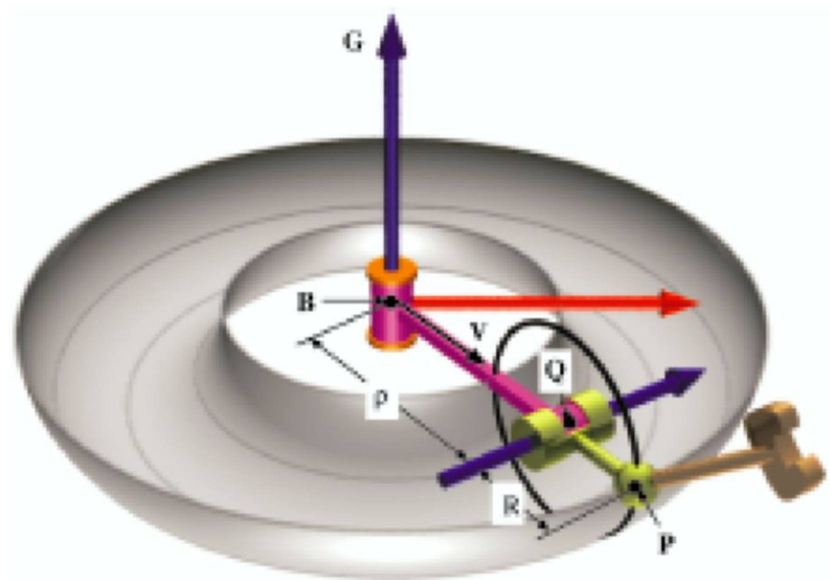


Figure 4.7: The circular torus traced by the wrist center of a "right" RRS serial chain.

H.J. Su. *Computer-Aided Constrained Robot Design Using Mechanism Synthesis Theory.* PhD thesis, University of California, Irvine, 2004.

Design of Serial Chains III

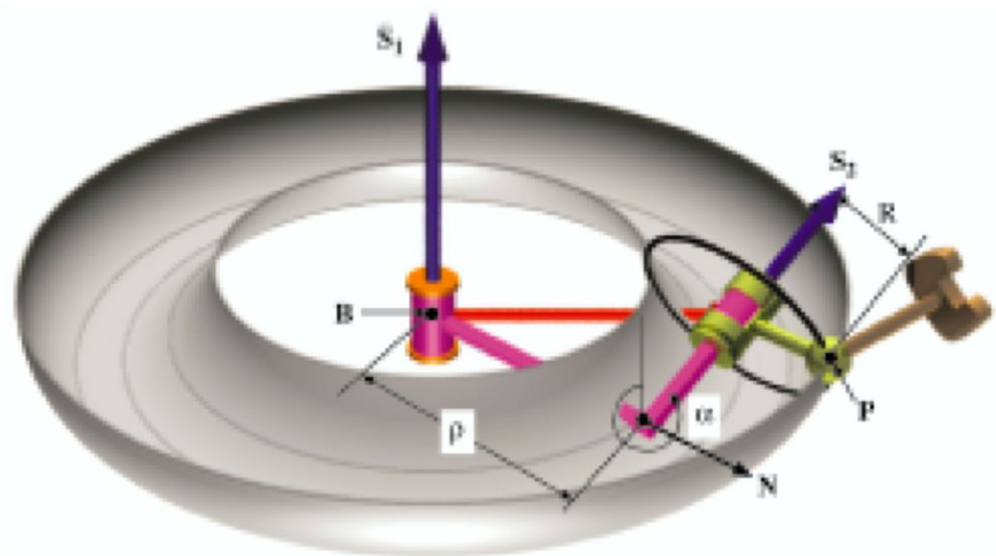


Figure 4.8: The general torus reachable by the wrist center of an RRS serial chain.

H.J. Su. *Computer-Aided Constrained Robot Design Using Mechanism Synthesis Theory*. PhD thesis, University of California, Irvine, 2004.

For more about these problems:

H.-J. Su and J.M. McCarthy: **Kinematic synthesis of RPS serial chains**. In the *Proceedings of the ASME Design Engineering Technical Conferences* (CDROM), Chicago, IL, Sep 2-6, 2003.

H.-J. Su, C.W. Wampler, and J.M. McCarthy: **Geometric design of cylindric PRS serial chains**. *ASME Journal of Mechanical Design* 126(2):269–277, 2004.

H.-J. Su, J.M. McCarthy, and L.T. Watson: **Generalized linear product homotopy algorithms and the computation of reachable surfaces**. *ASME Journal of Information and Computer Sciences in Engineering* 4(3):226–234, 2004.

Results on Mechanical Design Problems

joint with Yan Zhuang

	Bounds on #Solutions			Wall Time	
Surface	**Bézout**	**linear-product**	**Mixvol**	**our cluster**	**on argo**
elliptic cylinder	2,097,152	247,968	125,888	11h 33m	6h 12m
circular torus	2,097,152	868,352	474,112	7h 17m	4h 3m
general torus	4,194,304	448,702	226,512	14h 15m	6h 36m

Wall time for mechanism design problems on our cluster and argo.

- Compared to the linear-product bound, polyhedral homotopies cut the #paths about in half.
- The second example is easier (despite the larger #paths) because of increased sparsity, and thus lower evaluation cost.